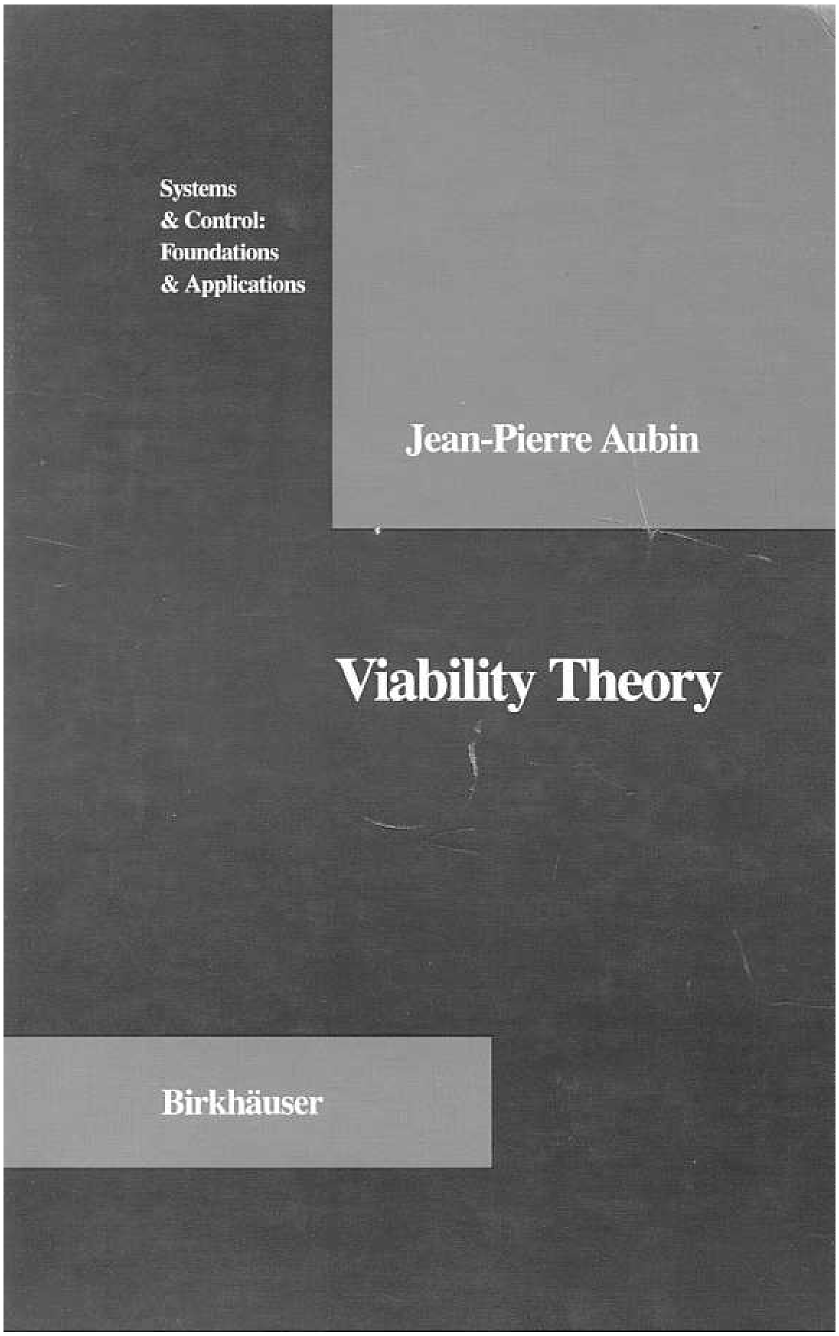
Systems
& Control:
Foundations
& Applications
Jean-Pierre Aubin
Viability Theory
Birkhäuser

This book is dedicated to
Hélène Frankowska

with love

Acknowledgments

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Epigraph

Viability theory is a mathematical theory that offers *mathematical metaphors*[2] *of evolution of macrosystems* arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory.

We shall specifically be concerned with three main common features:

- A nondeterministic (or contingent) engine of evolution, providing several (and even many) opportunities to explore the environment,

- Viability constraints that the state of the system must obey at each instant under "death penalty",

- An inertia principle stating that the "controls" of the system are changed only when viability is at stake.

The first two features are best summarized by the deeply intuitive statement attributed to Democritus by Jacques Monod: *"Everything that exists in the Universe is due to Chance and Necessity"*. The inertia principle is a mathematical formulation of the concept of *punctuated equilibrium* introduced recently in paleontology by Elredge and Gould. It runs against the teleological trend assigning aims to be achieved (in even an optimal way) by the state of the system and the belief that actors control the system for such purposes.

— **Nondeterminism:** We shall mean by this term that *les jeux ne sont jamais faits*, in the sense that at each instant, there are several available, or feasible, evolutions which depend upon the state, or even the history of the evolution of the state of the system up to this time. Therefore, the concept of evolution borrowed from Newtonian mechanics is no longer adequate for such systems. It has led to the misleading identification of mathematics with a *deterministic* paradigm, which implies that *the evolution of macrosystems can be predicted*. Even if we were to accept the existence

[2]Like other means of communications (languages, painting, music, etc.), mathematics provides *metaphors* that can be used to explain a given phenomenon by associating it with some other phenomenon that is more familiar, or at least is felt to be more familiar. This feeling of familiarity, individual or collective, inborn or acquired, is responsible for the inner conviction that this phenomenon is understood.

of deterministic mechanisms[3] underlying the evolution of biological, economic and social macrosystems, we know that such systems often can be inherently unstable - and this places the actual computation of their solutions beyond the capabilities of even the most sophisticated of present-day computers! To "run" models which have some inbuilt structural instability can serve no useful purpose.

