

VIABILITY KERNELS AND CAPTURE BASINS OF SETS UNDER DIFFERENTIAL INCLUSIONS*

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Abstract. This paper provides a characterization of viability kernels and capture basins of a target viable in a constrained subset as a *unique* closed subset between the target and the constrained subset satisfying tangential conditions or, by duality, normal conditions. It is based on a method devised by H el ene Frankowska for characterizing the value function of an optimal control problem as generalized (contingent or viscosity) solutions to Hamilton–Jacobi equations. These abstract results, interesting by themselves, can be applied to epigraphs of functions or graphs of maps and happen to be very efficient for solving other problems, such as stopping time problems, dynamical games, boundary-value problems for systems of partial differential equations, and impulse and hybrid control systems, which are the topics of other companion papers.

Key words. differential inclusion, control system, viability kernel, capture basin, Hamilton–Jacobi equations, local viability, backward invariance

AMS subject classifications. 49A52, 49J24, 49K24, 49L25

PII. S036301290036968X

1. Introduction. We consider in this paper a differential inclusion $x' \in F(x)$ (summarizing the dynamics of a control system) and two subsets C and K of a finite dimensional vector space X such that $C \subset K$. Here, K is regarded as a *constrained subset* in which the solution must evolve until possibly reaching the subset C regarded as a *target*.

DEFINITION 1.1.

1. The subset $\text{Viab}_F(K)$ of initial states $x_0 \in K$ such that at least one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is viable in K for all $t \geq 0$ is called the viability kernel of K under F . A subset K is a repeller under F if its viability kernel is empty.
2. The subset $\text{Capt}_F^K(C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is called the viable-capture basin of C in K , and $\text{Capt}_F(C) := \text{Capt}_F^K(C)$ is said to be the capture basin of C .
3. The subset

$$\text{Viab}_F(K, C) := \text{Viab}_F(K \setminus C) \cup \text{Capt}_F^K(C)$$

of initial states $x_0 \in K$ such that at least one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the viability kernel of K with target C under F .

A subset $C \subset K$ is said to be isolated in K by F if it coincides with its viability kernel K with target C :

$$\text{Viab}_F(K, C) = C.$$

*Received by the editors March 20, 2000; accepted for publication (in revised form) November 20, 2000; published electronically September 28, 2001.

<http://www.siam.org/journals/sicon/40-3/36968.html>

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The subset $\text{Env}_F(C) := \text{Capt}_{-F}(C)$ is known under various names such as *invariance envelope* or *accessibility map* or *controlled map* of C . (See [45] for properties of invariance envelopes under Lipschitz maps and [6, 8, 9] for Marchaud maps.) Henri Poincaré introduced the concept of *shadow* (in French, *ombre*) of K , which is the set of initial points of K from which (all) solutions leave K in finite time. It is thus equal to the complement $K \setminus \text{Viab}_F(K)$ of the viability kernel of K , which has been introduced in the context of differential inclusions in [1]. The concept of *viability kernel with a target* by a Lipschitz set-valued map has been introduced and studied in [48], where the viability kernel algorithm designed in [50] (see also the survey [31]) has been extended for approximating the viability kernel with a target.

One could regard the viability kernel $\text{Viab}(K)$ of K as the viability kernel $\text{Viab}(K, \emptyset)$ of K with the empty set as a target:

$$\text{Viab}(K) = \text{Viab}(K, \emptyset) \text{ and } \text{Capt}^K(\emptyset) = \emptyset.$$

Therefore, the viability kernel $\text{Viab}(K, C)$ of K with target C coincides with the capture basin $\text{Capt}^K(C)$ of C viable in K whenever the viability kernel $\text{Viab}(K \setminus C)$ is empty, i.e., whenever $K \setminus C$ is a repeller:

$$\text{Viab}(K \setminus C) = \emptyset \Rightarrow \text{Viab}(K, C) = \text{Capt}^K(C).$$

This happens, in particular, when K is a repeller, or when the viability kernel $\text{Viab}(K)$ of K is contained in the target C .

Consequently, the concept of viability kernel with a target allows us to study *both the viability kernel of a closed subset and the viable-capture basin of a target*.

These subsets can be characterized in diverse ways through tangential conditions. We recall that the contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in L$ for every n .

One of our objectives is to prove the following characterizations of the viability kernels and capture basins.

THEOREM 1.2. *Let us assume that F is Marchaud and that the target $C \subset K$ and K are closed. The viability kernel $\text{Viab}_F(K, C)$ of the subset K with target C is*

1. *the largest closed subset D satisfying $C \subset D \subset K$ and*

$$D \setminus C \text{ is locally viable under } F (\forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset).$$

2. *If, furthermore, K is assumed to be backward invariant under F and F to be Lipschitz, the viability kernel $\text{Viab}_F(K, C)$ is the unique closed subset $D \subset K$ satisfying the following.*

$$(1.1) \quad \left\{ \begin{array}{l} \text{(i)} \quad D \setminus C \text{ is locally viable under } F (\forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset), \\ \text{(ii)} \quad D \text{ is backward invariant under } F (\forall x \in D, F(x) \subset -T_D(x)), \\ \text{(iii)} \quad K \setminus D \text{ is a repeller.} \end{array} \right.$$

The uniqueness properties of the viability kernel and the viable-capture basins are obtained thanks to the *Frankowska method*, consisting in introducing (local) backward invariance together with (local) forward viability of subsets. Indeed, H el ene Frankowska did point out in [39, 40] the backward invariance and local forward viability properties of the epigraph of the value function of an optimal control problem. She proved that the epigraph of the value function of an optimal control problem—assumed to be only lower semicontinuous—is backward invariant and viable under a

(natural) auxiliary system. It allowed her to characterize the value functions as unique solutions of contingent inequalities and, by duality, to obtain lower semicontinuous (or bilateral) solutions to Hamilton–Jacobi partial differential equations, obtained by other methods in [19]. (See also [18] for more details on this point of view.) Furthermore, when the value function is continuous, she proved that its epigraph is viable and its hypograph invariant [35, 36, 37, 38]. By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton–Jacobi equation in the sense of Crandall and Lions in [32]. This *epigraphical approach* in the field of Hamilton–Jacobi equations has since been taken up by other authors.

Actually, we can spare the assumption that K is backward invariant in the above theorem if we are ready to trade the property that D is backward invariant with the weaker property that D satisfies $\text{Capt}_F^K(D) = D$. Indeed, we shall derive this theorem from Theorem 4.4 below, which does not assume that K is backward viable.

Not only is the concept of the viability kernel naturally important in the framework of economic models and biological problems having motivated viability theory in the first place, but it happens that the notions of equilibria, of absorbing sets, of basins of absorption, of attractors, of “permanence,” of “fluctuation,” of “Lyapunov stability,” of optimal Lyapunov functions, and of value function of an intertemporal optimization problem as well as other dynamical features can be studied by using the concept of the *viability kernel* as a mathematical tool (see [2, 3, 4, 5] for applications and further references).

The concept of the viable-capture basin also plays a fundamental role for solving first-order partial differential equations (see [12, 13, 14, 15, 16], chapter 8 of [2], [11] without boundary conditions, and [6, 7, 8, 9] for the Dirichlet boundary-value problems for such systems). Finally, the viability kernel algorithm allows us to compute the viability kernel (see [31, 47, 50]). Nonemptiness of the viability kernel is studied in [22, 23, 24]. Extension of this concept to impulse and hybrid control systems can be found in [17].

We shall conclude this paper by describing (without proofs that will be given in a forthcoming companion paper) an application of these results to optimal discounted intertemporal control. Consider the evolution of a control system with (multivalued) feedbacks:

$$\begin{cases} \text{(i)} & x'(t) = f(x(t), u(t)), \\ \text{(ii)} & u(t) \in P(x(t)), \end{cases}$$

where the state $x(\cdot)$ ranges over a finite dimensional vector-space X and the control $u(\cdot)$ ranges over another finite dimensional vector-space \mathcal{M} . The problem is to minimize a functional of the form

$$\begin{cases} J(t, x; (x(\cdot), u(\cdot))) \\ := e^{\int_0^t \mathbf{m}(x(s), u(s)) ds} \mathbf{c}(T - t, x(t)) + \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), u(s)) ds} \mathbf{1}(x(\tau), u(\tau)) d\tau \end{cases}$$

over the set $\mathcal{S}(x)$ of solutions $(x(\cdot), u(\cdot))$ to a control system

$$V(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{S}(x)} \inf_{t \in [0, T]} J(t, x; (x(\cdot), u(\cdot)))$$

or,

$$W(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{S}(x)} \sup_{t \in [0, T]} J(t, x; (x(\cdot), u(\cdot))).$$

The connection between these problems and the basic viability theorems is simple. For instance, the epigraph of the value function is the capture basin of the epigraph of the cost function \mathbf{c} under the auxiliary control system governed by the dynamics

$$g(x, y, u) = (f(x, u), -\mathbf{m}(x, y)y - \mathbf{l}(x, u)),$$

viable in the epigraph of an adequate function. This being checked, it will be sufficient to translate the properties of capture basins stated in Theorem 1.2 in terms of value functions, the tangential conditions characterizing capture basins becoming the Hamilton–Jacobi–Bellman variational inequalities of which the value function is an (adequately generalized) solution. It is enough to observe that the contingent cone to the epigraph of a function is, by definition, the epigraph of the contingent epiderivative of this function.

When we are studying the viability kernels with targets under differential inclusions, we observe that they are not specific to differential inclusions. They involve only few properties¹ of the solution map \mathcal{S} associating with any initial state x the set $\mathcal{S}(x)$ of pairs $t \mapsto (x(t), u(t))$ that are solutions to the above control system starting at x at initial time 0. These properties of the solution map are common to other control problems, such as

1. control problems with memory (see the contributions of [42, 43], some of them being presented in [2])—previously known under the name of functional control problems, the new fashion calling them “path dependent control systems,”
2. parabolic type partial differential inclusions (see the contributions of [51, 52, 53, 54, 55], some of them being presented in [2])—also known as distributed control systems;
3. “mutational equations” governing the evolution in metric spaces, including “morphological equations” governing the evolution of sets (see [4], for instance).

Although these problems are not covered in this paper by lack of place, we shall make another step in abstraction by gathering these common properties of the solution map under the name of *evolutionary systems* and study the properties of viability kernels with targets in this general framework. In the case of differential inclusions, we shall use the viability and invariance theorems for characterizing them in terms of tangential conditions.

The paper is organized as follows. Section 2 introduces evolutionary systems. The third section defines hitting and exit functions. Viability kernels and capture basins are defined and characterized in section 4 for general evolutionary systems. Their characterizations in terms of tangential conditions or, by duality, in terms of normal conditions, are provided in the fifth section. The sixth provides useful stability results. The last section summarizes the applications of the above theorems to optimal control and stopping time problems.

2. Evolutionary systems.

2.1. Definition of evolutionary systems. The following results dealing with viability kernel and capture basins are valid for any *evolutionary system* described by a set-valued map \mathcal{S} mapping some topological space X (most often, a topological

¹These are the translation and concatenation properties of the set-valued map $x \rightsquigarrow \mathcal{S}(x)$, as well as continuity properties of this set-valued map.

vector-space) to the space $\mathcal{C}(0, \infty; X)$ of continuous functions $x(\cdot)$ from \mathbf{R}_+ to X , supplied with the topology of uniform convergence on compact intervals.

