



*“Il pourrait au contraire sembler à quelques uns qu’étant donnée cette complexité des phénomènes économiques, l’économie mathématique est justement beaucoup trop simple. Ceux-là n’auraient qu’à se rappeler que les premières propositions de la géométrie sont, elles aussi, très simples, sans que cette simplicité interdise en rien les complications ultérieures.*

*Enfin, si quelques-uns, tout en reconnaissant que la méthode mathématique n’est pas superflue pour constituer la science de l’économie publique, et tout en reconnaissant sans doute que cette science n’atteint pas toute la complexité du réel vivant, mais que ces résultats cependant sont toujours sous-jacents, pour ainsi dire, à ce réel, se défiaient de cette science elle-même, et de son importance, et craignaient qu’on ne s’y enfermât un peu complaisamment, ceux-là n’auraient pas été frappés des admirables paroles, et décisives, où le savant fait lui-même la part de la science : la réforme sociale doit procéder à la fois du sentiment socialiste et de la science économique.”.*

CHARLES PÉGUY

*Un économiste socialiste : Léon Walras  
La Revue Socialiste, 1897*



**This book is dedicated**  
**to Noël Bonneuil, Jean Cartelier, Richard Day and Donald Saari**

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## Preface

This book is intended to provide theoretical economists with mathematical tools necessary to handle the concepts of **evolution** and **adaptation** arising in economics, pursuing the Arrow-Debreu-Hahn legacy<sup>1</sup>.

It may be now the time to answer the wish J. von Neumann and O. Morgenstern expressed in 1944 at the end of the first chapter of their monograph *“Theory of Games and Economic Behavior”*:

*“Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore, preferable...”*

*“Our static theory specifies equilibria ... A dynamic theory, when one is found — will probably describe the changes in terms of simpler concepts.”*

Economies in the real world are normally “out of equilibrium”. Actually, this does not signify much, because the concept of equilibrium is polysemous. The mathematical one, which we adopt here, expresses stationary — time independent — evolution, that is, no evolution. This is the static concept specified by von Neumann and Morgenstern. The other meaning is connoted by the **viability constraints** to which the state of the system is subjected, and that it is not allowed to violate during its evolution. Namely, the articulation between dynamics and constraints means here that from any initial state starts at least one evolution satisfying the constraints at each instant (such an evolution is called viable). An equilibrium may be viable or not viable, these are two independent issues.

The fact that many scarcity constraints are presented in terms of “balance”, such as the balance of payments, may be at the root of the misunderstanding. Indeed, the image of the balance conveys both the concept of equalization of opposite forces, hence of constraints, and the resulting stationarity — often called the “stability”, again, a dangerous word connoting too many different meanings. Therefore, speaking of disequilibrium does not necessarily mean putting at stakes the viability of a system. Rather, “disequilibrium” situations — nonstationary viable situations — appear to induce more or less efficient reactions of economic agents to adapt to viability constraints, the condition *sine qua non* on the economic evolution (Schumpeter has put particular emphasis on the role of entrepreneurs in this process).

It becomes then natural to specify the “minimal” conditions under which an econ-

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<sup>1</sup>However, this book does not provide the well established results on dynamical systems motivated by physical sciences, such as stability of equilibria, bifurcation, chaos, complex dynamics, that are covered in many excellent texts, and does not cover macro-economic models.

omy can work and to specify classes — as large as possible — of reasonable economies whose evolution does not violate these viability conditions (as well as other specifications). This is not always possible, and a preliminary task is to find — and compute — the **viability kernel**, which is the set of states from which starts at least one viable solution.

In my opinion, when one has to design a mathematical metaphor for an evolutionary model of economic variables, one should start by gathering the constraints of these variables which cannot — or should not — be violated. Usually, there are few disputes among “modelers” when they are listing these constraints. Serious disagreements may begin when behavioral assumptions have to be made. However, the problem of choice of a behavior of the agents is well circumscribed: one can propose, describe or suggest such and such shape of a change function and check whether or not a representative of this class belongs to an “envelope” of dynamical systems under which the constraints are viable. In particular, as in optimization theory, the theory allows us to associate with viability constraints dynamical systems involving **viability multipliers**, ranging over the same space than the Lagrange multipliers, which can be regarded as prices as well as other regulatory controls, called *regulees*.