Thus, we suppose here that the dynamics responsible for the evolution are not deterministic. This lack of determinism has many different features: it may be due to nonstochastic "uncertainty"[4], to "disturbances"and "perturbations" of various kinds, or to errors in modeling due to the impossibility of a comprehensive description of the dynamics of the system.

In several instances, the dynamics of the system are related to certain "controls", which, in turn, are restricted by state-dependent constraints (closed systems.) Such controls, which we do not dare to call *regulees* instead of controls, are typically

1. *prices or other fiduciary goods* in economics (when the evolution of commodities and services is regulated by Adam Smith's invisible hand or the market, the planning bureau, . . .),
2. *genotypes* or *fitness matrices* in genetics and population genetics (when the evolution of *phenotypes* of a population is regulated by sexual reproduction and mutations),
3. *conceptual controls* or *synaptic matrices* in pattern recognition mechanisms and neural networks (when the sensory-motor state is regulated by learning processes),
4. *affinity matrices* in immunological systems,
5. *strategies* in differential games (when the state of the system is regulated by the decision rules for the players),
6. *coalitions* in cooperative games,
7. *cultural codes* in sociology (when the evolution of societies is regulated by every individual believing and obeying such codes), etc..

[3] And now we discover that some of our "perfectly deterministic" models can exhibit all sorts of different trajectories. These are *chaotic* systems, making prediction virtually impossible.

[4] No a priori knowledge of an underlying probability law on the state of events is made. *Fuzzy viability* provides models where the available velocities can be ranked through a membership cost function to take into account that some velocities are more likely to be chosen than others.

— **Viability:** For a variety of reasons, not all evolutions are possible. This amounts to saying that the state of the system must obey constraints, called *viability constraints*. These constraints include homeostatic constraints in biological regulation, scarcity constraints in economics, state constraints in control, power constraints in game theory, ecological constraints in genetics, sociability constraints in sociology, etc. Therefore, the goal is to select solutions which are *viable in the sense that they satisfy, at each instant, these constraints.*

Viability theorems thus yield selection procedures of viable evolutions, i.e., characterize the connections between the dynamics and the constraints for guaranteeing the existence of at least one viable solution starting from any initial state. These theorems also provide the *regulation processes (feedbacks[5])* that maintain viability, or, even as time goes by, *improve* the state according to some *preference relation*.

Contrary to *optimal control theory*, viability theory does not require any single decision-maker (or actor, or player) to "guide" the system by optimizing an *intertemporal* optimality criterion[6].

Furthermore, the choice (even conditional) of the controls is not made *once and for all* at some initial time, but *they can be changed at each instant so as to take into account possible modifications of the environment of the system*, allowing therefore for *adaptation* to viability constraints.

Finally, by not appealing to intertemporal criteria, *viability theory does not require any knowledge of the future[7]* (even of a stochastic nature.) This is of particular importance when experimentation[8] is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated.*

Hence, *forecasting or prediction of the future are not the issues which we shall address in this book.*

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some

[5]thus providing the central concept of cybernetics as a *solution* to the regulation problem.

[6]the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

[7]Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

[8]Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to "know" the future evolution of the system.

behaviors which do not fit any reasonable optimality criterion.

Therefore, instead of using intertemporal optimization[9] that involves the future, viability theory provides selection procedures of *viable evolutions* obeying, at each instant, state constraints which depend upon the *present or the past*. (This does not exclude *anticipations*, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

Nonetheless, selection through viability constraints may not be discriminating enough. Starting from any state at any instant, several viable solutions may be implemented by the system, including equilibria, which are stationary evolutions[10].

Thus further selection mechanisms need to be devised or discovered. We advocate here a third feature to which a selection procedure must comply, the *Inertia Principle*.

— **Inertia Principle:** which states that *"the controls are kept constant as long as viability of the system is not at stake".*

Indeed, as long as the state of the system lies in the interior of the viability set (the set of states satisfying viability constraints), any regularity control will work. Therefore, the system can maintain the control inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system may evolve while the control remains constant, it may reach the viability boundary with an "outward" velocity. This event corresponds to a period of *crisis*: To survive, the system must find another regulatory control such that the new associated velocity forces the solution back inside the viability set. (See Figure 1.) Alternatively, if the viability constraints can evolve, another way to resolve the crisis is to relax the constraints so that the state of the system lies in the interior of the new viability set. When this is not possible, *strategies for structural change fail*: by design, this means the solution leaves the viability set and "dies".

Naturally, there are several procedures for selecting a viable control when viability is at stake. For instance, the selection at each instant of the controls providing viable

[9] which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

[10] This touches on another aspect of viability theory - that concerned with complexity and robustness: It may be observed that the state of the system becomes increasingly robust the further it is from the boundary of the viability set. Therefore, after some time has elapsed, only the parts of the trajectories furthest away from the viability boundary will remain. This fact may explain the apparent discontinuities ("missing links") and hierarchical organization arising from evolution in certain systems.

evolutions with *minimal velocity* is an example that obeys this inertia principle. They are called *"heavy" viable evolutions*[11] in the sense of heavy trends in economics.

Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of *punctuated equilibrium*[12] introduced recently in paleontology by Elredge and Gould.

