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Optima and Equilibria

An Introduction to Nonlinear Analysis

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НЕЛИНЕЙНЫЙ АНАЛИЗ и его экономические приложения

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COLLECTION MATHÉMATIQUES APPLIQUÉES POUR LA MAÎTRISE
SOUS LA DIRECTION DE P.G. CIARLET ET J.-L. LIONS
Exercices d'analyse non linéaire
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Introduction

This is a book on nonlinear analysis and its underlying motivations in economic science and game theory. It is entitled *Optima and Equilibria* since, in the final analysis, response to these motivations consists of perfecting mechanisms for selecting an element from a given set. Such *selection mechanisms* may involve either

- *optimisation of a criterion function* defined on this set (or of several functions, in the case of multi-criterion problems in game theory), or
- searching in this set for an *equilibrium of a given* undelying dynamical system*, which a stationary solution, of this dynmical system@.*

The mathematical techniques used have their origins in what is known is nonlinear analysis, and in particular, in convex analysis.

Progress in nonlinear analysis has proceeded hand in hand with that in the theory of economic equilibrium and in game theory; there is interaction between each of these areas, mathematical techniques are applied in economic science which, in turn, motivates new research and provides mathematicians with new challenges.

In the course of the book we shall have occasion to interrupt the logical course of the exposition with several historical recollections. Here, we simply note that it was Léon Walras who, at the end of the last century, suggested using mathematics in economics, when he described certain economic agents as automata seeking to optimise evaluation functions (utility, profit, etc.) and posed the problem of economic equilibrium. However, this area did not blossom until the birth of nonlinear analysis in 1910, with Brouwer's fixed-point theorem, the usefulness of which was recognised by John von Neumann when he developed the foundations of game theory in 1928. In the wake of von Neumann came the works of John Nash, Kakutani, Aumann, Shapley and many others which provided the tools used by Arrow, Debreu, Gale, Nikaïdo et al. to complete Walras's construction, culminating in the 1950s in the proof of the existence of economic equilibria. Under pressure from economists, operational researchers and engineers, there was stunning progress in optimisation theory, in the area of linear programming after the Second World War and following the work of Fenchel, in the 1960s in convex analysis. This involved the courageous step of differentiating nondifferentiable functions by Moreau and Rockafellar at the dawn of the 60's, and set-valued maps tenety years later,@ albeit in a different way and for different reasons than in distribution theory discovered by Laurent Schwartz in the 1950s. (see for instance (Aubin and Frankowska 1990) and (Rockafellar and Wets 1997)).@ These works provided for use of the rule hinted at by Fermat more than three hundred years ago, namely that the derivative of a function is zero at points at which the function attains its optimum, in increasingly complicated problems of the calculus of variations and optimal control theory. The 1960s also saw a re-awakening of interest in nonlinear analysis for the different problem of solving nonlinear, partial-differential equations. A profusion of new results were used to clarify many questions and simplify proofs, notably using an inequality discovered in 1972 by Ky Fan.

At the time of writing, at the dawn of the 1980s, it is appropriate to take stock and draw all this together into a homogeneous whole, to provide a concise and self-contained appreciation of the fundamental results in the areas of nonlinear analysis, the theory of economic equilibrium and game theory.

Our selection will not be to everyone's taste: it is partial. For example, in our description of the theory of economic equilibrium, we do not describe consumers in terms of their utility functions but only in terms of their demand functions. A minority will certainly hold this against us. However, conscious of the criticisms made of the present-day formalism of the Walrasian model, we propose an alternative which, like Walras, retains the explanation of prices in terms of their decentralising virtues and also admits dynamic processing.

Our succinct introduction to game theory is not orthodox, in that we have included the theory of cooperative games in the framework of the theory of fuzzy games.

In the book we accept the shackles of the static framework that are at the origin of the inadequacies and paradoxes which serve as pretexts for rejection of the use of mathematics in economic science. J. von Neumann and O. Morgenstern were also aware of this when, in 1944, at the end of the first chapter of *Theory of Games and Economic Behaviour*, they wrote:

'*Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and, therefore, preferable. But there is ample evidence from other branches of science that it is futile to try to build one as long as the static side is not thoroughly understood...*'

'*Finally, let us note a point at which the theory of social phenomena will presumably take a very definite turn away from the existing patterns of mathematical physics.*

This is, of course, only a surmise on a subject where much uncertainty and obscurity prevail...'