The next task is to derive from the confrontation of the dynamics and the constraints the concealed regulation mechanisms governing viable evolutions, and to select some of them according to some further principle. This allows us to **derive “adjustment laws” instead of founding the modelization process based on such a law**. This goes against the tradition of theoretical economy, where the adjustment of some variables for reaching an equilibrium plays a basic and prominent role. The so-called “law of supply and demand” states that prices react in a determined direction in response to a difference between supply and demand in the market: the price of a particular commodity is assumed to vary according to the sign of the excess demand of this commodity. Instead of reasoning with a law of adjustment *a priori* given, and which does not produce viable evolutions, we shall build “dedicated” laws of supply and demand which shall provide viable solutions. In some way, this reverse approach allows us to “explain” *a posteriori* the role of such an adjustment law instead of scrutinizing the consequences of an *a priori* given law for possible justifications.

As a consequence, whether in a time-independent environment an evolution — even viable — converges to an equilibrium is not the major issue that I shall address in this book. Instead, *the main purpose of the viability approach to dynamical economics is to explain the evolution of a system, governed by given nondeterministic dynamics and*

*viability constraints, to reveal the concealed regulation laws which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an “opportunistic” and “conservative” behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

The reader may agree or not with the proposed mathematical metaphors for explaining the evolution of a decentralized dynamic economic system, comparable to but different from the standard tâtonnement and nontâtonnement models. Relevant or not<sup>2</sup>, the point is to illustrate how these new tools can be used to journey from a static analysis built on concepts of optima, equilibria and attractors to a contingent dynamic framework.

Naturally, many attempts had already been made to introduce time and evolution in economic models, in the framework of market dynamics (one of the main motivations of this book), as well as sustainable development (regarded here as viable development), economic growth theory<sup>3</sup> or the theory of business cycles, which are mentioned in the bibliography. Let me just mention here the recent achievements of “evolutionary economics”<sup>4</sup>. Dynamical models have also been used in the framework of game theory: let us quote for instance the theory of repeated games on the discrete side<sup>5</sup>, differential and dynamical games on the continuous side<sup>6</sup>, evolutionary game theory<sup>7</sup> for which there are already several available books and an extensive literature.

However, I have attempted to comply to the request of J. von Neumann and O. Morgenstern by proposing other (and sometimes new) tools to tackle these issues which elude a stationary context. Actually, both economic evolution — as it is presented in

<sup>2</sup>I naturally hope that the approach I suggest is so. If it is unfortunately a fallacy, I wish that it meets at least this Vilfredo Pareto criterion : *“Give me a fruitful error anytime, full of seeds, bursting with its own corrections. You can keep your sterile truth for yourself.”* To which Johann von Goethe added however that *“if it becomes a kind of credo admitting no doubt or scrutiny, that is the real evil, one which has endured through the centuries.* I may borrow to Denis Diderot the lines with which he begins his *Pensées philosophiques* : *“Je compte sur peu de lecteurs, et n’aspire qu’à quelques suffrages. Si ces pensées ne plaisent à personne, elles ne pourront qu’être mauvaises ; mais je les tiens pour détestables si elles plaisent à tout le monde”.*

<sup>3</sup>dealing with growth of outputs, since the pioneer studies of Harrod, Domar and Solow after World War II.

<sup>4</sup>after the publication of the book [?, Nelson & Winter].

<sup>5</sup>See for instance [?, Fudenberg & Tirole].

<sup>6</sup>See for instance [?, Cardaliaguet, Quincampoix & Saint-Pierre].

<sup>7</sup>We refer for instance to [?, Hofbauer & Sigmund] and the popular book [?, Sigmund]

this book — and Darwinian type of evolution in various domains of biology and neurobiology motivated at the end of the seventies my own involvement and my contributions to what became to be known under the name of viability theory and the differential calculus of set-valued maps, together with many other colleagues, and distinctively among them, H el ene Frankowska<sup>8</sup>.

The tools of set-valued analysis and viability theory thus supplement the theory of nonlinear dynamical systems, whose economic applications were initiated by R.H. Day and which are exposed in his recent books [?, ?, Day].