In a nutshell, *the main purpose of viability theory is to explain the evolution of a system, determined by given nondeterministic dynamics and viability constraints, to reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

On the mathematical side, viability theory contributed to vigorous renewed interest in the field of "differential inclusions", as well as an engine for the development of a differential calculus of set-valued maps[13]. Indeed, as it often occurs in mathematics, these techniques have already found applications to other domains, for instance, to nonlinear systems theory (tracking, zero dynamics, local controllability and observability[14], control under state constraints, etc.) and Artificial Intelligence (qualitative physics, learning processes, etc.) These techniques can be efficiently used as mathematical tools and have been related to other questions (such as Lyapunov's second

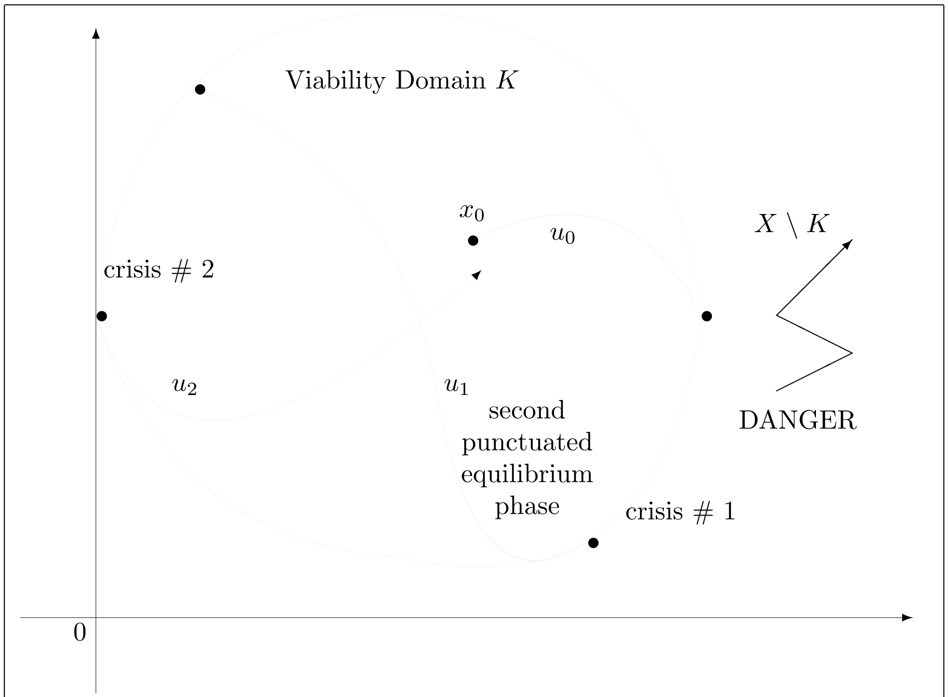
[11]When the controls are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

[12]Excavations at Kenya's Lake Turkana have provided clear evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolution journey that seems to have proceeded in fits and starts. Examination of more than 3,000 fossils by P. Williamson showed how 13 species evolved. The record indicated that the animals stayed much the same for immensely long stretches of time. But twice, about two million years ago and then, 700,000 years ago, the pool of life seemed to explode — set off, apparently, by a drop in the lake's water level. Intermediate forms appeared very quickly, new species evolving in 5,000 to 50,000 years, after millions of years of constancy, leading paleontologists to challenge the accepted idea of continuous evolution.

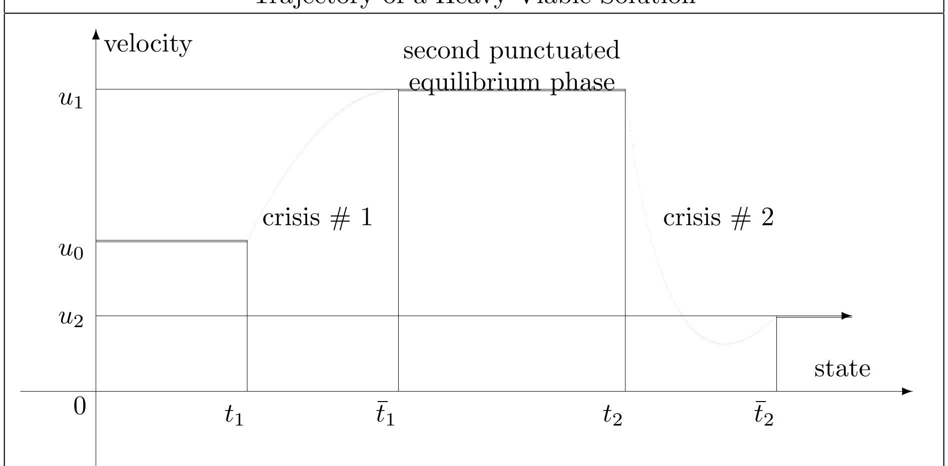
[13]One can say that by now the main results of functional analysis have their counterpart in what can be called *Set-Valued Analysis*. Only the results needed in this book will be presented. An exposition of Set-Valued Analysis can be found in the companion monograph SET-VALUED ANALYSIS by Hlne Frankowska and the author.

[14]These topics will be not developed here. The forthcoming monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hlne Frankowska provides an exhaustive treatment of Control Theory using set-valued analysis and differential inclusions.

Figure 1: Heavy Viable Solutions



Trajectory of a Heavy Viable Solution



Punctuated Evolution of Velocities. Starting from x_0 with the constant velocity u_0, the solution evolves in K until time t_1, (first *punctuated equilibrium phase*) when the velocity. Then a *crisis* happens during which velocities also evolve (as slowly as possible) to maintain viability, until time $\bar{t}_1$ where the velocities are contingent to K during the *second punctuated equilibrium phase.*

method, variational differential equations, etc..)