'*Our static theory specifies equilibria... A dynamic theory, when one is found – will probably describe the changes in terms of simpler concepts.*'

Thus, this book describes the static theory and the tool which may be used to develop it, namely nonlinear analysis.

It is only now that we can hope to see the birth of a dynamic theory calling upon all other mathematical techniques (see (Aubin and Cellina 1984), (Aubin 1991) and (Aubin 1997)).@ But, as in the past, so too now, and in the future, the static theory must be placed in its true perspective, even though this may mean questioning its very foundations, like March and Simon (who suggested replacing optimal choices by choices that are only satisfactory) and many (less fortunate) others. Imperfect yet perfectible, mathematics has been used to put the finishing touches to the monument the foundation of which was laid by Walras. Even if this becomes an historic monument, it will always need to be visited in order to construct others from it and to understand them once constructed.

Of course, the book only claims to present an *introduction* to nonlinear analysis which can be read by those with the basic knowledge acquired in a first-level university mathematics course. It only requires the reader to have mastered the fundamental notions of topology in metric spaces and vector spaces. Only Brouwer's fixed-point theorem is assumed.

This is a book of *motivated mathematics*, i.e. a book of mathematics motivated by economics and game theory, rather than a book of mathematics *applied* to these fields. We have included a *Foreword* to take up this issue which deals with pure, applied and motivated mathematics. In our view, this is important in order to avoid setting too great store by the importance of mathematics in its interplay with social sciences.

The book is divided into two parts. Part I describes the theory, while Part II is devoted to exercises, and problem statements and solutions. The book ends with an Appendix containing a *Compendium of Results*.

Foreword

By Way of Warning

As in ordinary language, metaphors may be used in mathematics to explain a given phenomenon by associating it with another which is (or is considered to be) more familiar. It is this sense of familiarity, whether individual or collective, innate or acquired by education, which enables one to convince oneself that one has understood the phenomenon in question.

Contrary to popular opinion, mathematics is not simply a *richer* or *more precise* language. Mathematical reasoning is a separate faculty possessed by all human brains, just like the ability to compose or listen to music, to paint or look at paintings, to believe in and follow cultural or moral codes, etc.

But it is impossible (and dangerous) to compare these various faculties within a *hierarchical* framework; in particular, one cannot speak of the *superiority* of the language of mathematics.

Naturally, the construction of mathematical metaphors requires the autonomous development of the discipline to provide theories which may be substituted for or associated with the phenomena to be explained. This is the domain of pure mathematics. The construction of the mathematical corpus obeys its own logic, like that of literature, music or art. In all these domains, an aesthetic satisfaction is at once the objective of the creative activity and a signal which enables one to recognise successful works. (Likewise, in all these domains, fashionable phenomena – reflecting social consensus – are used to develop aesthetic criteria).

That is not all. A mathematical metaphor associates a mathematical theory with another object. There are two ways of viewing this association. The first and best-known way is to search for a theory in the mathematical corpus which corresponds as precisely as possible with a given phenomenon. This is the domain of *applied mathematics*, as it is usually understood. But the association is not always made in this way;

the mathematician should not be simply a purveyor of formulae for the user. Other disciplines, notably physics, have guided mathematicians in their selection of problems from amongst the many arising and have prevented them from continually turning around in the same circle by presenting them with new challenges and encouraging them to be daring and question the ideas of their predecessors. These other disciplines may also provide mathematicians with metaphors, in that they may suggest concepts and arguments, hint at solutions and embody new modes of intuition. This is the domain of what one might call *motivated mathematics* from which the examples you will read about in this book are drawn.

You should soon realize that the work of a *motivated* mathematician is *daring*, above all where problems from the *soft* sciences, such as social sciences and, to a lesser degree, biology, are concerned. Many hours of thought may very well only lead to the mathematically obvious or to problems which cannot be solved in the short term, while the same effort expended on a structured problem of pure or applied mathematics would normally lead to visible results.