In the extent where the latter is confined to the study of stability or chaotic behavior around attractors, time and evolution are actually pushed aside, since one ignores the **transient** states before they converge to an attractor. If this makes sense in some physical problems where the transient stage is very short by human standards, economic evolutions remain transient for our time scale. What happens when time approaches infinity may not be much more relevant than looking for stationary solutions — equilibria. “In the long term, we are all dead”, Keynes reminded us. Local stability analysis gives poor intuitions about the effective evolutions and asymptotic properties are not sufficient for economists interested in the evolution *hic et nunc* of the economy. Stability properties of equilibria, which are too special evolutions, although important, are not the end of the story. In contrast to traditional dynamical economical analysis where local stability and asymptotic properties of models are the heart of the matter, the viability approach is concerned with contingent evolutions over time. This is one point.

Another one deals with the fact that until now the field of nonlinear systems and chaotic behavior deals with **deterministic systems** which are quite **unstable** and sensitive to initial conditions on some invariant domains (the attractor, for instance). For many problems arising in economics, I believe we face a completely orthogonal situation, with **nondeterministic systems** (caused by deeply ignorant and/or non rational economic agents, and the immense variety of possible decisions to be taken at each instant by economic agents). Nondeterministic (contingent) dynamical economic systems have actually to produce evolutions as **regular or stable** (in a very loose sense) as possible for the sake of adaptation and viability.

Contingent uncertainty does not mean stochastic uncertainty. Expectations of economic agents do not result from a cold probabilistic calculation<sup>9</sup>. Following Keynes,

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<sup>8</sup>See the bibliographical comments at the end of the book,

<sup>9</sup>The theory of probabilities provides only one manner to translate mathematically some aspects of chance and uncertainty. Besides contingent uncertainty, that we advocate in this book, minimizing on a topological space  $X$  the sum  $x \mapsto \varphi(x) + W(x)$  where  $\varphi$  and  $W$  are nonnegative extended inf-compact functions plays an analogous role to integrating on  $\Omega$  the function  $\int_{\Omega} \psi(x) d\mu(x)$ . In this case,

“the outstanding fact is the extreme precariousness of the basis of knowledge” on which expectations are made. In order to capture the kind of uncertainty relevant for economics, it seems better to rely upon non-deterministic non-stochastic models.

I also advocate in this book another distinctive approach by not appealing to intertemporal criteria, *which involve some knowledge of the future*<sup>10</sup> (even of a stochastic nature.) This is of particular importance when experimentation is not possible<sup>11</sup> or when the phenomenon under study is not cyclic or periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated.*

Hence, forecasting or prediction of the future are not the issues which we shall address here, whenever the forecasted future is not provided by an extrapolation mechanism of past history<sup>12</sup>. *La prévision est un rêve duquel l'événement nous tire*, wrote Paul Valéry.

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some behaviors which do not fit any reasonable optimality criterion. Indeed, (economic) history shows that most often **right timing is more relevant than optimality**, that decisions not taken at the wrong moment — be they optimal — can no longer be efficient or viable, advocating a dynamical approach to decision sciences.

The complexity<sup>13</sup> of dynamic economic systems stems from such a non teleological

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the map  $M : K \mapsto M(K) := \inf_K W(x)$  from the closed subsets  $K \subset X$  to the “(min, +)-algebra  $(\mathbf{R}_+, \min, +)$  — called the **Maslov measure** plays the role of the measure  $A\mu(A) := \int_A d\mu(x)$  mapping a  $\sigma$ -algebra of measurable subsets to  $(\mathbf{R}_+, +, \cdot)$ . The Cramèr transform maps nonnegative measures to nonnegative extended lower semicontinuous convex functions, Dirac measures of elements to indicators of the same elements, Gaussians to quadratic functions, etc. For a short account on Maslov measures and  $\sigma$ -algebra, see for instance [?, Aubin & Dordan], [?, Aubin] and [?, Bacelli, Cohen, Olsder & Quadrat]. I am personally convinced that the use of Maslov measures for encapsulating some aspects of uncertainty can be as relevant to most economic problem than the use of classical probability theory.

<sup>10</sup>Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

<sup>11</sup>Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to “know” the future evolution of the system.