It can be the solution map associated with a differential inclusion $x' \in F(x)$ on a finite dimensional vector space X , with a differential inclusion with memory $x'(t) \in F(T(t)x)$ or with a mutational equation $\dot{x} \ni f(x)$ on metric spaces.

DEFINITION 2.1. *An evolutionary system is a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ satisfying the following.*

1. *The translation property. Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the function $y(\cdot)$ defined by $y(t) := x(t + T)$ is a solution $y(\cdot) \in \mathcal{S}(x(T))$ starting at $x(T)$.*
2. *The concatenation property. Let $x(\cdot) \in \mathcal{S}(x)$ and $T \geq 0$. Then for every $y(\cdot) \in \mathcal{S}(x(T))$, the function $z(\cdot)$, defined by*

$$z(t) := \begin{cases} x(t) & \text{if } t \in [0, T], \\ y(t - T) & \text{if } t \geq T, \end{cases}$$

belongs to $\mathcal{S}(x)$.

We shall associate with \mathcal{S} its backward evolutionary system $\mathcal{S}_- : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ defined by $y(\cdot) \in \mathcal{S}_-(x)$ if and only if there exists a solution $z(\cdot) \in \mathcal{S}(x)$ such that for every $T \geq 0$, the function $x(\cdot)$, defined by

$$x(t) := \begin{cases} y(T - t) & \text{if } t \in [0, T], \\ z(t - T) & \text{if } t \geq T, \end{cases}$$

belongs to $\mathcal{S}(x)$.

We observe that $\mathcal{S}_- \circ \mathcal{S}_- = \mathcal{S}$.

The viability and capturability issues use the notion of evolutions viable in a subset.

DEFINITION 2.2. *Let $K \subset X$ be a subset of X . A function $t \in [0, T] \mapsto x(t) \in X$ is said to be viable in K on $[0, T]$ if*

$$\forall t \in [0, T], x(t) \in K,$$

and viable in K if $T = +\infty$.

The following results dealing with these issues shall use only the translation and concatenation properties and topological properties such that the upper semicompactness² and/or lower semicontinuity of the evolutionary system $\mathcal{S} : x \in X \rightsquigarrow \mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$.

Before proceeding further, let us recall that differential inclusions provide examples of evolutionary systems.

2.2. Evolutionary systems associated with differential inclusions. Let $X := \mathbf{R}^n$ be a finite dimensional vector space, and let $F : X \rightsquigarrow X$ be a strict³

²A set-valued map $F : X \rightsquigarrow Y$ is said to be *upper semicompact* at x if for every sequence x_n converging to x and for every sequence $y_n \in F(x_n)$, there exists a subsequence y_{n_p} converging to some $y \in F(x)$. It is said to be *lower semicontinuous* at x if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in \text{Dom}(F)$ converging to x , there exists a sequence of elements $y_n \in F(x_n)$ converging to y .

³This means that for every $x \in X$, $F(x) \neq \emptyset$. We denote by

$$\text{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

the *graph* of a set-valued map $F : X \rightsquigarrow Y$ and by $\text{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$ its *domain*.

set-valued map. We denote by $\mathcal{S}_F(x) \subset \mathcal{C}(0, \infty; X)$ the set of *absolutely continuous functions* $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)),$$

starting at time 0 at x : $x(0) = x$. The set-valued map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the *solution map* (or the set-valued flow) associated with F .

Without assumptions, the solution map \mathcal{S}_F may have empty values. However, whenever the solution map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ associated with the differential inclusion $x' \in F(x)$ is strict, it obviously satisfies the *translation property* and the *concatenation property*.

One can also observe that the backward evolutionary system \mathcal{S}_{F_-} is the solution map \mathcal{S}_{-F} associated with $-F$.

2.2.1. Marchaud differential inclusions.

DEFINITION 2.3 (Marchaud map). *We shall say that F is a Marchaud map if*

$$\left\{ \begin{array}{l} \text{(i)} \quad \text{the graph and the domain of } F \text{ are nonempty and closed,} \\ \text{(ii)} \quad \text{the values } F(x) \text{ of } F \text{ are convex,} \\ \text{(iii)} \quad \text{the growth of } F \text{ is linear:} \\ \quad \exists c > 0 \mid \forall x \in X, \quad \|F(x)\| := \sup_{v \in F(x)} \|v\| \leq c(\|x\| + 1). \end{array} \right.$$

We recall the following version of the important Theorem 3.5.2 of [2] stating that the solution map is strict and upper semicontact.

STATEMENT 1. *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map \mathcal{S}_F is an upper semicontact evolutionary system from X into the space of continuous functions supplied with the topology of uniform convergence on compact intervals.*

2.2.2. Lipschitz differential inclusions.

DEFINITION 2.4. *The set-valued map F is said to be Lipschitz if there exists a constant $\lambda > 0$ such that*

$$\forall x, y \in X, \quad F(x) \subset F(y) + B(0, \lambda\|x - y\|).$$

The Filippov theorem (see Theorem 5.3.1 of [2]) implies that whenever F is Lipschitz, the associated evolutionary system is lower semicontinuous.

STATEMENT 2. *Assume that $F : X \rightsquigarrow X$ is Lipschitz. Then the solution map \mathcal{S}_F is a lower semicontinuous evolutionary system from X into the space of continuous functions supplied with the topology of uniform convergence on compact intervals.*

3. Exit and hitting time functions. We shall associate with an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ the concepts of upper exit time function of a subset K and the lower hitting function (or minimal time function) of a target and study their continuity (actually, semicontinuity) properties.

DEFINITION 3.1. *Let $K \subset X$ be a subset. The functional $\tau_K : \mathcal{C}(0, \infty; X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its exit time $\tau_K(x(\cdot))$ defined by*

$$\tau_K(x(\cdot)) := \inf \{t \in [0, \infty[\mid x(t) \notin K\}$$

is called the exit functional.

Let $C \subset K$ be a target. We introduce the (constrained) hitting functional $\varpi_{(K,C)}$ defined by

$$\varpi_{(K,C)}(x(\cdot)) := \inf \{t \geq 0 \mid x(t) \in C \text{ and } \forall s \in [0, t], x(s) \in K \}$$

associating with $x(\cdot)$ its hitting time, introduced in [29]). When $K := X$, we set

$$\varpi_C(x(\cdot)) = \varpi_{(X,C)}(x(\cdot)) : \mathcal{C}(0, \infty; X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$$

and call it the hitting functional (or minimal time functional).

We use the convention $\inf\{\emptyset\} := +\infty$, and we observe that

$$\text{if } \varpi_{(X,C)}(x(\cdot)) < +\infty, \text{ then } \varpi_C(x(\cdot)) = \varpi_{(X,C)}(x(\cdot)) \leq \tau_C(x(\cdot)).$$

We also note that

$$(3.1) \quad \forall s \in [0, \varpi_C(x(\cdot))], \quad \varpi_C(x(\cdot + s)) = \varpi_C(x(\cdot)) - s$$

and that if $K_1 \subset K_2$ and $D_1 \supset D_2$, then $\varpi_{(K_1,D_1)}(x(\cdot)) \leq \varpi_{(K_2,D_2)}(x(\cdot))$. Let us point out that

$$\varpi_{\bigcup_{i=1}^n D_i}(x(\cdot)) = \min_{i=1, \dots, n} \varpi_{D_i}(x(\cdot)).$$

Therefore,

$$\forall x \in K \setminus C, \quad \tau_{K \setminus C}(x(\cdot)) = \min(\varpi_C(x(\cdot)), \tau_K(x(\cdot)))$$

since

$$\tau_{K \setminus C}(x) = \varpi_{X \setminus (K \setminus C)}(x(\cdot)) = \varpi_{C \cup (X \setminus K)} = \min(\varpi_C(x(\cdot)), \varpi_{X \setminus K}(x(\cdot))).$$

DEFINITION 3.2. Consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$. Let $C \subset K$ and K be two subsets.

The function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\sharp(x) := \sup_{x(\cdot) \in \mathcal{S}(x)} \tau_K(x(\cdot))$$

is called the upper exit function.

The function $\varpi_{(K,C)}^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\varpi_{(K,C)}^\flat(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$

is called the lower constrained hitting function, and the function

$$\varpi_C^\flat(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \varpi_C(x(\cdot))$$

is called the lower hitting function.

STATEMENT 3. Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a strict upper semicompact map, and let C and K be two closed subsets such that $C \subset K$. Then the hitting function $\varpi_{(K,C)}^\flat$ is lower semicontinuous and the exit function τ_K^\sharp is upper semicontinuous. Furthermore, for any $x \in \text{Dom}(\varpi_{(K,C)}^\flat)$, there exists at least one solution $x^\flat(\cdot) \in \mathcal{S}(x)$ which hits C as soon as possible before possibly leaving K ,

$$\varpi_{(K,C)}^\flat(x) = \varpi_{(K,C)}(x^\flat(\cdot)),$$

and for any $x \in \text{Dom}(\tau_K^\sharp)$, there exists at least one solution $x^\sharp(\cdot) \in \mathcal{S}(x)$ which remains viable in K as long as possible:

$$\tau_K^\sharp(x) = \tau_K(x^\sharp(\cdot)).$$

This statement is a consequence of the more general Theorem 6.2 dealing with upper hypolimits of upper exit functions and epilimits of lower constrained epifunctions of subsets that is proved below. See also [29, 30].

4. Viability kernels and capture basins. We shall answer in this section questions such as the following.

- Starting from K , is it possible to remain viable in K (as long as possible)?
- Starting outside of a target $C \subset K$, is it possible to reach it (as fast as possible) while being viable in the subset K ?

These two very natural questions lead to the introduction of the following concepts.

4.1. Viability kernels with targets. We now define the viability kernels, the capture basins, and the viable-capture basins of a subset under a set-valued map.

DEFINITION 4.1. *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a set-valued evolutionary system, and let $C \subset K \subset X$ be two subsets, C being regarded as a target and K as a constrained set.*

1. *The subset K is said to be locally viable under \mathcal{S} if from any initial state $x \in K$ starts at least one solution viable in K on a nonempty interval and viable if this solution is viable on $[0, +\infty[$. We shall say that K captures the target C if from any initial state $x \in K$ starts at least one solution viable in K until it may reach the target C , and we say that K finitely captures the target C if it reaches it in finite time.*
2. *The subset $\text{Viab}(K, C)$ of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the viability kernel of K with target C under \mathcal{S} . A subset $C \subset K$ is said to be isolated in K by \mathcal{S} if it coincides with its viability kernel:*

$$\text{Viab}(K, C) = C.$$

3. *The subset $\text{Capt}^K(C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the viable-capture basin of C in K , and*

$$\text{Capt}(C) := \text{Capt}^X(C)$$

is said to be the capture basin of C .

4. *When the target $C = \emptyset$ is the empty set, we set*

$$\text{Viab}(K) := \text{Viab}(K, \emptyset) \text{ and } \text{Capt}^K(\emptyset) = \emptyset,$$

and we say that $\text{Viab}(K)$ is the viability kernel of K . A subset K is a repeller under \mathcal{S} if its viability kernel is empty, or, equivalently, if the empty set is isolated in K .