Motivated mathematicians must possess a sound knowledge of another discipline and have an adequate arsenal of mathematical techniques at their fingertips together with the capacity to create new techniques (often similar to those they already know). In a constant, difficult and frustrating dialogue they must investigate whether the problem in question can be solved using the techniques which they have at hand or, if this is not the case, they must *negotiate* a deformation of the problem (a possible restructuring which often seemingly leads to the original model being forgotten) to produce an *ad hoc* theory which they sense will be useful later. They must convince their colleagues in the other disciplines that they need a very long period for learning and appreciation in order to grasp the language of a given theory, its foundations and main results and that the proof and application of the simplest, the most naive and the most attractive results may require theorems which may be given in a number of papers over several decades; in fact, one's comprehension of a mathematical theory is never complete. In a century when no more cathedrals are being built, but impressive skyscrapers rise up so rapidly, the profession of the motivated mathematician is becoming rare. This explains why users are very often not aware of how mathematics could be used to improve aspects of the questions with which they are concerned. When users are aware of this, the intersection of their central areas of interest with the preoccupations of mathematicians is often small – users are interested in *immediate* impacts on their problems and not in the mathematical techniques that could be used and their relationship with the overall mathematical structure.

It is these constraints which distinguish mathematicians from researchers in other disciplines who use mathematics, with a *different time constant*. It is clear that the slowness and the esoteric aspect of the work of mathematicians may lead to impatience amongst those who expect them to come up with rapid responses to their problems. Thus, it is vain to hope to *pilot* the mathematics *downstream* as those who believe that scientific development may be *programmed* (or worse still, *planned*) may suggest.

In Part I, we shall only cover aspects of pure mathematics (optimisation and nonlinear analysis) and aspects of mathematics motivated by economic theory and game theory. It is still too early to talk about *applying* mathematics to economics. Several fruitful attempts have been made here and there, but mathematicians are a long way from developing the mathematical techniques (the domains of pure mathematics) which are best adapted to the potential applications.

However, there has been much progress in the last century since pioneers such as Quesnais, Boda, Condorcet, Cournot, Auguste and Léon Walras@, despite great opposition, dared to use the tools of mathematics in the economic domain. Brouwer, von Neumann, Kakutani, Nash, Arrow, Debreu, Scarf, Shapley, Ky Fan and many others all contributed to the knowledge you are about to share.

You will surely be disappointed by the fact that these difficult theorems have little relevance to the major problems facing mankind. But, please don't be impatient, like others, in your desire for an overall, all-embracing explanation. Professional mathematicians must be very humble and modest.

It is this modesty which distinguishes mathematicians and scientists in general from prophets, ideologists and modern system analysts. The range of scientific explanations is reduced, hypotheses must be contrasted with logic (this is the case in mathematics) or with experience (thus, these explanations must be falsifiable or@ refutable). Ideologies are free from these two requirements and thus all the more seductive.

But what is the underlying motivation, other than to contribute to an explanation of reality? We are brains which perceive the outside world and which intercommunicate in various ways, using natural language, mathematics, bodily expressions, pictorial and musical techniques, etc.

It is the consensus on the consistency of individual perceptions of the environment, which in some way measures the degree of reality in a given social group.

Since our brains were built on the same model, and since the ability to believe in explanations appears to be innate and universal, there is a very good chance that a social group may have a sufficiently broad consensus that its members share a common concept of reality. But prophets and sages often challenge this consensus, while high priests and guardians of the ideology tend to dogmatise it and impose it on the members of the social group. (Moreover, quite often prophets and sages themselves become the high priests and guardians of the ideology.) This continual struggle forms the framework for the history of science.

Thus, research must contribute to the evolution of this consensus, teaching must disseminate it, without dogmatism, placing knowledge in its relative setting and making you take part in man's struggle, since the day when *Homo sapiens, sapiens* ... But we do not know what happened, we do not know when, why or how our ancestors sought to agree on their perceptions of the world to create myths and theories, when why or how they transformed their faculty for exploration into an insatiable curiosity, when, why or how mathematical faculties appeared, etc.

It is not only the utilitarian nature (in the short term) which has motivated mathematicians and other scientists in their quest. We all know that without this permanent, free curiosity there would be no technical or technological progress.

Perhaps you will not use the techniques you will soon master and the results you will learn in your professional life. But the hours of thought which you will have devoted to understanding these theories will (subtly and without you being aware) shape your own way of viewing the world, which seems to be the hard kernel around which knowledge organizes itself as it is acquired. At the end of the day, it is at this level that you must judge the relevance of these lessons and seek the reward for your efforts.