This is a book of *motivated mathematics*[15], which searches for new sources of mathematical metaphors.

Unfortunately, the length of the theoretical part of viability theory did not allow us to include in this volume the discussion of the motivating problems. Some problems arising in Artificial Intelligence, economics, game theory, biology, cognitive sciences, etc., which have spawned many of the mathematical questions treated below, will be investigated in forthcoming additional volumes.

By looking at common features of otherwise very different systems and looking at shared consequences, it was necessary to set our mathematical metaphors at a fairly high level of abstraction, yielding an amount of information inversely proportional to the height of this level so to speak.

For the time being at least, this theory is still far from providing an ideal description of the evolution of macrosystems. Some potential users (economists, biologists, ...) should not be disappointed or discouraged by the results obtained so far — for it is too early for such a theory to be "applied" in the engineering sense.

However, the available results may explain a portion of "reality" in the extent where *the degree of reality for a social group at a given time is understood in terms of the consensus*[16] *interpretations of the group member's perceptions of their physical,*

[15] We have already mentioned a mathematical metaphor as a means of associating a particular mathematical theory with a certain observed phenomenon. This association can arise in two different ways. The first possibility is to look for an existing mathematical theory which seems to provide a good explanation of the phenomenon under consideration. This is usually regarded as the domain of applied mathematics. However, it is also possible to approach the problem from the opposite direction. Other fields provide mathematicians with metaphors, and this is the domain of what can be called "motivated mathematics".

The ancients divided *analysis* into two forms: *zetetic*, which corresponds to what we mean by motivated mathematics or modeling, and *poristic*, which corresponds to applied mathematics, a procedure by which the validity of the model is confirmed. It is much later, in 1591, that F. Viète added a third form, *rhetic* or *exegetic*, which would correspond to our pure mathematics.

[16] Since our brains are built according to the same biological blueprint, and since the general acceptance of local cultural codes seems to be an innate and universal phenomenon, it is highly probable that the individuals comprising a social group arrive at a consensus wide enough for a reasonably believable concept of reality to emerge. However, the prophets and scholars of each group continually question the validity of the metaphors on which this consensus is based, while the high priests and other guardians of ideological purity ultimately try to transform it into dogma and impose it on the other members of the group. (It often happens that the prophets and scholars themselves eventually become high priests ; movement in the reverse direction is much less common.) It is through

biological, social and cultural environments.

I hope that this book may help readers from different scientific areas to find a common ground for comparing the behaviors of the systems they study and for asking new questions. Anyhow, whatever the ultimate outcome, the motivation provided by the viability problems has already benefited mathematics by suggesting new concepts and lines of argument, by giving some inkling of possible solutions, or by developing new modes of intuition, leading many mathematicians to revive and enrich the theory of dynamical systems and set-valued analysis. The history of mathematics is full of instances in which mathematical techniques motivated by problems encountered in one scientific field have found applications in many others. *It is this "universality" which renders mathematics so fascinating.*

Jean-Pierre Aubin
Paris, May 12, 1990

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this permanent struggle that knowledge evolves. But there is an important difference between the metaphors of science and those of, say, religion or ideology : a metaphor that claims scientific validity must be limited, even narrow, in scope. The more "applied" a scientific study, the narrower it must necessarily be. Scientific theories — scientific metaphors — must be capable of logical refutation (as in mathematics) or of experimental falsification (which of course requires that theories be falsifiable.) Ideologies escape these requirements : the "broader" they are, the more seductive they appear, the more dangerous they can be.

Introduction

Consider the evolution of a control system with (multivalued) feedbacks:

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in U(x(t)) \end{cases}$$

where the state $x(\cdot)$ ranges over a finite dimensional vector-space X and the control $u(\cdot)$ ranges over another finite dimensional vector-space Z. Here, the first equation describes how the control — regarded as an *input* to the system — yields the state of the system[17] — regarded as an *output* — whereas the second inclusion shows how the state-output "feeds back" to the control-input. The set-valued map $U : X \rightsquigarrow Z$ may be called an "a priori feedback". It describes the *state-dependent constraints on the controls*. A solution to this system is a function $t \to x(t)$ satisfying this system for some control $t \to u(t)$.

Viability constraints are described by a closed subset[18] K of the state space: These are intended to describe the "viability" of the system because outside of K, the state of the system is no longer viable.

A subset K *is viable* under the control system described by f and U if for every initial state $x_0 \in K$, there exists at least one solution to the system starting at x_0 which is *viable* in the sense that

$$\forall\, t \geq 0, \quad x(t) \in K$$

The first task is to characterize the subsets having this property. To be of value, this task must be done without solving the system and then checking the existence of viable solutions for each initial state.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the viability set, where the viability of the system is at stake, there should exist a velocity which is in some sense *tangent* to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the Viability Theorem states. But, first, the mathematical implementation of the concept of tangency must be made.

[17] once the initial state is fixed.

[18] We shall naturally investigate in the book the cases when K depends upon the time, the state, the history of the evolution of the state. We shall also cover the case of solutions *which improve a reference preorder when time evolves*.

We cannot be content with viability sets that are smooth manifolds, because inequality constraints would thereby be ruled out. So, we need to "implement" the concept of a direction v tangent to K at $x \in K$, which should mean that starting from x in the direction v, we do not go too far from K.