<sup>12</sup>Indeed, using such an extrapolation mechanism provides history-dependent — now called track-dependent — dynamical models, that will be studied in this book. With such approaches, uncertainty is transferred to the choice of the extrapolation mechanism, opening the way to many other issues, such as learning procedures, involving mimetic and herd behaviors, issues that we shall slightly touch upon later.

<sup>13</sup>This word is used here with its usual meaning. We shall however investigate one aspect of complexity, *connectionist complexity*, as one mechanism to adapt to more and more viability constraints,

collective evolution of the set of agents, even though many individual economic agents think of themselves as being pursuing definitive and rational aims, instead of **adapting permanently** to the many viability constraints (among which scarcity constraints) they face under uncertainty, either stochastic or contingent. This theme has been introduced and studied in economic theory under the name of “bounded rationality”. Indeed economic agents are humans, not computers, are seldom rational, obey inertia principles, are poor forecasters, actually very myopic. They base many decisions not on rational grounds, but on faith, beliefs and bets, rules of thumb, moods, rumors, and the like. They are more inductive than deductive in their learning processes. Actually, we can adopt Peirce’s terminology and look at them as “abductive”, i.e., as making conjectures rather than predictions<sup>14</sup>. They are most often more conservative than innovative, afraid of changes when they are not perceived as improving their situation. They may prefer to adopt a herd behavior instead of choosing dissident ways opening new avenues.

Even though we are far away of being able to capture mathematically the indomitable psychological, sociological, political and cultural<sup>15</sup> behavior of human beings which warns their purely economic behavior<sup>16</sup>, some recent concepts were diffidently introduced during the last decade for analyzing these features, some of them simple despite their novelty. Most of them were discovered or redirected for answering questions of economic theory in an evolutionary perspective. Among these, let us mention a sample: differential inclusions, differential inclusions with memory, viability theorems for differential inclusions and stochastic differential inclusions governing the evolution under constraints, impulse differential inclusion and hybrid control systems<sup>17</sup>, mutational equations for studying the evolution of sets (such as production sets), differential calculus of set-valued maps and set-valued analysis for many reasons, nonsmooth analysis for studying stability problems by the Lyapunov method or for revealing concealed utility functions, boundary-value problems for systems of first-order partial differential equations or inclusions, etc. Some of these theories are exposed in scattered journals and books, most often for a mathematical audience. We shall survey them in the third

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which implies the emergence of links between the components of a system and their evolution, together with — or instead of — using decentralizing “messages”. We shall compare it with other dynamical decentralization mechanisms, such as price decentralization. The emergence of the cyberspace may give a new life to connectionist mechanisms, since any two agents would soon be able to communicate at each instant without the mediation of a price or other messages providing the necessary information.

<sup>14</sup>Non mathematical accounts of such questions can be found in [?, Aubin].

<sup>15</sup>in which the science and technology components of culture are included.

<sup>16</sup>restricted to production, distribution and consumption of commodities and services.

<sup>17</sup>Hybrid control systems, a hot topic by now in control theory, have been introduced independently under the name of **multiple-phase economic dynamics** by Richard Day in [?, Day].

part of this book — the Mathematical Tool-Box — without reproducing the proofs when they appeared elsewhere and can be easily retrieved.

Naturally, getting apprised with these mathematical tools requires from the non-professional mathematician a huge investment in time and efforts, since most concepts and results are too recent to be familiar and accepted by a clear consensus. However, I share the conviction of some of us that these tools — and others to be invented — will be mandatory for tackling evolutionary issues in economics (and social sciences). If we accept that physics studies much simpler phenomena than the ones investigated by social and biological sciences, and that for this very purpose, they motivated and used a more and more complex mathematical apparatus, we have to accept also that social sciences require a new and dedicated mathematical arsenal which goes beyond what is presently available. It has been said that physics could be defined as the part of the environment which can be understood through mathematical metaphors. By dropping the adjective “political” which characterized so well their domain of investigation and by looking for mathematical metaphors, some economists wished to transform their science into a physical — hard ? — one. We would rather hope that in return, some mathematicians would add qualifying adjectives such as political, social, human, to their domain of investigation to encourage them to direct their quest to the analysis of human and social behavior. Paradoxically, the very fact that the mathematical tools useful for social sciences are and have to be quite sophisticated impairs their acceptance by many social scientists and economists, and the gap menaces to widen.