Outline of the Book

In the first three chapters, we discuss the existence of solutions minimising a function, in the general framework (Chapter 1) and in the framework of convex functions (Chapter 3). Between times, we prove the projection theorem (on which so many results in functional analysis are based) together with a number of separation theorems and we study the duality relationship between convex functions and their conjugate functions.

The following three chapters are devoted to Fermat's rule which asserts that the gradient of a function is zero at any point at which the function attains its minimum. Since convex functions are not necessarily differentiable in the customary sense, the notion of the 'differential' had to be extended for Fermat's rule to apply. The simple, but unfamiliar idea consists of replacing the concept of gradient by that of subgradients, forming a *set* called a *subdifferential*. We describe a subdifferential calculus of convex functions in Chapter 4 and in Chapter 5, we exploit Fermat's rule to characterise the solutions of minimisation problems as solutions of a many-valued equation (called an *inclusion*) or as the subdifferential of another function.

In Chapter 6, we define the notion of the generalised gradient of a locally Lipschitz function, as proposed by F. Clarke in 1975. This enables us to apply Fermat's rule to functions other than differentiable functions and convex functions. It will be useful in the study of cooperative games.

Chapters 7 and 8 are devoted to the theory of two-person games; here, we prove two fundamental minimax theorems due to von Neumann (1928) and Ky Fan (1962).

In Chapter 9, we use Ky Fan's inequality to prove the existence theorems for solutions of the inclusion

$$0 \in C(\overline{x})$$

(where C is a set-valued map) together with the fixed-point theorems which we shall use to prove the existence of economic equilibria and non-cooperative equilibria in the theory of n-person games.

In Chapter 10, we provide two explanations of the role of prices in a decentralisation mechanism which provides economic agents with access to sufficient information for them to take their decisions without knowing the global state of the economic system or the decisions of other agents. The first explanation is provided by the Walrasian model, as formalised since the fundamental work of Arrow and Debreu in 1954. The second explanation is compatible with dynamic models which go beyond the scope of this book and for which we refer to (Aubin, 1997)@.

Chapter 11 is devoted to a study of the von Neumann growth model and provides us with the opportunity to prove the Perron–Frobenius theorem on the eigenvalues of positive matrices.

In Chapter 12 we adapt the concepts introduced in Chapter 7 for 2-person games to study n-person games.

Chapter 13 deals with standard cooperative games (using the behaviour of coalitions of players) and fuzzy cooperative games (involving fuzzy coalitions of players).

The collection of 165 exercises and 48 problems with solutions in Part II has two objectives in view. Firstly, it will provide the reader of Part I with the wherewithal to practise the manipulation of the new concepts and theorems which he has just read about.

Whilst, once assimilated, the mathematics may appear simple (and even self-evident), a great deal of time (and energy) is needed to familiarise oneself with these new cognitive techniques.

If a passive approach is taken, the assimilation will be difficult; for, strange as it may seem, emotional mechanisms (or, in the terminology of psychologists, motivational mechanisms) play a crucial role in the acquisition of these new methods of thinking. This mathematics book should be read (or skimmed through) quickly when the reader is looking for a piece of information which is indispensable to the solution of problem which is occupying his mind day and night!

Thus, it is best to approach this work as dispassionately as possible. You will then realise how easy it is to acquire a certain mastery of the subject. You will also see that old knowledge takes on a new depth, when it is replaced in a new perspective. You will improve (or at least modify) your understanding of aspects you thought you had already understood, since there is no end to understanding, either in the theory of mathematics or in other areas of knowledge. That is why we advise the reader to skim through the book to determine what it is about. You will then begin to understand it in a more active way by proving for yourself the results listed for each chapter of Part I at the beginning of the relevant section of the Exercises (Chapter 14). Both the pleasure of success and the lessons of partial failure will help you to overcome the difficulties you encounter. The pleasure of discovery is not a vain sentiment; the more ambitious is the challenge, the more intense is the pleasure.

These exercises (and above all the solutions) were also designed to provide the reader with additional information which could not be given in an introductory text. The results which the reader will discover will convince him of the richness of nonlinear analysis.

The exercises (Chapter 14) are grouped according to chapters and follow the order of Part I. Except for certain exceptions (which are explicitly mentioned), they only use results that have already been proved. However, some exercises do assume that one or two immediately preceding exercises have been solved.