To convert this intuition into mathematics, we shall choose from among the many ways there are to translate what it means to be "not too far" the one suggested by Bouligand fifty years ago: a direction v is *contingent to K at $x \in K$* if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such directions, which are in some sense "inward", constitutes a closed cone $T_K(x)$, called the *contingent cone*[19] to K at x. Naturally, except if K is a smooth manifold, we lose the fact that the set of contingent vectors is a vector-space.

We then associate with the dynamical system (described by f and U) and with the viability constraints (described by K) the *(set-valued) regulation map R_K*. It maps any state x to the subset $R_K(x)$ consisting of controls $u \in U(x)$ which are *viable* in the sense that

$$f(x,u) \quad \text{is contingent to } K \text{ at } x$$

If, for every $x \in K$, there exists at least one viable control $u \in R_K(x)$, we then say that K is a *viability domain* of the control system with dynamics described by both f and U.

The Viability Theorem we mentioned earlier holds true for a rather large class of systems, called *Marchaud systems*: Beyond imposing some weak technical conditions, the only severe restriction is that, for each state x, the set of velocities $f(x,u)$ when u ranges over $U(x)$ is *convex*[20]. From now on, we assume that the systems under investigation are Marchaud systems.

The basic viability theorem states that for such systems,

a closed subset K is viable under a Marchaud system
if and only if K is a viability domain of this system.

Many of the traditional interesting subsets such as *equilibrium points, trajectories of periodic solutions, the ω-limit sets of solutions, are examples of closed viability domains.*

[19]replacing the linear structure underlying the use of tangent spaces by the contingent cone is at the root of *Set-Valued Analysis*.

[20]This happens for the class of control systems of the form

$$x'(t) \;=\; f(x(t)) + G(x(t))u(t)$$

where $G(x)$ are linear operators from the control space to the state space and when the control set U (or the images $U(x)$) are convex.

Actually, equilibrium points $\overline{x}$, which are solutions to

$$f(\overline{x}, \overline{u}) = 0 \text{ for some } \overline{u} \in U(\overline{x})$$

are the smallest viability domains, the ones reduced to a single point. This is because being *stationary states*, the velocities $f(\overline{x}, \overline{u})$ are equal to zero. Furthermore, there exists a basic and curious link between viability theory and general equilibrium theory:

every compact convex viability domain
contains an equilibrium point.

This statement is an equivalent version of the 1910 *Brouwer Fixed Point Theorem*, the cornerstone of nonlinear analysis, which finds here a particularly relevant formulation (viability implies stationarity.)

What happens if a closed subset K is not a viability domain?

First, we characterize the points of the boundary from which some, or all solutions enter or leave the subset (anatomy of a set).

Second, we also look for closed subsets of K which are viability domains. We shall prove that

there exists a largest closed viability domain contained in K.

This domain will be denoted $\mathrm{Viab}(K)$ and called the *viability kernel*[21] of K. It may be empty (in this case, the subset K is some kind of "repeller".) Furthermore, every closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope*.

Third, one can also keep the set of constraints and change the dynamics, as it is done in mechanics of unilateral constraints (variational differential equations).

The Viability Theorem also provides a *regulation law* for regulating the system in order to maintain the viability of a solution: The viable solutions $x(t)$ are regulated by viable "open loop controls" $u(t)$ through the regulation law:

$$\text{for almost all } t, \quad u(t) \in R_K(x(t))$$

The multivaluedness of the regulation map (this means that several controls $u(t)$ may exist in $R_K(x(t))$) is an indicator of the "robustness" of the system: *The larger*

[21] This concept of viability kernel happens to be a quite efficient mathematical tool that we shall use often.

It is also closely related to the concept of *zero dynamics* introduced recently by Byrnes and Isidori in control theory.

the set $R_K(x(t))$, the larger the set of disturbances which do not destroy the viability of the system !

Observe that solutions to a control system are solutions to the differential inclusion $x'(t) \in F(x(t))$ where, for each state x, $F(x) := f(x, U(x))$ is the subset of feasible velocities. Conversely, a differential inclusion is an example of a control system in which the controls are the velocities ($f(x, u) = u$ & $U(x) = F(x)$.)

As far as servomechanisms are concerned, the question arises of how to build mechanisms for selecting a *unique* control $\widehat{u}(x)$ in $R_K(x)$ for each state x. Such a map $\widehat{u}(\cdot)$, associating with every x a single control $\widehat{u}(x)$ is called a *closed loop* control (or single-valued feedback.) This is because it allows the system to *automatically associate with any state $x(t)$ the control $\widehat{u}(x(t))$* which produces a viable solution through the differential equation

$$x'(t) \;=\; f(x(t), \widehat{u}(x(t)))$$

An interesting example of closed loop control is provided by *slow solutions*. These are the solutions regulated by the controls $u^0(x) \in R_K(x)$ with minimal norm. Despite the fact that $u^0(\cdot)$ is not necessarily continuous, we shall prove that the above differential equation still has solutions. For instance, when the controls are the velocities of the system, viable solutions with *velocities of minimal norm* are implemented by such a selection procedure. This is why they are called *slow solutions*.

Such selection procedures by closed loop controls answer many engineering control problems. But they may not be adequate for the type of systems arising in economic, social, biological and cognitive sciences, as well as in some areas of engineering where the controls must evolve continuously. Here, we are looking for selection procedures which obey the *inertia principle*: keep the control constant as long as the viability of the system is not at stake.

We can reformulate the inertia principle by saying that if the derivative of a viable open loop control $u(\cdot)$ is equal to 0, then this control is the one which is chosen and implemented.

This raises several questions.