Jean-Pierre Aubin

Paris, October 19, 1999

## Outline of the Book

I have tried to find an organization with increasing mathematical difficulty. Actually, difficulties, if any, are due more to novel mathematical concepts and viewpoints than to intrinsic technical vicissitudes. The reader is invited to a new conceptual world that requires some habitude and familiarization, and thus, some investment in time. Such a bottom-up outline, as we say now for induction, demands many repetitions that a concise top-down — deductive — organization would have avoided, to a heavy pedagogical price. Hence, most of the technicalities are relegated to the “Tool-Box”. This goes also the logics of economics. Except for Chapter 1, exposing how the basic principles of viability theory change the standard presentation by deriving adjustment laws from basic facts instead of starting with the Walrasian tâtonnement, an adjustment law, to the concept of Walrasian equilibrium, and Chapter 3, where we develop slowly, step by step, the design of dynamical economies in the simple framework of allocation of scarce resources, economic and financial examples are presented as soon as the mathematical material is presented.

I this open this book with a plea<sup>18</sup> for decentralized “ecodynamics”, or rather, “eco<sup>2</sup>-dynamics, in order to “house”<sup>19</sup> both economical and ecological dynamical analysis under a same roof. This conceptual chapter raises the issues in the Arrow-Debreu framework of Walrasian economics, but in an evolutionary perspective, comparing the approach when the Walras equilibrium is derived from an *a priori* law of supply and demand and the viability approach where “viable” laws of supply and demand are derived from a dynamical description of the behavior of consumers faced to scarcity

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<sup>18</sup>written in collaboration with Jean Cartelier.

<sup>19</sup>The Greek root “oiko” means house. The word “economy” appeared in French in 1615 in the title of the treatise *Traité d'économie politique* by Antoine de Montchrestien, in opposition to the word “mesnagerie” meaning at the time private economy. The word “ecology” has been coined by Ernst Haeckel in 1866, defined as : “...the global science of relations between organisms and their environment, in which we include the living conditions”. The choice of the term “eco-viability” has been suggested by Luc Doyen and Philippe Cury.

constraints.

The second chapter — of a visual, intuitive and non rigorous nature — presents low dimensional examples of viability kernels of several explicit models obtained either analytically or by computer programs using the Saint-Pierre viability kernel algorithm. It illustrates how the concept of viability kernel can bring different informations in the dynamical analysis of such systems, supplementing the study of stability properties of their equilibria and their asymptotic behavior.

The second part of this book entitled “ecodynamics” is then devoted to the introduction of the theorems dealing with evolution under nonstochastic (also called contingent) uncertainty and viability issues in the dynamic version of the basic problem of resource allocations and more general dynamical economies.

Chapter 3 states the main viability theorems proved later in the case of allocation of scarce resources. Its purpose is to accustom the reader with the viewpoints suggested in this book in the familiar framework of the Arrow-Debreu world. The main idea is to “differentiate” so to speak the set of allocations of available commodities among the consumers, so that we can obtain the “envelope” of all dynamical systems compatibles with the viability constraints imposed by the scarcity of the available commodities. Among them, we introduce a class of dynamical systems parametrized by messages such as prices. We look for prices such as the associated dynamics belongs to this envelope: This provides the “regulation law” associating with any allocation the set of prices which “regulate” the dynamical allocation of scarce resources. The dual version of the regulation law reveals “budgetary rules” allowing the consumers to share the available commodities at each instant. Among them, one find the dynamical version of the Walras law stating that it is forbidden to spend more monetary units than received during the transactions. We then show how to integrate time both in the dynamics and the constraints, how to “govern” the evolution of the sets of resources through “mutational equations” and how to incorporate the past both in the dynamics and the constraints.

Chapter 4 then shows how to regard the problem of dynamical resource allocation as a particular case of “dynamical economies” governing viable solutions, for which “planning” mechanisms are devised. Among them, we single out “slow evolutions” which are the evolutions regulated by prices with minimal norms. It is shown that in some cases, they are the same than evolutions governed by the “least complex” connection matrices, illustrating mathematically that in this case, the decentralization role of prices corresponds to minimal complexity of interactions between agents.