In other words, the viability kernel $\text{Viab}(K)$ is the subset of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$. Furthermore, we observe that

$$(4.1) \quad \text{Viab}(K, C) = \text{Viab}(K \setminus C) \cup \text{Capt}^K(C).$$

Therefore, the viability kernel $\text{Viab}(K, C)$ of K with target C coincides with the capture basin $\text{Capt}^K(C)$ of C viable in K whenever the viability kernel $\text{Viab}(K \setminus C)$ is empty, i.e., whenever $K \setminus C$ is a repeller:

$$(4.2) \quad \text{Viab}(K \setminus C) = \emptyset \Rightarrow \text{Viab}(K, C) = \text{Capt}^K(C).$$

Consequently, the concept of the viability kernel with a target allows us to study both the viability kernel of a closed subset and the viable-capture basin of a target.

Remark. If subsets K_i capture a given target $C \subset K_i$ for all $i \in I$, so does their union $\bigcup_{i \in I} K_i$. However, the intersection of two subsets K_1 and K_2 capturing a same target C does not necessarily capture C , since starting from a state of $K_1 \cap K_2$, there may exist two different solutions that are viable in K_1 or in K_2 but no solution viable in $K_1 \cap K_2$. \square

We observe that the *viability kernel* is characterized by

$$\text{Viab}(K) := \{x \in K \mid \tau_K^\sharp(x) = +\infty\}$$

and that the *viable-capture basin*

$$\text{Capt}^K(C) := \{x \in K \mid \varpi_{(K,C)}^b(x) < +\infty\}$$

is the domain of the constrained hitting function $\varpi_{(K,C)}^b$.

To say that K is a repeller under \mathcal{S} amounts to saying that the exit function τ_K^\sharp is finite on K , and to say that $K \setminus C$ is a repeller amounts to saying that all solutions $x(\cdot) \in \mathcal{S}(x)$ starting from $x \in K \setminus C$ reach C or leave K in finite time, i.e., satisfy $\tau_{K \setminus C}(x(\cdot)) = \min(\varpi_C(x(\cdot)), \tau_K(x(\cdot))) < +\infty$.

The viability kernel $\text{Viab}(K, C)$ of K with target C captures C .

PROPOSITION 4.2. *The viability kernel $\text{Viab}(K, C)$ of K with target C is the largest subset of K capturing C , and the viability kernel $\text{Viab}(K)$ of K is the largest viable subset of K .*

Proof. First, any subset D such that $C \subset D \subset K$ capturing C is obviously contained in the viability kernel $\text{Viab}(K, C)$ with target C .

For proving that the viability kernel $\text{Viab}(K, C)$ with target C captures the target C , take $x_0 \in \text{Viab}(K, C)$, and prove that there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $\text{Viab}(K, C)$ until it possibly reaches C . Indeed, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ viable in K until some time $T \geq 0$, either finite when it reaches C or infinite. Then for all $t \in [0, T[$, the function $y(\cdot)$ defined by $y(\tau) := x(t + \tau)$ is a solution $y(\cdot) \in \mathcal{S}(x(t))$ starting at $x(t)$ and viable in K until it reaches C at time $T - t$. Hence $x(t)$ does belong to $\text{Viab}(K, C)$ for every $t \in [0, T[$. \square

Furthermore, we derive from Proposition 4.2 and Theorem 6.4 below the following proposition.

PROPOSITION 4.3. *Let us assume that the map \mathcal{S} is upper semicompact and that $C \subset K$ and K are closed. Then the viability kernel $\text{Viab}(K, C)$ with a target is the largest closed subset of K capturing C , and the viability kernel $\text{Viab}(K)$ is the largest viable closed subset of K .*

4.2. Characterization of the viability kernel with a target. The first characterization is stated in the following theorem.

THEOREM 4.4. *Let us assume that \mathcal{S} is upper semicompact and that the subsets $C \subset K$ and K are closed. The viability kernel $\text{Viab}(K, C)$ of a subset K with target C under \mathcal{S} is the unique closed subset satisfying $C \subset D \subset K$, and*

$$(4.3) \quad \begin{cases} \text{(i)} & D \setminus C \text{ is locally viable under } \mathcal{S}, \\ \text{(ii)} & D \text{ is isolated in } K \text{ by } \mathcal{S} \text{ (} \text{Viab}(K, D) = D \text{)}. \end{cases}$$

It follows from Theorem 4.6 characterizing the viability kernel as the largest closed subset $D \subset K$ such that $D \setminus C$ is locally viable and from Theorem 4.7 characterizing the viability kernel as the smallest subset D isolated in K .

We begin with necessary conditions.

PROPOSITION 4.5. *Let us consider a closed subset C of K . Then the following hold.*

1. *If $D \supset C$ captures C , then $D \setminus C$ is locally viable under \mathcal{S} .*
2. *If $D_1 \supset C$ captures C and $D_2 \supset D_1$ captures D_1 , then D_2 captures C (transitivity of the capturability property).*

Consequently, the viability kernel $\text{Viab}(K, C)$ of a subset K with target C under \mathcal{S} satisfies the following properties:

- (i) $\text{Viab}(K, C) \setminus C$ is locally viable and $\text{Viab}(K)$ is viable under \mathcal{S} ,
- (ii) $\text{Viab}(K, C)$ is isolated in K by \mathcal{S} ($\text{Viab}(K, \text{Viab}(K, C)) = \text{Viab}(K, C)$).

Proof. For proving the first statement, take $x_0 \in D \setminus C$, and prove that there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $D \setminus C$ on a nonempty interval. Indeed, since C is closed, there exists $\eta > 0$ such that $B(x_0, \eta) \cap C = \emptyset$, so that $x(t) \in B(x_0, \eta) \cap D \subset D \setminus C$ on some nonempty interval.

For proving that D_2 captures C , take any $x_0 \in D_2$. There exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ viable in D_2 forever or else, until it possibly reaches the subset D_1 of D_2 at some finite time $T > 0$ at $x(T) \in D_1$. In this case, for any $t \geq T$, $x(t)$ remains in D_1 , and thus, in D_2 , until it possibly reaches C . Hence D_2 captures C .

In particular, if we take $D_1 := \text{Viab}(K, C)$ and $D_2 := \text{Viab}(K, \text{Viab}(K, C))$, we infer that $D_1 = D_2$ since D_1 is the largest subset of K capturing the target C . \square

We now proceed with the proof of the sufficiency.

THEOREM 4.6. *Assume that \mathcal{S} is upper semicompact. Let $C \subset K$ be closed subsets.*

Then the viability kernel $\text{Viab}(K, C)$ of K with target C under \mathcal{S} is the largest closed subset $D \subset K$ containing C such that $D \setminus C$ is locally viable.

In particular, K captures C if and only if $K \setminus C$ is locally viable.

Proof. When $C = \emptyset$, this is Proposition 4.3. Otherwise, Theorem 6.4 and Proposition 4.5 imply that the viability kernel $\text{Viab}(K, C)$ of K with target C under \mathcal{S} is a closed subset such that $\text{Viab}(K, C) \setminus C$ is locally viable.

Let $D \subset K$ containing C such that $D \setminus C$ is locally viable. Since $C \subset \text{Viab}(K, C)$, let us take x in $D \setminus C$ and show that it belongs to $\text{Viab}(K, C)$. Either there exists a solution $x(\cdot) \in \mathcal{S}(x)$ viable in $D \subset K$ forever or, if not, by Statement 3, there exists a solution $x^\sharp(\cdot) \in \mathcal{S}(x)$ that maximizes $\tau_D(x(\cdot))$,

$$\tau_D^\sharp(x) := \sup_{x(\cdot) \in \mathcal{S}(x)} \tau_D(x(\cdot)) = \tau_D(x^\sharp(\cdot)),$$

and thus that leaves D at $x^\sharp := x^\sharp(\tau_D^\sharp(x)) \in D$. Actually, this point belongs to C .

Otherwise, since $D \setminus C$ is locally viable, one could associate with $x^\sharp \in D \setminus C$ a solution $y(\cdot) \in \mathcal{S}(x^\sharp)$ and $T > 0$ such that $y(\tau) \in D \setminus C$ for all $\tau \in [0, T]$. Concatenating this solution to $x^\sharp(\cdot)$, we obtain a solution viable in D on an interval $[0, \tau_D^\sharp(x) + T]$, which contradicts the definition of $x^\sharp(\cdot)$. \square

THEOREM 4.7. *Let $C \subset K$. Then the viability kernel $\text{Viab}(K, C)$ is the smallest subset D between C and K isolated in K by \mathcal{S} .*

Proof. Proposition 4.5 implies that the viability kernel $\text{Viab}(K, C)$ is isolated in K by \mathcal{S} . Conversely, since D is isolated in K by \mathcal{S} , we infer that $\text{Viab}(K, C) \subset \text{Viab}(K, D) = D$. \square

4.3. Isolated subsets. We need to characterize further isolated subsets for enriching the above characterization theorem.

First, we point out the following.

PROPOSITION 4.8. *Let C and K be two subsets such that $C \subset K$. Then the following properties are equivalent.*

1. C is isolated in K by \mathcal{S} : $\text{Viab}(K, C) = C$.
2. For all $x \in K \setminus C$, all solutions reach $X \setminus K$ in finite time before (possibly) hitting C .
3. $\text{Viab}(K) = \text{Viab}(C)$, and $\text{Capt}^K(C) = C$.
4. $K \setminus C$ is a repeller and $\text{Capt}^K(C) = C$.

Isolated subsets enjoy local backward invariance properties discovered by Hélène Frankowska in her studies of Hamilton–Jacobi equations associated with value functions of optimal control problems under state constraints that play a crucial role in the characterization of viability kernels with a target. Indeed, there is a close connection between isolation in K and local backward invariance relatively to K .

DEFINITION 4.9. *We shall say that a subset $C \subset K$ is locally backward invariant relatively to K under \mathcal{S} if for every $x \in C$, all backward solutions starting from x and viable in K on an interval $[0, T]$ are viable in C on $[0, T]$, i.e., if for every $x \in C$, for every $t_0 \in]0, +\infty[$, and for all solutions $x(\cdot)$ arriving at x at time t_0 such that there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, then $x(\cdot)$ is viable in C on the same interval.*

Naturally, if $C \subset K$ is locally backward invariant, it remains locally backward invariant relatively to K . If K is itself locally backward invariant, any subset locally backward invariant relatively to K is locally backward invariant.

If $C \subset K$ is locally backward invariant relatively to K , then $C \cap \text{Int}(K)$ is locally backward invariant, and from any $x \in C \cap \partial K$, all backward solutions $y(\cdot) \in \mathcal{S}_-(x)$ satisfy

$$\left\{ \begin{array}{l} \text{either } \exists T > 0 \text{ such that } \forall t \in [0, T], x(t) \in C, \\ \text{or } \exists t_n \rightarrow 0+ \mid y(t_n) \in X \setminus K. \end{array} \right.$$

THEOREM 4.10. *A closed subset $C \subset K$ is locally backward invariant relatively to K if and only if $\text{Capt}^K(C) = C$.*

Proof. Assume that C is locally backward invariant relatively to K , and consider $x \in \text{Capt}^K(C) \setminus C$. There exists a solution $x(\cdot) \in \mathcal{S}(x)$ viable in K until it reaches C at time $T := \varpi_C(x(\cdot)) \geq 0$ at $c = x(\varpi_C(x(\cdot)))$. Since C is closed, then $T > 0$ is positive. Let $z(\cdot) \in \mathcal{S}_-(x)$, and let $y(\cdot)$ be the function defined by

$$y(t) := \begin{cases} x(T - t) & \text{if } t \in [0, T], \\ z(t - T) & \text{if } t \geq T. \end{cases}$$

Then $y(\cdot) \in \mathcal{S}_-(c)$ and is viable in K on the interval $[0, \varpi_C(x(\cdot))]$. Since C is assumed to be locally backward invariant relatively to K , then $y(t) \in C$ for all $t \in [0, \varpi_C(x(\cdot))]$, and, in particular, $y(T) = x$ belongs to C . We have obtained a contradiction.