— The first one concerns controls which are smooth (at least, differentiable almost everywhere.) (This issue may be relevant for engineering problems, where the lack of continuity of controls $u(t) := \widehat{u}(x(t))$ can be damaging.)

— The second one deals with the problem of differentiating the regulation law.

— The third is to find selections (called *dynamical closed loops*) of the derivative of the regulation map, with which we obtain a system of differential equations which govern the *smooth* viable evolution of both the state and the control.

— The fourth is to find some feedback controls as solutions to systems of first-order partial differential inclusions.

We see at once that this programme requires a concept of derivative of a set-valued map and a chain rule formula in order to differentiate the regulation law.

The idea behind the construction of a differential calculus of set-valued maps is simple and goes back to the very origins of differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function:

the tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \mapsto f'(x)u$

Consider now a set-valued map $F : X \rightsquigarrow Y$, which is characterized by its graph (the subset of pairs (x, y) such that y belongs to $F(x)$.)

The contingent cone to the graph of F at the point (x, y) of its graph is the graph of the contingent derivative of the set-valued map F at a point (x, y)

The contingent derivative at (x, y) is a set-valued map from X to Y denoted by $DF(x, y)$.

Contingent derivatives keep enough properties of the derivatives of smooth functions to be quite efficient. They enjoy a rich calculus, and they enable such basic theorems of analysis as the inverse function theorem to be extended to the set-valued case.

The chain rule is an example of a property which is still true in this framework: Assume that we start from a "smooth state", producing a viable solution $x(t)$ and a viable control $u(t)$ which are both differentiable (almost everywhere.) Then we can "differentiate" the regulation law to obtain a "first order regulation law":

$$\text{for almost all } t, \quad u'(t) \ \in \ DR_K(x(t), u(t))(x'(t))$$

Heavy viable solutions are the ones regulated by the controls whose velocities have minimal norm in the set

$$DR_K(x(t), u(t))(f(x(t), u(t)))$$

For instance, when the controls are the velocities of the system, we choose viable solutions with *acceleration of minimal norm*, i.e., accelerations with maximum inertia. This is why these solutions are called *heavy solutions*. This point of view leads to the introduction of *viability niches* $N(u)$ associated with controls u. These are (possibly empty) subsets consisting of states x such that the zero velocity belongs to $DR_K(x,u)(f(x,u))$. In such a *viability niche* $N(u)$*, the state can evolve while being regulated by the stationary control* u*.*

Finally, using the concept of contingent derivative, we can obtain feedbacks as solutions of partial differential inclusions.

Let us conclude this introduction with some motivational comments.

In economics, the viability constraints are the scarcity constraints. We can replace the fundamental Walrasian model[22] of resource allocations by a decentralized dynamical model in which the role of the controls is played by the prices[23] (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of the allocations. It is possible that among these viable prices, the market (or even a planning bureau) would have a tendency to choose heavy solutions.

In cooperative games, coalitions of player may play the role of controls: each coalition acts on the environment by changing it through a dynamical system. Here, a coalition is described by the players's rate of participation, positive or negative, according to their cooperative or anti-cooperative behavior. The regulation law provides, in this case, an explanation of the evolution of coalitions and alliances.

In noncooperative games, viability constraints describe power relations among players. Each players associates with each state a subset in which the other players are confined to choosing their own states. Strategies take the role of controls, through which the players act on the state according to some differential equations. We often observe that the inertia principle is operative. The choice of viable strategies (or of their velocities) can be made, at each instant and in a myopic way, by standard game theoretical mechanisms, in such a way as to comply with the inertia principle.

[22] Most static models of mathematical economics are based in the last analysis on *general equilibrium theory*. They can be reformulated in a dynamical framework, by slightly changing the underlying dynamical system. (Walrasian tâtonnement, which does not produce viable solutions, except when they reach an equilibrium.)

[23] and other fiduciary goods for which the scarcity constraint can be transgressed. Unlike physical goods, they are limited only by measures dictated by the trust (or, rather, the tolerance) of the agents. Any disequilibrium that cannot exist in physical goods can then be transferred to the fiduciary goods.

In genetics and population genetics, the viability constraints are the ecological constraints, the state describes the phenotype and the controls are genotypes or fitness matrices. The regulation law may explain the evolution of genotypes or fitness matrices derived from the dynamics and the ecological constraints.

In sociology, a society can be interpreted as a set of individuals subject to viability constraints. They correspond to what is necessary to the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. These cultural codes play the role of controls. The regulation law may represent the evolution of cultural codes for maintaining society's viability, the evolution of which obeys the inertia principle. Such a metaphor may account for the small number of them and the robustness of religions, ideologies and scientific paradigms. It may also explain the phenomena of massive conversions to new cultural codes.

In cognitive sciences, the state describes the sensory-motor couple of the cognitive system, while the control translates into what could be called a conceptual control (which is the synaptic matrix in neural networks.) The state and control are related by a pattern recognition mechanism, which recognizes the (variations of) the perception of the action of the automaton on the environment. The regulation law provides a learning process, that goes beyond simple stimulus-response processes: it associates with each sensory-motor state a subset of (learned) conceptual controls. It seems that in this case, again, the inertia principle applies.

Outline of the Book

Instead of beginning with viability theorems for differential inclusions, we prefer to sketch in Chapter 1 the role of the concept of viability domain in the much simpler case of differential equations. (The first viability theorem was proved in 1942 by Nagumo.)