The problems (Chapter 15) use a priori all the material in Part I and are largely grouped according to topic.

The first nine problems concern various topological properties of set-valued maps. The description of the notion of set-valued maps and their properties given in Part I is a bare minimum and is insufficient for profound applications of nonlinear analysis. The tenth problem generalises Banach's theorem (closed graph or open image) either to the case of continuous linear operators defined on a closed convex cone or to that of set-valued maps (Robinson–Ursescu theorem). It goes together with Problem 14 which extends the inverse function theorem to set-valued maps and which thus plays an important role in applications. Problem 11 returns to the proof of Ekeland's theorem in the very instructive context of discrete dynamical systems. Problems 12, 13, 14 and 28 provide applications of Ekeland's theorem, which turns out to be the most manageable and the most effective theorem in the whole family of results equivalent to the fixed-point theorem for contractions. This is complemented by a fixed-point theorem for non-expansive mappings (Problem 16) which uses an interesting notion (the asymptotic centre of sequences, which is a sort of virtual limit) which is the subject of Problem 15.

The solution of Problem 17 on the properties of orthogonal projectors onto convex closed cones (discovered by Jean-Jacques Moreau, co-founder with R.T. Rockafellar of convex analysis) is indispensable. Problem 18 studies a class of functions with properties analogous to those of convex functions.

A continuous mapping is 'proper' if it transforms closed sets to closed sets and if its inverse has compact images. As one might imagine, such functions play an important role. Their properties are the subject of Problem 19. Problems 20, 21, 23 and 26 are designed to extend the results of Chapters 3 to 5 for the functions $x \to f(x)+g(Ax)$ to the functions $x \to L(x, Ax)$; they will help the reader to assimilate the above chapters properly. Problem 24 is devoted to the application of Chapter 5 to linear programming. Variational principles form the subject of Problems 26, 27, 45 and 46; these last two problems use Ky Fan's inequality.

The graph of a continuous linear operator is a closed vector subspace. The set-valued maps analogous to continuous linear operators are set-valued maps with graphs a convex closed cone. These are known as 'closed convex processes' and inherit numerous properties of continuous linear operators, as Problems 10 (closed graph) and 29 (transposition) show.

Since the derivatives of differentiable mappings are continuous linear operators, we might expect to look for candidates for the role of the derivative of a set-valued map among such closed convex processes. It is sufficient to return to the origins, that is to say to Pierre de Fermat who introduced the notion of the tangent to a curve. This idea is taken up in Problem 33, which provides an introduction to the differential calculus of set-valued maps. Over recent years, this latter has become the subject of intense activity, because of its intrinsic attraction and its numerous potential applications. This 'geometric' view of the differential calculus is taken up again in Problem 34 to complete the study of subdifferentials of convex functions, whilst Problem 35 leads to a very elegant formula for calculating the subdifferential of a marginal function. This differential calculus of set-valued maps is the topic of (Aubin and Frankowska 1990) which contains a thorough investigation of set-valued maps. Problems 36, 37, 38, 39 and 40 describe refinements of the minimax inequalities of von Neumann and Ky Fan which are very useful in infinite-dimensional spaces. Problems 41 and 48 provide variants and applications of the Gale–Nikaïdo–Debreu theorem, whilst Problem 42 shows how to trade the compactness of the domain of a set-valued map for 'coercive' properties. The existence of eigenvectors of set-valued maps forms the subject of Problems 43 (general case) and 44 (positive set-valued maps).

Problem 47 provides an introduction to maximum monotonic set-valued maps and their numerous properties.

We could have included many other problems, but forced ourselves to make a difficult selection. One area of applications of nonlinear analysis, namely the calculus of variations and optimal control, is not touched on by this collection of problems, although it is a most rich and exciting area which remains the subject of active research.

This requires a reasonable mastery of topological vector spaces (weak topologies) and of function and distribution spaces (Sobolev spaces) which is not demanded of the reader (Aubin 1979a). If the latter has a knowledge of the basic tools of convex analysis, non-regular analysis and nonlinear analysis, he will be well equipped to tackle these theories effectively.

It remains to wish the reader (in fact, the explorer) deserved success in mastering this exciting area of mathematics, nonlinear analysis.