The converse statement follows from the next theorem.

PROPOSITION 4.11. *The viability kernel $\text{Viab}(K, C)$ of K with a target $C \subset K$ and the viable-capture basin $\text{Capt}^K(C)$ are locally backward invariant relatively to K . Consequently, every subset $C \subset K$ isolated in K is locally backward invariant relatively to K .*

Proof. Let us consider $x \in \text{Viab}(K, C)$ and $z(\cdot) \in \mathcal{S}(x)$ viable in K until it possibly reaches C . Let us consider a backward solution $y(\cdot) \in \mathcal{S}_-(x)$ viable in K

such that $\tau_K(y(\cdot)) > 0$. (This is always the case whenever $x \in \text{Int}(K)$.) For every $T \in [0, \tau_K(y(\cdot))]$, we associate with it the solution $x(\cdot) \in \mathcal{S}(x(T))$ defined by

$$x(t) := \begin{cases} y(T-t) & \text{if } t \in [0, T], \\ z(t-T) & \text{if } t \geq T, \end{cases}$$

starting at $y(T) \in K$ viable in K until it possibly reaches C . This means that $y(T) \in \text{Viab}(K, C)$ for every $T \in [0, \tau_K(y(\cdot))]$, i.e., that the backward solution $y(\cdot) \in \mathcal{S}_-(x)$ is viable in $\text{Viab}(K, C)$ on the interval $[0, \tau_K(y(\cdot))]$.

In other words, for every $x \in \text{Viab}(K, C)$, every backward solution viable in K on some time interval is actually viable in $x \in \text{Viab}(K, C)$ on the same interval. \square

We derive the following characterization.

PROPOSITION 4.12. *Let us consider a closed subset $C \subset K$. Then C is isolated in K by \mathcal{S} if and only if*

1. C is locally backward invariant relatively to K , and
2. $K \setminus C$ is a repeller.

Putting together these results, we obtain Theorem 4.13, characterizing viability kernels with targets, and Theorem 4.14, characterizing capture basins.

THEOREM 4.13. *Let us assume that \mathcal{S} is upper semicompact and that the subsets $C \subset K$ and K are closed. The viability kernel $\text{Viab}(K, C)$ of a subset K with target C under \mathcal{S} is the unique closed subset satisfying $C \subset D \subset K$ and*

$$(4.4) \quad \begin{cases} \text{(i)} & D \setminus C \text{ is locally viable under } \mathcal{S}, \\ \text{(ii)} & D \text{ is locally backward invariant relatively to } K \text{ under } \mathcal{S}, \\ \text{(iii)} & K \setminus D \text{ is a repeller under } \mathcal{S}. \end{cases}$$

Therefore, Theorem 4.13 and (4.2) imply that when $K \setminus C$ is a repeller, the above theorem implies a characterization of the viable-capture basins.

THEOREM 4.14. *Let us assume that \mathcal{S} is upper semicompact and that a closed subset $C \subset K$ satisfies the property*

$$(4.5) \quad \text{Viab}(K \setminus C) = \emptyset.$$

Then the viable-capture basin $\text{Capt}^K(C)$ is the unique closed subset D satisfying $C \subset D \subset K$ and

$$(4.6) \quad \begin{cases} \text{(i)} & D \setminus C \text{ is locally viable under } \mathcal{S}, \\ \text{(ii)} & D \text{ is locally backward invariant relatively to } K \text{ under } \mathcal{S}. \end{cases}$$

4.4. Viability kernels of backward invariant sets. We obtain further properties when K is backward invariant under \mathcal{S} . To begin with, *the capture basin $\text{Capt}(C) := \text{Capt}^X(C)$ is contained in K and equal to $\text{Capt}^K(C)$, so that*

$$\text{Viab}(K, C) = \text{Viab}(K \setminus C) \cup \text{Capt}(C).$$

THEOREM 4.15. *A subset K is invariant under a set-valued map \mathcal{S} if and only if its complement $X \setminus K$ is backward invariant under \mathcal{S} .*

Proof. To say that K is not invariant under \mathcal{S} amounts to saying that there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ and $T > 0$ such that

$$x(0) \in K \ \& \ x(T) \in X \setminus K.$$

Let $z(\cdot) \in \mathcal{S}_-(x_0)$ be a backward solution, and define the function $y(\cdot)$ by

$$y(t) = \begin{cases} x(T-t) & \text{if } t \in [0, T], \\ z(t-T) & \text{if } t \geq T. \end{cases}$$

It is a backward solution starting at $y(0) = x(T) \in X \setminus K$ and satisfying $y(T) = x_0 \in K$. This amounts to saying that the complement $X \setminus K$ of K is not backward invariant. \square

We then derive the following theorem.

THEOREM 4.16. *Assume that \mathcal{S} is upper semicompact, that $C \subset K$ and K are closed, and that K is backward invariant under \mathcal{S} . Then the viability kernel $\text{Viab}(K, C)$ of K with target C under \mathcal{S} is the unique closed subset D satisfying $C \subset D \subset K$ and*

- (i) $D \setminus C$ is locally viable under \mathcal{S} ,
- (ii) D is backward invariant under \mathcal{S} (or, equivalently, $X \setminus D$ is invariant under \mathcal{S}),
- (iii) $K \setminus D$ is a repeller under \mathcal{S} .

Proof. To say that K is backward invariant amounts to saying that the complement of K is invariant thanks to Theorem 4.15. Therefore, $\text{Viab}(K, C)$ being isolated, all solutions starting from $K \setminus \text{Viab}(K, C)$ leave K in finite time before possibly hitting C . Actually, they never reach C because the complement $X \setminus K$ is invariant. Hence we have checked that the complement $X \setminus \text{Viab}(K, C)$ of the viability kernel of K with target C is invariant. Theorem 4.15 implies that the viability kernel $\text{Viab}(K, C)$ of K with target C is backward invariant. \square

4.5. The barrier property. The boundary of the viability kernel satisfies the barrier property.

DEFINITION 4.17. *If $D \subset K$, the boundary $\partial_K(D)$ of D relative to K is the subset*

$$\partial_K(D) := \overline{D} \cap \overline{(K \setminus D)},$$

and the subset $\partial D := \partial_X(D)$ is called the boundary of D . We shall say that a subset $D \subset K$ enjoys the barrier property relative to K under \mathcal{S} if its boundary $\partial_K(D)$ of D relative to K is locally invariant with respect to D : For every $x \in \partial_K(D)$, all solutions starting from x viable in D are actually viable in the boundary $\partial_K(D)$ of D relative to K until they reach the boundary of K .

We see at once that

$$\partial_K(D) \cap \text{Int}(K) = \partial D \cap \text{Int}(K)$$

and that

$$\text{if } D \subset \text{Int}(K), \text{ then } \partial_K(D) = \partial D.$$

Remark on the barrier property. The ‘‘barrier property’’ of the viability kernel of a closed subset has been discovered by Marc Quincampoix in [46] and generalized by Pierre Cardaliaguet in [25, 26, 27, 28] for differential games. It plays an important role in control theory and the theory of differential games, because every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or, equivalently, no solution starting from outside the viability

kernel can cross its boundary. Such solutions can remain only on the boundary of the viability kernel, or leave it.

This is a semipermeability property of the viability kernel, which is very important in terms of interpretation. Viability is indeed a very fragile property, which cannot be reestablished from the outside. *In other words, love it or leave it.* \square

THEOREM 4.18. *If \mathcal{S} is upper semicompact and lower semicontinuous, then the viability kernel $\text{Viab}(K, C)$ of a closed subset K with a closed target $C \subset K$ under \mathcal{S} enjoys the barrier property relative to K .*

Proof. Let x belong to $\partial_K(\text{Viab}(K, C))$, and let $x(\cdot) \in \mathcal{S}(x)$ be a solution viable in K forever ($\varpi_{(K,C)}^\sharp(x(\cdot)) = +\infty$) or until it reaches C at finite time $\varpi_{(K,C)}^\sharp(x(\cdot)) < +\infty$. Let $x_n \in K \setminus \text{Viab}(K, C)$ converge to x . Since \mathcal{S} is lower semicontinuous by Statement 3, there exists a solution $x_n(\cdot) \in \mathcal{S}(x_n)$ converging to $x(\cdot)$ uniformly over compact intervals. Since $\text{Viab}(K, C)$ is isolated, we know that for every n ,

$$\forall t \leq \tau_K(x_n(\cdot)), \quad x_n(t) \in K \setminus \text{Viab}(K, C).$$

Since $\varpi_{\partial K}(x_n(\cdot)) \leq \tau_K(x_n(\cdot))$ and since the functional $x(\cdot) \mapsto \varpi_{\partial K}(x(\cdot))$ is lower semicontinuous, we infer that for every $t < \varpi_{\partial K}(x(\cdot))$ there exists $N > 0$ such that for any $n \geq N$,

$$t < \varpi_{\partial K}(x_n(\cdot)) \leq \tau_K(x_n(\cdot)),$$

and thus that $x_n(t)$ belongs to $K \setminus \text{Viab}(K, C)$. Taking the limit, we infer that $x(t)$ belongs to $\overline{K \setminus \text{Viab}(K, C)}$. Hence $x(t)$ belongs to the boundary $\partial_K(\text{Viab}(K, C))$ of the viability kernel relative to K whenever $t < \varpi_{\partial K}(x(\cdot))$. \square

5. Frankowska’s and viscosity property of viability kernels. We restrict now our study to the case of viability kernels with targets under evolutionary systems defined by the solution maps of differential inclusions $x' \in F(x)$. In this case, the viability and invariance theorems characterize the viability and invariance properties by tangential conditions, as it was mentioned in the introduction, or, equivalently,⁴ by normal conditions. We recall that the (regular) *normal cone*⁵ $N_L(x) := T_L(x)^\circ$ to a subset L at $x \in L$ is the polar cone to the contingent cone $T_L(x)$ (see, for instance, [10] or [49] for more details). We denote by

$$\forall p \in X^*, \quad \sigma(K, p) := \sup_{x \in K} \langle p, x \rangle$$

the *support function* of K .

$$\forall x \in K \setminus R^{-1}(K), \quad F(x) \cap T_K(x) \neq \emptyset.$$

5.1. The basic viability and invariance theorems.

STATEMENT 4. *Assume that F is Marchaud. The two following statements hold true.*

1. *If K is closed, then K is (globally) viable under F if and only if*

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset,$$

⁴The equivalence between tangential and normal conditions was first noticed in a different context in [41]. A simpler proof of this fact was given by H el ene Frankowska and appeared in [14] and in Theorem 3.2.4 of [2]. Other proofs were provided later in [24] and [57].