For a variety of reasons, an important example of a viability set is the probability simplex. Whenever the state of a system is difficult to model mathematically, one way to overcome this difficulty is to deal with probabilities, frequencies, concentrations, proportions, etc., and the probability simplex then naturally appears. Systems controlled by scalar controls (called flux) of the form

$$x_i'(t) \quad = \quad x_i(t)(f_i(x(t)) - x(t)u), \quad i = 1, \ldots, n$$

are called *replicator systems*. They are encountered in such diverse fields as biochemistry (Eigen & Schuster's hypercycle), ethology (Maynard-Smith's game for behavioral strategies), population dynamics (Fisher's model of the evolution of genes in a population), ecology, etc. These examples are presented in the first chapter.

We also included in this chapter viability and invariance theorems for *stochastic differential equations*, which provide another way to treat uncertainty.

This chapter can be bypassed by readers mainly interested in differential inclusions and control systems.

Chapter 2 deals with the minimal information about set-valued maps that is needed to prove the viability theorems for differential inclusions. Upper and lower semicontinuous set-valued maps are defined. Then our basic result, *the Convergence Theorem*, is proved. Since this involves convex-valued maps, some results on support functions of convex subsets are recalled in this chapter. Closed convex processes, which are the set-valued analogues of continuous linear operators, enjoy most of the properties of linear operators, including Banach's closed graph theorem and the uniform boundedness theorem. These results are reviewed, because contingent derivatives of set-valued maps being closed processes, they will be used later.

Chapter 3 is basic: it states and proves the main viability theorems (in locally compact, open and closed viability sets respectively) and shows that the solution map is upper semicontinuous. We also prove a *stability result:* (upper) limits of viability domains are still viability domains and we show that ω*-limit sets* of solutions, limits of solutions when the time goes to infinity (equilibria), trajectories of periodic solutions are examples of closed viability domains.

We adapt Saari's principle on the chaotic behavior of discrete systems to the case of differential inclusions. The viability domain is divided into a number of cells in such a way that each of them can be "visited" in any given way by at least one trajectory of a differential inclusion.

We then proceed in Chapter 4 with further properties of the *viability kernels* of closed subsets: There exists a largest closed viability domain contained in a closed subset, called the viability kernel, which enjoys many properties which are investigated in this chapter. Important concepts of biomathematics such as *permanence* and *fluctuation* can be defined in terms of viability kernels. On the other hand, each closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope*.

The analysis is refined by introducing *exit time functions* associating with each initial state the first instant at which at least one solution starting from this state leaves the viability set. Viability kernels are the subsets of states with infinite exit time. We then introduce *exit tubes*, which are the subsets of states from which at least one solution satisfies the viability constraints for a prescribed length of time.

We then study the *anatomy of a set* by distinguishing inward and outward areas of the boundary of a set. It is also shown that the boundary of a viability kernel is also a viability domain.

These facts among others are used to study several *viability kernel algorithms*, including the *zero dynamics algorithm*, which converge to viability domains and/or kernels.

We devote the fifth chapter to the study of *invariant subsets*, which are sets K with the property that *all solutions* to a differential inclusion starting from a state in K are viable in K.

We need for that purpose more information on contingent cones, which are involved in a crucial way in the characterization of the viability and invariance properties. For this reason, we review some results about these cones before proceeding any further. We recall some useful formulas of the calculus of contingent cones (proved in the fourth chapter of *Set-Valued Analysis*.)

Several characterizations of invariance are provided, one of which is based on the fundamental Filippov Theorem dealing with differential inclusions with Lipschitz right-hand sides. It implies that the solution map is lower semicontinuous. This latter property is crucial to prove the existence of *invariance kernels*, which are the largest closed invariant domains contained in closed subsets.

It also implies the *semi-permeability* property of the boundary of the viability kernel of a closed subset, which states that no solution can cross the boundary to enter the interior of the viability kernel.

These invariance and viability kernels are needed to define *defeat and victory domains of a target.*

We illustrate these results in the case of *linear differential inclusions*, which are differential inclusions whose right-hand sides are closed convex processes. In this framework, we show that the concepts of invariance and viability domains are dual.

We tackle in Chapter 6 the problem of *regulating control systems by closed loop controls (single-valued feedback controls.)* The problem we have to solve is that of finding selections of the regulation map, possibly continuous. The latter are provided by Michael's Theorem, but in a non constructive way. Hence we have to design *selection procedures* which yield explicit selections, which may not be continuous, but still provide viable solutions when fed back to the differential equation governing the evolution of the control system. These selection procedures provide in particular *slow viable solutions* regulated by controls with minimal norm. For that purpose we need to complete our study of lower semicontinuous maps and provide lower semicontinuity criteria for finite and infinite intersections of lower semicontinuous maps.

Chapter 7 deals with *the inertia principle, heavy viable solutions and "punctuated equilibria", ramp controls, etc.*, which constitute the main motivations of viability theory.

At this point, we need to differentiate the regulation map. Hence this chapter starts with the shortest introduction to derivatives of set-valued maps needed to proceed. It continues with the construction of regulation maps providing viable controls that are almost everywhere differentiable.

Once we know the regulation maps yielding differentiable controls, we can differentiate the regulation law and discover the system of differential inclusions which governs the evolution of both the state and the control of the system. Then, by using the selection procedures introduced in the preceding chapter, we are able to define dynamical closed loops and, among them, the ones which provide heavy viable solutions. Viability problems for second order differential inclusions, which are first order systems in disguise, are also investigated in this chapter.