Chapter 5 uses the dual characterization of the regulation law to characterize prices

as “viability multipliers” associated with the constraints, as in optimization theory under constraints. This allows us to design dynamical economies for which the viability is guaranteed.

These three chapters — using somewhat more mathematical technicalities — motivate a further step forward in the details of viability theory. Hence, in Chapter 6, we shall in particular answer two basic and natural questions: When a constrained subset is not viable,

1. what is the set of initial states from which starts at least one viable solution? We shall call this set the **viability kernel**.
2. what is the set of initial states from which starts a least one solution reaching the constrained set in finite time? We shall call this set the **capture basin**.

These concepts are central for the viability analysis of many problems.

Some of them are needed for implementing mathematically in Chapter 7 the “inertia principle” and the concepts of “viability niches”, “punctuated equilibria” and “heavy solutions”, as well as to derive the “adjustment laws” such as the law of supply and demand from the confrontation of a dynamical economy with constraints. They are obtained by “differentiating” the regulation law, so to speak. When several regulatees are involved in a regulation mechanism, we advocate the selection of evolutions in the context of “hierarchical dynamical games” in which the different regulatees obey a cascade of different hierarchical inertia.

Chapter 8 is devoted to “dynamical inequalities” involving inequalities that must be always satisfied, which are viability constraints of a special type in disguise. They involve the study and the construction of “utility functions” which increase along at least one evolution, the “Lyapunov functions” so useful for studying asymptotic stability, the “exit time function” associating with any initial state the largest time interval in which one solution obeys the constraints, the “crisis function” measuring the time spent violating the constraints and the “value functions” of intertemporal optimal control problems. Being viability constraints in disguise, these dynamical inequalities are amenable to viability theorems.

The ecodynamics part of the book is concluded in Chapter 9 by a very short introduction to a viability approach of the difficult topic of dynamical games, in which the role of two players acting on two different regulatees is either agonistic, as in comprehensive economies, or antagonistic, as in dynamical games against nature.

The third part of the book presents a “Mathematical Tool Box”. It surveys the available tools that I believe to be relevant and useful for investigating the issues which

make the topic of this monograph. It is more “mathematically technical” and can be used mostly as an enriched “mathematical glossary”.

Chapter 10 is devoted to the concepts of pseudo inverses of linear operators which are needed for the characterization of specific regulation laws such as slow or heavy solutions and tensor products of linear operators necessary to deal with the evolution of connection and technological matrices.

Chapter 11 gives an account of the definitions of set-valued analysis and Chapter 12 surveys the main results of viability, invariance and equilibrium theorems in the framework of differential inclusions, of time-dependent differential inclusions, of differential inclusions with memory and mutational equations.

Chapter 13 and 14 summarizes classical results on convex optimization and equilibrium theory, with a double objective: Give to the reader not familiar with static mathematical economics the basic instruments of convex and nonlinear analysis on one hand, provide a short account of some results of general equilibrium theory on the other.

Chapter 15 is a gate to future investigations, because it presents the stochastic version of some of the basic viability theorems, where both the drift and the diffusion are set-valued. These are tools needed in some domains of finance and actuarial analysis, where the present consensus admits that uncertainty is governed by stochastic processes satisfying almost surely given constraints at each instant. It also attempts to reconcile two mathematical representations of uncertainty through stochastic differential inclusions: a “stochastic uncertainty”, driven by a Wiener process, and a “contingent uncertainty”, driven by a set-valued map<sup>20</sup>. But the price to pay in mathematical technicality may become prohibitive.

Usually, there are few disputes among “modelers” when they are listing the constraints looming in an economic problem. Serious disagreements may begin when behavioral assumptions have to be made.

Using a dynamical approach such as viability theory instead of a static one such as theory optimization or equilibrium theory is a first element of controversy. Utilizing dynamical games against nature or stochastic differential equation is another element of contention.

In order to weaken such controversies, or to prorogue the ultimate choice of a behavioral description of the economic agents, I have tried to offer in this book an array of options for modelers, knowing too well that this is too short a list and that

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<sup>20</sup>Not to mention another approach dealing with “fuzzy viability” which is not considered in this book.

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too ineffective are its items.