⁵One can replace if wished this normal cone $N_L(x)$ by the smaller subset $x - \Pi_L(x)$ of *normal proximals* to L at x , where $\Pi_L(x)$ denotes the set of best approximations of x by elements of L .

or, equivalently, in dual form, if and only if

$$\forall x \in K, \forall p \in N_K(x), \sigma(F(x), -p) \geq 0.$$

2. If $C \subset K$ is closed, then K captures C by F if and only if

$$\forall x \in K \setminus C, F(x) \cap T_K(x) \neq \emptyset,$$

or, equivalently, in dual form, if and only if

$$\forall x \in K \setminus C, \forall p \in N_K(x), \sigma(F(x), -p) \geq 0.$$

STATEMENT 5. Assume that F is Lipschitz. The two following statements hold true.

1. If K is closed, then K is (globally) invariant under F if and only if

$$\forall x \in K, F(x) \subset T_K(x),$$

or, equivalently, in dual form, if and only if

$$\forall x \in K, \forall p \in N_K(x), \sigma(F(x), p) \leq 0.$$

2. If $C \subset K$ is closed, then K absorbs C by F if and only if

$$\forall x \in K \setminus C, F(x) \subset T_K(x),$$

or, equivalently, in dual form, if and only if

$$\forall x \in K \setminus C, \forall p \in N_K(x), \sigma(F(x), p) \leq 0.$$

3. If $C \subset K$ is closed, then C is backward invariant under F relatively to K if and only if

$$(5.1) \quad \begin{cases} \text{(i)} & \forall x \in C \cap \text{Int}(K), -F(x) \subset T_C(x), \\ \text{(ii)} & \forall x \in C \cap \partial K, -F(x) \subset T_C(x) \cup T_{X \setminus K}(x), \end{cases}$$

or, equivalently, in normal form, if and only if

$$(5.2) \quad \begin{cases} \text{(i)} & \forall x \in C \cap \text{Int}(K), \forall p \in N_C(x), \sigma(F(x), -p) \leq 0, \\ \text{(ii)} & \forall x \in C \cap \partial K, \forall p \in N_C(x) \cap N_{\overline{X \setminus K}}(x), \sigma(F(x), -p) \leq 0. \end{cases}$$

5.2. Tangential and normal characterizations of viability kernels with targets. Using the viability theorem, Statement 1, and the invariance theorem, Statement 5, we deduce that the viability kernels and the viable-capture basins enjoy tangential and normal characterizations.

For that purpose, we introduce the following Frankowska property.

DEFINITION 5.1. Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K . We shall say that a subset D between C and K satisfies the Frankowska property with respect to F if

$$(5.3) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset, \\ \text{(ii)} & \forall x \in D \cap \text{Int}(K), -F(x) \subset T_D(x), \\ \text{(iii)} & \forall x \in D \cap \partial K, -F(x) \subset T_D(x) \cup T_{X \setminus K}(x), \end{cases}$$

or, equivalently, by duality, satisfying the “normal conditions”

$$(5.4) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, \forall p \in N_D(x), \sigma(F(x), -p) \geq 0, \\ \text{(i)} & \forall x \in D \cap \text{Int}(K), \forall p \in N_D(x), \sigma(F(x), -p) \leq 0, \\ \text{(ii)} & \forall x \in D \cap \partial K, \forall p \in N_D(x) \cap N_{\overline{X \setminus K}}(x), \sigma(F(x), -p) \leq 0. \end{cases}$$

When K is assumed further to be backward locally invariant, the above conditions (5.3) and (5.4) boil down to

$$(5.5) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset, \\ \text{(ii)} & \forall x \in D, -F(x) \subset T_D(x), \end{cases}$$

and

$$(5.6) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, \forall p \in N_D(x), \sigma(F(x), -p) = 0, \\ \text{(ii)} & \forall x \in D, \forall p \in N_D(x), \sigma(F(x), -p) \leq 0, \end{cases}$$

respectively.

We deduce from the characterization theorem, Theorem 4.13, its tangential and normal formulations.

THEOREM 5.2. *Let us assume that F is Marchaud and that $C \subset K$ and K are closed. The viability kernel $\text{Viab}_F(K, C)$ of the subset K with target C under F is*

1. the largest closed subset D of K satisfying

$$\forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset.$$

2. When F is assumed to be also Lipschitz, the viability kernel $\text{Viab}_F(K, C)$ is the unique closed subset $D \subset K$ satisfying
 - (a) the Frankowska property (5.3) (or its dual formulation (5.4));
 - (b) $K \setminus D$ is a repeller.

As a consequence, we obtain the following tangential characterization of viable-capture basins.

THEOREM 5.3. *Let us assume that F is Marchaud, that K is closed, and that a closed subset C satisfies $\text{Viab}_F(K \setminus C) = \emptyset$. Then the viable-capture basin $\text{Capt}_F^K(C)$ is*

1. the largest closed subset D satisfying $C \subset D \subset K$ and

$$(5.7) \quad \forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset.$$

2. If F is Lipschitz, the viable-capture basin $\text{Capt}_F^K(C)$ is the unique closed subset D satisfying the Frankowska property (5.3) (or its dual formulation (5.4)).

We now define the following “viscosity property.”

DEFINITION 5.4. *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K . We shall say that a subset D between C and K satisfies the viscosity property with respect to F if*

$$(5.8) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset, \\ \text{(ii)} & \forall x \in \overline{X \setminus D}, F(x) \subset T_{\overline{X \setminus D}}(x), \end{cases}$$

and, in normal form,

$$(5.9) \quad \begin{cases} \text{(i)} & \forall x \in D \setminus C, \forall p \in N_D(x), \sigma(F(x), -p) \geq 0, \\ \text{(ii)} & \forall x \in \overline{X \setminus D}, \forall p \in N_{\overline{X \setminus D}}(x), \sigma(F(x), p) \leq 0, \end{cases}$$

respectively.

When $C = \emptyset$, we recognize the definition of a discriminating kernel of K of the Hamiltonian $H(x, p) := \sigma(F(x), -p)$ given in [26], for instance.

THEOREM 5.5. *Let us assume that F is Marchaud and Lipschitz, that $C \subset K$ and K are closed, and that K is backward invariant. The viability kernel $\text{Viab}_F(K, C)$ of the subset K with the target C under F is the unique closed subset $D \subset K$ satisfying the following:*

1. the viscosity property (5.8) (or its dual formulation (5.9));
2. $K \setminus D$ is a repeller.

6. Stability properties. Consider two sequences of subsets $C_n \subset C$ and $K_n \subset X$ and their Painlevé–Kuratowski upper limits

$$C^\# := \text{Limsup}_{n \rightarrow +\infty} C_n \ \& \ K^\# := \text{Limsup}_{n \rightarrow +\infty} K_n.$$

Recall that the upper limit of a sequence of constant subsets C is the closure of C .

DEFINITION 6.1. *We define the hypolimit $\lim_{\downarrow n \rightarrow \infty} \tau_{K_n}^\#$ of upper exit functions $\tau_{K_n}^\#$ whose hypograph is the upper limit of the hypographs of the functions $\tau_{K_n}^\#$*

$$\mathcal{H}p \left(\lim_{\downarrow n \rightarrow \infty} \left(\tau_{K_n}^\# \right) \right) := \text{Limsup}_{n \rightarrow \infty} \mathcal{H}p(\tau_{K_n}^\#).$$

It is the upper hypolimit of the functions $\tau_{K_n}^\#$, equal to

$$\left(\lim_{\downarrow n \rightarrow \infty} \tau_{K_n}^\# \right) (x_0) = \limsup_{n \rightarrow \infty, x_n \rightarrow_{K_n} x_0} \tau_{K_n}^\#(x_n).$$

In the same way, we define the upper epilimit $\lim_{\uparrow n \rightarrow \infty} \varpi_{(K_n, C_n)}^\flat$ whose epigraph is the upper limit of the epigraphs of the functions $\varpi_{(K_n, C_n)}^\flat$

$$\mathcal{E}p \left(\lim_{\uparrow n \rightarrow \infty} \varpi_{(K_n, C_n)}^\flat \right) := \text{Limsup}_{n \rightarrow \infty} \mathcal{E}p \left(\varpi_{(K_n, C_n)}^\flat \right).$$

It is the upper epilimit of the functions $\varpi_{(K_n, C_n)}^\flat$, equal to

$$\left(\lim_{\uparrow n \rightarrow \infty} \varpi_{(K_n, C_n)}^\flat \right) (x_0) = \liminf_{n \rightarrow \infty, x_n \rightarrow_{K_n} x_0} \varpi_{(K_n, C_n)}^\flat(x_n).$$

We have to prove this very useful stability result.

THEOREM 6.2. *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a strict upper semicompact map. Consider two sequences of subsets $C_n \subset C$ and $K_n \subset X$ and their Painlevé–Kuratowski upper limits*

$$C^\# := \text{Limsup}_{n \rightarrow +\infty} C_n \ \& \ K^\# := \text{Limsup}_{n \rightarrow +\infty} K_n.$$

Then

1. *the upper hypolimit of the upper exit functions of a sequence of subsets K_n is smaller than or equal to the upper exit function of their upper limit:*

$$\left(\lim_{\downarrow n \rightarrow \infty} \tau_{K_n}^\# \right) (x) \leq \tau_{K^\#}^\#(x);$$

2. the upper epilimit of the lower constrained hitting functions of a sequence of subsets $C_n \subset K_n$ is larger than or equal to the lower constrained hitting function of their upper limit:

$$\left(\lim_{\#}^{\dagger} \varpi_{(K_n, C_n)}^b\right)(x) \geq \varpi_{(K^\#, C^\#)}^b(x).$$

Proof. Let us begin by proving the first inequality, which can be translated in the form of the inclusion

$$\text{Limsup}_{n \rightarrow \infty} \mathcal{H}p\left(\tau_{K_n}^\#\right) \subset \mathcal{H}p\left(\tau_{K^\#}^\#\right).$$

For that purpose, let us take a sequence $(T_n, x_n) \in \mathcal{H}p(\tau_{K_n}^\#)$ converging to (T, x) and check that this limit belongs to the hypograph of $\tau_{K^\#}^\#$. By definition, there exists a solution $x_n(\cdot) \in \mathcal{S}(x_n)$ starting at x_n such that, for every $t \in [0, T_n[$, $x_n(t)$ belongs to K_n . Since \mathcal{S} is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some solution $x(\cdot) \in \mathcal{S}(x)$ starting at x . Take $t < T$ and n large enough for having $t < T_n$. In this case, $x_n(t)$ belongs to K_n and, passing to the limit, $x(t)$ belongs to $K^\#$. This implies that

$$T \leq \tau_K(x(\cdot)) \leq \tau_{K^\#}^\#(x).$$

Taking $K_n := K$, $x_n := x \in K$, and $T_n < \tau_K^\#(x)$ converging to $\tau_K^\#(x)$, we infer that the solution $x(\cdot)$ obtained above achieves the supremum.