The *tracking problem*, as well as *observability, decentralization, hierarchical* issues, are studied in Chapter 8 in the framework of viability theory. The common thread of these problems is the connection between two dynamical systems through an *observation map*: Are some or all solutions to these differential inclusions linked by this observation map, in the sense that its graph is a viable or invariant manifold ? The viability theorems applied to the graphs of the observation maps imply that such observation maps are solutions to some *systems of first-order partial differential inclusions*, where the derivatives are taken in the contingent sense.

Derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions to partial differential equations and inclusions. Contingent derivatives offer another way to weaken the required properties of a derivative, loosing the linear character of the differential operator, but allowing a pointwise definition. They provide a convenient way to treat hyperbolic problems and also allow us to look for solutions among set-valued maps, since we know how to differentiate them. Set-valued solutions constitute a useful framework to describe shocks for instance.

We study the existence of both single-valued and set-valued solutions to such partial differential inclusions, as well as a variational principle.

Differential inequalities, Lyapunov functions and related matters can also be analyzed in terms of special viability problems where the viability sets are epigraphs of functions or, more generally, graphs of preorders. This allows us to include, among the candidates that enjoy Lyapunov-type inequalities, not only differentiable functions but also lower semicontinuous functions. Thus we derive from viability theorems several generalizations of classical results. Applying to this situation the concept of viability kernel, we infer the existence of the *smallest Lyapunov function larger than a given one*.

Asymptotic stability is treated here in the framework of viability theory. These are explained in Chapter 9.

Chapter 10 gathers miscellaneous issues, such as *variational differential inequalities*. The question is the following: If we take a differential inclusion and a closed subset which is not a viability domain, can we modify the set-valued map F in such a way that K becomes a viability domain for the new map? The method is straightforward: we project the images $F(x)$ onto the contingent cone $T_K(x)$ (and obtain, when K is convex, variational differential inequalities.) By doing so, we lose both the convexity of the images and the upper semicontinuity. However, it is still possible to prove the existence of the projected system and even, under stronger assumptions, the existence of slow solutions.

The second section of chapter 10 deals with *fuzzy differential inclusions*. The right-hand sides of such differential inclusions are *fuzzy* subsets, whose membership functions are cost functions taking their values in $[0, \infty]$ (instead of $[0, 1]$ for membership functions of usual fuzzy sets). The concept of uncertainty involved in differential inclusions becomes more refined, by allowing the velocities not only to depend in a plain multi-valued way upon the state of the system, but also in a fuzzy way.

The viability theorems are adapted to fuzzy differential inclusions and to sets of state constraints which are either usual or fuzzy. The existence of a largest closed fuzzy viability domain contained in a given closed fuzzy subset is also provided.

The third section presents a very short introduction to some numerical aspects of differential inclusions. The convergence of solutions to implicit and projected explicit finite-difference schemes to viable solutions of a differential inclusion is proved.

The fourth section deals with the adaptation of continuous Newton's methods for finding an equilibrium of a set-valued map: it happens that this is also a viability problem.

Chapter 11 is devoted to *time-dependent viability sets* $t \rightsquigarrow P(t)$, naturally called *tubes*. Tubes which contain at least one viable solution[24] starting from any initial state $x_0 \in P(t_0)$ at any initial time are viability tubes. These are solutions to a set-valued differential inclusion of the form $F(x) \cap DP(t, x)(1) \neq \emptyset$.

We will study the Cauchy problem, where we look for minimal viability tubes satisfying an initial condition.

One can show that their "limits"[25] when $t \to \infty$ are viability domains, and actually, attractors. If we use such viability tubes to guide a solution towards a target, we see that a necessary condition for a subset to be an asymptotic target is that it is a viability domain.

Of much greater importance for systems arising in biology, economics and cognitive sciences is the case when *both the velocity and viability sets depend upon the history of*

[24]in the sense that $x(t) \in P(t)$ for all t.

[25]in the sense of upper limits. When the tube $P(t) := \{x(t)\}$ is single-valued, this upper limit boils down to the ω-limit set.

the evolution of the state. Delays

$$\forall\, t \geq 0,\;\; x(t) \;\in\; M(x(t-\theta_1), \ldots, x(t-\theta_p))$$

accumulated consequences of past evolution

$$\forall\, t \geq 0,\;\; x(t) \;\in\; M\left(\int_{-\infty}^{t} A(t-s)x(s)ds\right)$$

all these features fall under the case called *functional viability*. Here, functional viability sets $\mathcal{K}$ are subsets consisting of time-dependent functions, and viable solutions are the solutions which evolve in such function subsets in the sense that for all $t \geq 0$, $x(t+\cdot) \in \mathcal{K}$. It is the topic of Chapter 12.

Can viability theorems be extended to partial differential equations and inclusions? The answer is positive, at least for elliptic and parabolic type inclusions, as is shown in chapter 13. In this case, viability sets are comprised of spatial functions (functions depending upon the space variable.) The situation becomes more complex, because we have to work with unbounded operators on Hilbert spaces, but still, the statements which are expected to be true can be proved.

Chapter 14 treats differential games, where the controls are regarded as *strategies* used by the players to govern the evolution of the states of the game. Here, intertemporal criteria involved classically in differential games are replaced by viability constraints representing *power relations* among players, describing the constraints imposed by one player on the other. We characterize winability, playability properties adequately defined by *contingent Isaacs' equations*.

We shall prove the existence of continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating in a myopic way optimal strategies in a cooperative framework or minimax strategies in a noncooperative environment.

In other words, *the players can implement playable feedbacks by playing for each state a static game on the strategies.*

We also provide closed loop decision rules, which *operate on the velocities of the strategies*, (regarded as *decisions*).