Let us prove now the second inequality, which can be translated in the form of the inclusion

$$\text{Limsup}_{n \rightarrow \infty} \mathcal{E}p\left(\varpi_{(K_n, C_n)}^b\right) \subset \mathcal{E}p\left(\varpi_{(K^\#, C^\#)}^b\right).$$

For that purpose, let us take sequences $(T_n, x_n) \in \mathcal{E}p(\varpi_{(K_n, C_n)}^b)$ converging to (T, x) and check that this limit belongs to the epigraph of $\varpi_{(K^\#, C^\#)}^b$.

For every $\varepsilon > 0$, there exist N such that for $n \geq N$, there exists a solution $x_n(\cdot) \in \mathcal{S}(x_n)$ and $t_n \leq T_n + \frac{\varepsilon}{2} \leq T + \varepsilon$ such that $x_n(t_n) \in C_n$, and for every $s < t_n$, $x_n(s) \in K_n$. Since \mathcal{S} is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some solution $x(\cdot) \in \mathcal{S}(x)$. Let us consider also a subsequence (again denoted by) t_n converging to some $T^* \leq T + \varepsilon$. By passing to the limit, we infer that $x(T^*)$ belongs to $C^\#$ and that, for any $s < T^*$, $x(s)$ belongs to $K^\#$. This implies that

$$\varpi_{(K^\#, C^\#)}^b(x) \leq \varpi_{(K^\#, C^\#)}^b(x(\cdot)) \leq T^* \leq T + \varepsilon.$$

We conclude by letting ε converge to 0. Taking $K_n := K$, $x_n := x \in K$, and $T_n < \tau_K^\#(x)$ converging to $\tau_K^\#(x)$, we infer that the solution $x(\cdot)$ obtained above achieves the supremum.

Taking $K_n := K$, $C_n := C$, $x_n := x \in K$, and $T_n \geq \varpi_{(K, C)}^b(x)$ converging to $\varpi_{(K, C)}^b(x)$, we infer that the solution $x(\cdot)$ obtained above achieves the infimum. \square

We derive stability properties of the viability kernels with targets.

THEOREM 6.3. *Let us assume that the map \mathcal{S} is upper semicompact.*

If a subset K captures a subset $C \subset K$ under \mathcal{S} , then its closure \bar{K} also captures the closure \bar{C} of the target C .

More generally, let us consider a sequence of subsets K_n and of targets $C_n \subset K_n$. If K_n captures C_n for every $n \geq 0$, then the upper limit $\text{Limsup}_{n \rightarrow +\infty} K_n$ captures the upper limit $\text{Limsup}_{n \rightarrow +\infty} C_n$ of the targets C_n .

Proof. Let us set

$$C^\sharp := \text{Limsup}_{n \rightarrow +\infty} C_n \text{ and } K^\sharp := \text{Limsup}_{n \rightarrow +\infty} K_n.$$

Let us consider the limit $x := \lim_{n \rightarrow +\infty} x_n \in K^\sharp$ of elements $x_n \in K_n$. Since C_n captures K_n , there exists a solution $x_n(\cdot) \in \mathcal{S}(x_n)$ viable in K_n until it possibly reaches C_n at time $t_n := \varpi_{C_n}(x_n(\cdot))$, finite or infinite.

Since \mathcal{S} is upper semicontact, a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.

Since $x_n(\cdot)$ is viable in K_n until it reaches C_n , we know that

$$\varpi_{C_n}(x_n(\cdot)) \leq \tau_{K_n}(x_n(\cdot)).$$

Either the limit x belongs to the viability kernel $\text{Viab}(K^\sharp)$ of the upper limit K^\sharp , or else this limit x does not belong to the viability kernel $\text{Viab}(K^\sharp)$ and we have to check that it belongs to the viable-capture basin $\text{Capt}_{K^\sharp}(C^\sharp)$. This means that $\tau_{K^\sharp}^\sharp(x)$ is finite. Since \mathcal{S} is upper semicontact, Theorem 6.2 implies that

$$\limsup_{n \rightarrow +\infty, x_n \rightarrow_{K_n} x} \tau_{K_n}^\sharp(x_n) \leq \tau_{K^\sharp}^\sharp(x).$$

For n large enough, there exists $T_n < +\infty$ satisfying

$$\varpi_{C_n}(x_n(\cdot)) \leq T_n \leq \tau_{K_n}(x_n(\cdot)) \leq \tau_{K_n}^\sharp(x_n) \leq \tau_{K^\sharp}^\sharp(x) + 1 < +\infty.$$

Therefore, a subsequence (again denoted by) T_n converges to some $T^* \leq \tau_{K^\sharp}^\sharp(x) + 1$. Theorem 6.2 implies that

$$\varpi_{(K^\sharp, C^\sharp)}^\flat(x) \leq \varpi_{(K^\sharp, C^\sharp)}(x(\cdot)) \leq T^* \leq \tau_{K^\sharp}(x(\cdot)) \leq \tau_{K^\sharp}(x).$$

Hence from every $x \in K^\sharp$ starts a solution viable in K^\sharp until it possibly reaches C , so that K^\sharp captures C^\sharp . \square

As a consequence, we obtain the following theorem.

THEOREM 6.4. *Let us assume that the map \mathcal{S} is upper semicontact. Then*

$$\overline{\text{Viab}(K, C)} \subset \text{Viab}(\overline{K}, \overline{C}),$$

and thus, if $C \subset K$ and K are closed, so is the viability kernel $\text{Viab}(K, C)$ of K with target C .

More generally, let us consider a sequence of subsets K_n and of targets $C_n \subset K_n$ and their upper limits K^\sharp and C^\sharp . Then

$$(6.1) \quad \text{Limsup}_{n \rightarrow +\infty} \text{Viab}(K_n, C_n) \subset \text{Viab}(\text{Limsup}_{n \rightarrow +\infty} K_n, \text{Limsup}_{n \rightarrow +\infty} C_n).$$

THEOREM 6.5. *If the set-valued map \mathcal{S}_- is lower semicontinuous, then for any sequence of closed subsets C_n ,*

$$(6.2) \quad \text{Capt}(\text{Liminf}_{n \rightarrow +\infty} C_n) \subset \text{Liminf}_{n \rightarrow +\infty} \text{Capt}(C_n).$$

Proof. For proving that

$$\text{Capt}(\text{Liminf}_{n \rightarrow +\infty} C_n) \subset \text{Liminf}_{n \rightarrow +\infty} \text{Capt}(C_n),$$

let C^b denote the lower limit of the subsets C_n . Let us take $x \in \text{Capt}(C^b)$ and a solution $x(\cdot) \in \mathcal{S}(x)$ viable in K until it reaches the target C^b at time $T < +\infty$ at $c := x(T) \in C^b$. Hence the function $t \mapsto y(t) := x(T - t)$ is a solution $y(\cdot) \in \mathcal{S}_-(c)$. Let us consider a sequence of elements $c_n \in C_n$ converging to c .

Since \mathcal{S}_- is lower semicontinuous, there exist solutions $y_n(\cdot) \in \mathcal{S}_-(c_n)$ converging uniformly over compact intervals to $x(\cdot)$. Therefore, $x_n := y_n(T)$ converges to x . It is enough to observe that x_n belongs to $\text{Capt}(C_n)$ to conclude. \square

As a consequence, we obtain the following theorem.

THEOREM 6.6. *Let us consider a sequence of closed subsets C_n satisfying $\text{Viab}(K) \subset C_n \subset K$ and*

$$\text{Lim}_{n \rightarrow +\infty} C_n := \text{Limsup}_{n \rightarrow +\infty} C_n = \text{Liminf}_{n \rightarrow +\infty} C_n.$$

If the set-valued map \mathcal{S} is upper semicompact, if \mathcal{S}_- is lower semicontinuous, and if K is closed and backward invariant under \mathcal{S} , then

$$(6.3) \quad \text{Lim}_{n \rightarrow +\infty} \text{Capt}^K(C_n) = \text{Capt}^K(\text{Lim}_{n \rightarrow +\infty} C_n).$$

7. Optimal evolutionary control system. We devote this section to statements of applications to optimal control and stopping time problems. We refer to [8, 9] for applications to systems of first-order partial differential equations and inclusions.

7.1. Control evolutionary systems. We denote by $L^1(0, \infty; \mathcal{U})$ the space of measurable integrable functions from $[0, +\infty[$ to a finite dimensional vector-space \mathcal{U} , the *control space*. We shall supply it with the weakened topology.

DEFINITION 7.1. *Let us consider topological vector spaces X (the state space) and \mathcal{U} (the control space). A control evolutionary system is a set-valued map $\mathcal{C} : X \rightsquigarrow \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ associating with any x a set of state-control pairs $(x(\cdot), u(\cdot))$ satisfying the following.*

1. *The translation property. Let $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$. Then for all $T \geq 0$, the function $(y(\cdot), v(\cdot))$ defined by $y(t) := x(t + T)$ and $v(t) := u(t + T)$ is a solution $(y(\cdot), v(\cdot)) \in \mathcal{C}(x(T))$ starting at $x(T)$.*
2. *The concatenation property. Let $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$, and $T \geq 0$. Then for every $(y(\cdot), v(\cdot)) \in \mathcal{C}(x(T))$, the pair $(z(\cdot), w(\cdot))$ of functions defined by*

$$z(t) := \begin{cases} x(t) & \text{if } t \in [0, T], \\ y(t - T) & \text{if } t \geq T \end{cases}$$

and

$$w(t) := \begin{cases} u(t) & \text{if } t \in [0, T], \\ v(t - T) & \text{if } t \geq T \end{cases}$$

belongs to $\mathcal{C}(x)$.

We shall say that the control evolutionary system \mathcal{C} is upper semicompact if the set-valued map $x \rightsquigarrow \mathcal{C}(x)$ is upper semicompact from X to $\mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$.

Control evolutionary systems provide examples of evolutionary systems by setting

$$\mathcal{S}(x) := \bigcup_{\{u(\cdot) \mid (x(\cdot), u(\cdot)) \in \mathcal{C}(x)\}} \{x(\cdot)\}.$$

Usual control problems provide examples of control evolutionary systems.

7.2. Control systems. Let us consider a control problem (P, f) with a priori feedback map $P : X \rightsquigarrow \mathcal{U}$ from X to some finite dimensional vector space \mathcal{U} governing the evolution of $(x(\cdot), u(\cdot))$ according the system

$$(7.1) \quad \begin{cases} \text{(i)} & x'(t) = f(x(t), u(t)), \\ \text{(ii)} & u(t) \in P(x(t)). \end{cases}$$

Starting from x , we define $\mathcal{C}_{(P,F)}(x)$ as the set of pairs $(x(\cdot), u(\cdot)) \in \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ satisfying (7.1) for almost all $t \geq 0$ such that $x(0) = x$.

DEFINITION 7.2. We shall say that the control system (P, f) is

1. Marchaud if the set-valued map $P : X \rightsquigarrow \mathcal{U}$ is Marchaud, if $f : X \times \mathcal{U} \mapsto X$ is continuous and affine with respect to the control, and if f satisfies the growth condition

$$\forall (x, u) \in \text{Graph}(P), \quad \|f(x, u)\| \leq c(\|x\| + \|u\| + 1);$$

2. Lipschitz if the set-valued map $P : X \rightsquigarrow \mathcal{U}$ is Lipschitz and if $f : X \times \mathcal{U} \mapsto X$ is Lipschitz.

Therefore, a control system (P, f) provides an example of upper semicompact evolutionary systems \mathcal{S} if the control system (P, f) is Marchaud and an example of a lower semicontinuous evolutionary systems \mathcal{S} if the control system (P, f) is Lipschitz.

7.3. Optimal evolutionary control. Let us introduce the following two features:

1. a discount factor

$$\mathbf{m} : (x, u) \in X \times \mathcal{U} \mapsto \mathbf{m}(x, u) \in \mathbf{R},$$

2. an extended ‘‘Lagrangian’’

$$\mathbf{l} : (x, u) \in X \times \mathcal{U} \mapsto \mathbf{l}(x, u) \in \mathbf{R},$$

used to measure a cumulated cost over time.

We associate with them the auxiliary evolutionary control system \mathcal{R} defined by

$$\mathcal{R}(T, x, y) = \{(T - \cdot, x(\cdot), u(\cdot), y(\cdot))\}_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)},$$

where

$$(7.2) \quad y(t) \leq e^{-\int_0^t \mathbf{m}(x(s), u(s)) ds} \left(y - \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau)) d\tau \right).$$

7.4. Objective and constraints. Let us consider two nonnegative extended cost functions \mathbf{b} (constraint function) and \mathbf{c} (objective function) satisfying

$$\forall (t, x) \in \mathbf{R}_+ \times \mathbf{R}_+^{n+1}, \quad 0 \leq \mathbf{b}(t, x) \leq \mathbf{c}(t, x) \leq +\infty,$$

allowed to take infinite values in order to describe state constraints. We extend them as functions from $\mathbf{R} \times \mathbf{R}^{n+1}$ to $\mathbf{R}_+ \cup \{+\infty\}$ by setting

$$\forall (t, x) \notin \mathbf{R}_+ \times \mathbf{R}_+^{n+1}, \quad \mathbf{b}(t, x) = \mathbf{c}(t, x) = +\infty,$$

so that nonnegativity constraints on time and on the state variables are automatically taken into account. In particular, we shall denote by $\mathbf{0}$ the function defined by

$$\mathbf{0}(t, x) = \begin{cases} 0 & \text{if } t \geq 0, x \in \mathbf{R}_+^{n+1}, \\ +\infty & \text{if not.} \end{cases}$$

Several control problems, in particular, financial problems such as the valuation of options, are stated in the following fashion.

DEFINITION 7.3. *The two nonnegative extended constraint and objective functions being given, and given also a horizon time $T > 0$, the problem is to*

1. *find the valuation subset $\mathcal{V} \subset \mathbf{R}_+ \times \mathbf{R}^{n+1} \times \mathbf{R}$ of triples (T, x, y) made of the horizon time T , the initial state x , and the cost y such that there exists a control $t \in [0, T] \mapsto u(t) \in P(x(t))$ and a time $\varpi(T) \in [0, T]$ such that the solution to the system (7.3) satisfying $x(0) = x, y(0) = y$ and*

$$(7.3) \quad \begin{cases} \text{(i)} & \forall t \in [0, \varpi(T)], \quad y(t) \geq \mathbf{b}(T - t, x(t)), \\ \text{(ii)} & y(\varpi(T)) \geq \mathbf{c}(T - \varpi(T), x(\varpi(T))); \end{cases}$$

2. *associate with any initial price x the smallest cost*

$$(7.4) \quad V(T, x) := \inf_{(T, x, y) \in \mathcal{V}} y.$$

The function $(T, x) \mapsto V(T, x)$ is called the value function of the problem, i.e., the minimal initial cost y satisfying the two above constraints (7.3).

We observe at once the following property. The value function satisfies the initial condition

$$\forall x \in \mathbf{R}^n, \quad V(0, x) = \mathbf{c}(0, x).$$

We observe that the valuation subset \mathcal{V} is the viable-capture basin of the epigraph of \mathbf{c} viable in the epigraph of \mathbf{b} under the auxiliary evolutionary control system (7.3) because dynamical constraints (7.3) can be reformulated in the form

$$(7.5) \quad \begin{cases} \text{(i)} & \forall t \in [0, \varpi(T)], \quad (T - t, x(t), y(t)) \in \mathcal{E}p(\mathbf{b}), \\ \text{(ii)} & (T - \varpi(T), x(\varpi(T)), y(\varpi(T))) \in \mathcal{E}p(\mathbf{c}). \end{cases}$$

Therefore, we can reformulate the definition of the valuation subset \mathcal{V} and of the value function $(T, x) \mapsto V(T, x)$ in the following way.

PROPOSITION 7.4. *The valuation subset*

$$\mathcal{V} = \text{Capt}_{\mathcal{R}}^{\mathcal{E}p(\mathbf{b})}(\mathcal{E}p(\mathbf{c}))$$

is the viable-basin capture of the epigraph of the cost function \mathbf{c} under the auxiliary evolutionary control system (7.3) viable in the epigraph of the cost function \mathbf{b} .

We can prove that V is the concealed value function of an optimal evolutionary control system that we have to unearth. For that purpose, we associate with the function \mathbf{c} the cost functional

$$\begin{cases} J_{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) \\ := e^{\int_0^t \mathbf{m}(x(s), u(s)) ds} \mathbf{c}(T - t, x(t)) + \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau)) d\tau \end{cases}$$

(where t ranges over $[0, T]$), constituted by the sum of the discounted spot cost and the cumulated costs at time t of a solution to the control problem starting at x at the initial time. The controls—most often prices or other regulees in economics, portfolio in finance—appear both in the discount factor \mathbf{m} and the Lagrangian \mathbf{l} . In

the same way, we associate with the function \mathbf{b} the *cost functional* $J_{\mathbf{b}}$ and the maximal cumulated cost up to the current time t :

$$K_{\mathbf{b}}(t, x; (x(\cdot), u(\cdot))) := \sup_{s \in [0, t]} J_{\mathbf{b}}(s, x; (x(\cdot), u(\cdot))).$$

We next integrate this cumulated cost together with the former cost $J_{\mathbf{c}}(t, x; (x(s), u(s)))$ by introducing the new cost function

$$L_{\mathbf{b}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) := \max(K_{\mathbf{b}}(t, x; (x(\cdot), u(\cdot))), J_{\mathbf{c}}(t, x; (x(\cdot), u(\cdot)))).$$

The problem is now to minimize over all $t \in [0, T]$ and over all the solutions to the evolutionary control problem:

$$V_{\mathbf{b}}(\mathbf{c})(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{b}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))).$$

STATEMENT 6. *Let us assume that the extended functions \mathbf{b} and \mathbf{c} are nontrivial and nonnegative. The constrained discounted intertemporal value function $V_{\mathbf{b}}(\mathbf{c})$ is equal to the function V associated with the viable-capture basin $\text{Capt}_{\mathcal{R}}^{\mathcal{E}p(\mathbf{b})}(\mathcal{E}p(\mathbf{c}))$ of $\mathcal{E}p(\mathbf{c})$ under \mathcal{R} . Furthermore, any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from $x \in \text{Dom}(V_{\mathbf{b}}(\mathbf{c}))$ satisfying the inequality for every $t \in [0, \varpi(T, x(\cdot))]$*

$$(7.6) \left\{ \begin{array}{l} V_{\mathbf{b}}(\mathbf{c})(T, x) \\ \geq e^{\int_0^t \mathbf{m}(x(s), u(s)) ds} V_{\mathbf{b}}(\mathbf{c})(T - t, x(t)) + \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau)) d\tau \end{array} \right.$$

until the first time $\varpi(T, x(\cdot))$ when

$$V_{\mathbf{b}}(\mathbf{c})(T - \varpi(T, x(\cdot)), x(\varpi(T, x(\cdot)))) = \mathbf{b}(T - \varpi(T, x(\cdot)), x(\varpi(T, x(\cdot))))$$

is an optimal solution for the optimal time $\varpi(T, x(\cdot))$ and actually satisfies the equality

$$(7.7) \left\{ \begin{array}{l} \forall t \in [0, \varpi(T, x(\cdot))], V_{\mathbf{b}}(\mathbf{c})(T, x) \\ = e^{\int_0^t \mathbf{m}(x(s), u(s)) ds} V_{\mathbf{b}}(\mathbf{c})(T - t, x(t)) + \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), u(s)) ds} \mathbf{l}(x(\tau), u(\tau)) d\tau. \end{array} \right.$$

Finally, the value function is a solution \mathbf{v} to the two following functional equations stating that the functions $L_{\mathbf{b}}^{\mathbf{c}}$ and $L_{\mathbf{v}}^{\mathbf{c}}$ have the same infimum as $L_{\mathbf{b}}^{\mathbf{c}}$:

$$(7.8) \quad \left\{ \begin{array}{l} \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{v}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) \\ = \mathbf{v}(T, x) \\ = \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{b}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))). \end{array} \right.$$

We list a manifold of examples in classical optimal control (in the case when $\mathbf{m} = 0$), recalling that financial problems⁶ (in the case when $\mathbf{l} = 0$) also fit the above framework. Playing with the choice of the spot cost \mathbf{c} , we shall cover several examples.

⁶See [44] for a treatment of dynamical valuation of portfolios in the framework of dynamical games.

1. Taking $\mathbf{b} = 0$ and \mathbf{c} defined by

$$\mathbf{c}(t, x) := \begin{cases} \mathbf{u}(x) & \text{if } t = 0, \\ +\infty & \text{if } t > 0, \end{cases}$$

the above problem boils down to

$$V(\mathbf{c})(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} J_{\mathbf{c}}(T, x; (x(\cdot), u(\cdot))),$$

which is the *Bolza problem*

$$\inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \left(\mathbf{u}(x(T)) + \int_0^T \mathbf{l}(x(\tau), u(\tau)) d\tau \right)$$

and the *Mayer problem*

$$\inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \mathbf{u}(x(T))$$

when, furthermore, $\mathbf{l} = 0$.

2. Taking $\mathbf{b} = 0$ and $\mathbf{c}(t, x) := \mathbf{u}(x)$, we find the classical *stopping time problem*

$$\mathbf{F}(u)(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} \left(\mathbf{u}(x(t)) + \int_0^t \mathbf{l}(x(\tau), u(\tau)) d\tau \right)$$

associated with \mathbf{u} in control theory. The cost function \mathbf{l} can be regarded as a density of Maslov measures, $\mathbf{F}(\mathbf{u})$ being then the *mathematical faith* of \mathbf{u} introduced by Pierre Bernhard (with the minus sign, under the name of *mathematical fear*). They correspond to the *mathematical expectation* $\mathbf{E}(f)$ of densities f of probability measures.

3. Let us consider an extended function $\mathbf{b} : \mathbf{R}_+ \times X \mapsto \mathbf{R} \cup \{+\infty\}$ with which we associate the problem

$$W(\mathbf{b})(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \sup_{t \in [0, T]} J_{\mathbf{b}}(t, x; (x(\cdot), u(\cdot))).$$

$\mathbf{b} : \mathbf{R}_+ \times X \mapsto \mathbf{R} \cup \{+\infty\}$ is an extended cost function. We observe that $W(\mathbf{b}) = V_{\mathbf{b}}(\mathbf{c})$, where

$$\mathbf{c}(t, x) := \begin{cases} \mathbf{b}(0, x) & \text{if } t = 0, \\ +\infty & \text{if } t > 0. \end{cases}$$

Indeed, we see that $J_{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) = +\infty$ if $t < T$ and $J_{\mathbf{c}}(T, x; (x(\cdot), u(\cdot))) = J_{\mathbf{b}}(T, x; (x(\cdot), u(\cdot)))$. Therefore,

$$L_{\mathbf{b}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) := \begin{cases} K_{\mathbf{b}}(T, x; (x(\cdot), u(\cdot))) & \text{if } t = T, \\ +\infty & \text{if } t < T, \end{cases}$$

and thus, $W(\mathbf{b}) = V_{\mathbf{b}}(\mathbf{c})$.

7.5. Episolutions to Hamilton–Jacobi–Bellman inequalities. Let us consider the case when the evolutionary control system is associated with the control system (7.1) and apply Theorem 5.3 characterizing capture basins in terms of the tangential conditions. This allows us to relate the value function with generalized solutions to Hamilton–Jacobi–Bellman partial differential variational inequalities

$$\left\{ \begin{array}{l} \forall (t, x) \in \Omega(\mathbf{v}), \\ -\frac{\partial \mathbf{v}(t, x)}{\partial t} + \inf_{u \in P(x)} \left(\left\langle \frac{\partial \mathbf{v}(t, x)}{\partial x}, f(x, u) \right\rangle + \mathbf{l}(x, u) + \mathbf{m}(x, u)\mathbf{v}(t, x) \right) = 0 \end{array} \right.$$

on the subset

$$\Omega(\mathbf{v}) := \{(t, x) \in \mathbf{R}_+ \times X \mid \mathbf{b}(t, x) \leq \mathbf{v}(t, x) < \mathbf{c}(t, x)\}.$$

Let us recall that the *contingent epiderivative* $D_{\uparrow} \mathbf{v}(t, x)$ of \mathbf{v} at (t, x) is defined by

$$D_{\uparrow} \mathbf{v}(t, x)(\lambda, v) := \liminf_{h \rightarrow 0+, u \rightarrow v} \frac{\mathbf{v}(t + h\lambda, x + hu)}{h}$$

and that

$$\mathcal{E}p(D_{\uparrow} \mathbf{v}(t, x)) = T_{\mathcal{E}p(\mathbf{v})}(t, x, \mathbf{v}(t, x)).$$

The first part of Theorem 5.3 implies a characterization of the value function as a solution of Hamilton–Jacobi variational inequalities.

STATEMENT 7 (Frankowska). *Let us assume that the control system $(P, f, \mathbf{l}, \mathbf{m})$ is Marchaud and that the functions \mathbf{b} and \mathbf{c} are nontrivial, nonnegative, and lower semicontinuous.*

Then the value function $V_{\mathbf{b}}(\mathbf{c})$ is characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ satisfying for every $(t, x) \in]0, \infty[\times X$

$$\left\{ \begin{array}{l} \text{(i)} \quad \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x), \\ \text{(ii)} \quad \text{if } (t, x) \in \Omega(\mathbf{v}), \\ \quad \inf_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(-1, f(x, u)) + \mathbf{l}(x, u) + \mathbf{m}(x, u)\mathbf{v}(t, x)) \leq 0. \end{array} \right.$$

Let us set

$$\mathbf{R}(t, x) := \{u \in P(x) \mid D_{\uparrow} V_{\mathbf{b}}(\mathbf{c})(t, x)(-1, f(x, u)) + \mathbf{l}(x, u) + \mathbf{m}(x, u)V_{\mathbf{b}}(\mathbf{c})(t, x) \leq 0\}.$$

Knowing the value function, an optimal solution is obtained in the following way. Starting from x_0 such that $V_{\mathbf{b}}(\mathbf{c})(T, x_0) < \mathbf{c}(T, x_0)$, any solution $(x(\cdot), u(\cdot))$ to the control system

$$(7.9) \quad \left\{ \begin{array}{l} \text{(i)} \quad x'(t) = f(x(t), u(t)), \\ \text{(ii)} \quad u(t) \in \mathbf{R}(t, x(t)), \end{array} \right.$$

is an optimal solution, and the first time $\varpi(T, x(\cdot)) \geq 0$ when

$$V_{\mathbf{b}}(\mathbf{c})(T - \varpi(T, x(\cdot)), x(\varpi(T, x(\cdot)))) = \mathbf{c}(T - \varpi(T, x(\cdot)), x(\varpi(T, x(\cdot))))$$

is the optimal time.

The second part of Theorem 5.3 implies the characterization of the value function $V_{\mathbf{b}}(\mathbf{c})$ as a unique Frankowska *episolution* to the Hamilton–Jacobi–Bellman variational inequality.

STATEMENT 8 (Frankowska). *Let us assume that the control system (P, f) is Marchaud and Lipschitz and that \mathbf{b} and \mathbf{c} are nontrivial, nonnegative, and lower semicontinuous.*

Then the value function $V_{\mathbf{b}}(\mathbf{c})$ is the unique lower semicontinuous episolution \mathbf{v} to the system of differential inequalities: for every $(t, x) \in \text{Dom}(\mathbf{v})$,

$$(7.10) \quad \left\{ \begin{array}{l} \text{(i)} \quad \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x), \\ \text{(ii)} \quad \text{if } \mathbf{v}(t, x) < \mathbf{c}(t, x), \\ \quad \inf_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(-1, f(x, u)) + \mathbf{l}(x, u) + \mathbf{m}(x, u)\mathbf{v}(t, x)) \leq 0, \\ \text{(iii)} \quad \text{if } \mathbf{v}(t, x) > \mathbf{b}(t, x), \\ \quad \sup_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(1, -f(x, u)) - \mathbf{l}(x, u) - \mathbf{m}(x, u)\mathbf{v}(t, x)) \leq 0, \\ \text{(iv)} \quad \text{if } \mathbf{v}(t, x) = \mathbf{b}(t, x), \\ \quad \sup_{u \in P(x)} [\min(D_{\uparrow} \mathbf{v}(t, x)(1, -f(x, u)), D_{\downarrow} \mathbf{b}(t, x)(1, -f(x, u))) \\ \quad - \mathbf{l}(x, u) - \mathbf{m}(x, u)\mathbf{v}(t, x))] \leq 0. \end{array} \right.$$

Remark. Condition (7.10)(iv) is automatically satisfied whenever

$$\sup_{u \in P(x)} (D_{\downarrow} \mathbf{b}(t, x)(1, -f(x, u)) - \mathbf{l}(x, u)) - \mathbf{m}(x, u)\mathbf{v}(t, x) \leq 0,$$

i.e., whenever the epigraph of \mathbf{b} is locally backward invariant under the auxiliary system. \square

7.6. Bilateral and viscosity solutions to Hamilton–Jacobi–Bellman variational inequalities. We obtain by duality equivalent statements involving subdifferential and/or superdifferentials, involving the Hamiltonian $H : X \times \mathbf{R}_+ \times X^* \mapsto \mathbf{R} \cup \{+\infty\}$ associated with the control problem and the Lagrangian by

$$H(x, y, p) := \inf_{u \in P(x)} (\langle p, f(x, u) \rangle + \mathbf{l}(x, u) + \mathbf{m}(x, u)y)$$

and the horizon Hamiltonian $H^\infty : X \times X^* \mapsto \mathbf{R} \cup \{+\infty\}$, by

$$H^\infty(x, p) := \inf_{u \in P(x)} \langle p, f(x, u) \rangle.$$

We recall the definition of the subdifferential $\partial_- \mathbf{v}(t, x)$ and the horizon subdifferential $\partial^\infty \mathbf{v}(t, x)$ of the function \mathbf{v} at (t, x) :

$$\left\{ \begin{array}{l} \text{(i)} \quad (p_t, p_x) \in \partial_- \mathbf{v}(t, x) \text{ if } (p_t, p_x, -1) \in N_{\mathcal{E}_p(\mathbf{v})}(t, x, \mathbf{v}(t, x)), \\ \text{(ii)} \quad (p_t, p_x) \in \partial^\infty \mathbf{v}(t, x) \text{ if } (p_t, p_x, 0) \in N_{\mathcal{E}_p(\mathbf{v})}(t, x, \mathbf{v}(t, x)). \end{array} \right.$$

Let us recall that the horizon subdifferential $\partial^\infty \mathbf{v}(t, x) = (0, 0)$ whenever the domain of the contingent epiderivative $D_{\uparrow} \mathbf{v}(t, x)$ is dense in $\mathbf{R}_+ \times X$. This happens whenever \mathbf{v} is Lipschitz in a neighborhood of (t, x) .

STATEMENT 9 (Frankowska). *Under the assumptions of Statement 7, the value function $V_{\mathbf{b}}(\mathbf{c})$ is the smallest lower semicontinuous nonnegative function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ satisfying for every $(t, x) \in]0, \infty[\times X$*

$$\left\{ \begin{array}{l} \text{(i)} \quad \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x), \\ \text{(ii)} \quad \text{if } \mathbf{v}(t, x) < \mathbf{c}(t, x), \\ \quad \forall (p_t, p_x) \in \partial_- \mathbf{v}(t, x), \quad -p_t + H(x, \mathbf{v}(t, x), p_x) \leq 0, \\ \quad \forall (p_t, p_x) \in \partial^\infty \mathbf{v}(t, x), \quad -p_t + H^\infty(x, p_x) \leq 0. \end{array} \right.$$

Statement 8 can be stated in terms of subdifferentials, providing the existence and uniqueness of bilateral solutions proved independently by Barron and Jensen and Frankowska.

STATEMENT 10 (Barron–Jensen and Frankowska). *We posit the assumptions of Statement 8. Then the value function $V_{\mathbf{b}}(\mathbf{c})$ is the unique lower semicontinuous solution \mathbf{v} —also called bilateral solution—to the system of differential inequalities: for every $(t, x) \in \text{Dom}(\mathbf{v})$,*

$$(7.11) \quad \left\{ \begin{array}{l} \text{(i)} \quad \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x), \\ \text{(ii)} \quad \text{if } \mathbf{b}(t, x) < \mathbf{v}(t, x) < \mathbf{c}(t, x), \text{ the equations} \\ \quad \forall (p_t, p_x) \in \partial_- \mathbf{v}(t, x), \quad -p_t + H(x, \mathbf{v}(t, x), p_x) = 0, \\ \quad \forall (p_t, p_x) \in \partial_-^\infty \mathbf{v}(t, x), \quad -p_t + H^\infty(x, p_x) = 0, \\ \text{(iii)} \quad \text{if } \mathbf{v}(t, x) = \mathbf{c}(t, x), \text{ the boundary condition} \\ \quad \forall (p_t, p_x) \in \partial_- \mathbf{v}(t, x), \quad -p_t + H(x, \mathbf{v}(t, x), p_x) \geq 0, \\ \quad \forall (p_t, p_x) \in \partial_- \mathbf{v}^\infty(t, x), \quad -p_t + H^\infty(x, p_x) \geq 0, \\ \text{(iv)} \quad \text{if } \mathbf{b}(t, x) = \mathbf{v}(t, x), \text{ the boundary condition} \\ \quad \forall (p_t, p_x) \in \partial_-^\infty \mathbf{v}(t, x) \cap -\partial_+^\infty \mathbf{b}(t, x), \quad -p_t + H^\infty(x, p_x) = 0. \end{array} \right.$$

See [20, 21, 39, 40].

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