





Cet ouvrage de Jean-Pierre Aubin fait suite à son livre *Applied Abstract Analysis* ; mais, en fait, le nouvel ouvrage est accessible directement. Il est excellent. Il était difficile de donner un exposé aussi complet d'une branche essentielle des mathématiques utiles tout en étant aussi clair. Deux procédés concourent à cette réussite. D'une part, on se place dans le cadre des espaces de Hilbert. D'autre part, l'auteur a rapproché les applications des énoncés de la théorie.

Le livre est divisé en trois parties. Les chapitres 1-5 exposent les théorèmes généraux des espaces vectoriels, dans le cas des espaces de Hilbert. Applications : problèmes d'approximation, multiplicateurs de Lagrange, théorème minimax, optimum de Pareto, théorèmes de Lax-Milgram et de Lions-Stampacchia sur les problèmes variationnels. La seconde partie traite la transformation de Fourier et les espaces de Sobolev de fonctions et de distributions. La troisième partie est une ouverture vers des applications que l'auteur réussit à exposer d'une manière relativement élémentaire : théorie spectrale, produits tensoriels dans les espaces de Hilbert, problèmes de valeurs au bord des équations aux dérivées partielles elliptiques et paraboliques, notions sur le contrôle optimal. Il y a là une excellente introduction à l'analyse numérique moderne. L'ouvrage se termine par l'énoncé de deux cents exercices.

Écrit à l'occasion d'un cours Mathématiques de la décision à l'Université Paris-Dauphine, l'ouvrage est susceptible de constituer un ouvrage de base pour un large public.

Extrait de « La Recherche », n° 119, Février 1981.

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To Anne-Laure,  
 who studied the first edition of this book when she was a student,  
 to Henri-Jean and Marc,  
 who escaped this task,  
 and to Pierre-Cyril,  
 who may read this edition as a curiosity in twenty years.

## Preface

**Yet another book on functional analysis! Yabfa!**, would exclaim a computer scientist in his/her exotic language.

Why, 20 years after the first edition of *Applied Functional Analysis*, after so many other monographs on this basic topic, do I propose a second edition of this text devoted to an introduction — an induction? — to this seductive field ?

The mathematicians of my generation were lucky enough to receive as a dowry the tools of Functional Analysis created at the dawn of our finishing century by David Hilbert and Stefan Banach, just to name those two visionaries. With many other mathematicians, they offered us a formidable unifying framework and a panoply of tools for solving problems stemming in many different areas of knowledge, making a universe of a “multiverse” of motivating applications: *It is this universality of mathematical results, having their origin in one discipline and finding applications in others, which makes so fascinating functional analysis in particular, mathematics in general.*

The success of this machinery allowed thousands of mathematicians to use it in so many different directions that it is impossible to pursue the early Dunford-Schwartz or the Bourbaki attempts to expose an exhaustive state of the art. Many other books then evolved in a Darwinian way, exploring many specific and diverse directions, reflecting the experiences as well as the views on the purpose of mathematics of each author, eventually finding an adequate niche through the natural selection left to the care of the readers.

The first edition of this book reflected my personal experience at the time, derived from numerical analysis of partial differential equations and later, mathematical economics. Two decades passing by, my views evolved and my experience broadened, so

that my teaching of functional analysis to the students of Université Paris-Dauphine evolved years after years. I could not resist both the pleasure and the pain of offering young students the divulgation at their level of what was continuously going on the research front. I then felt it was time to write down an account of some recent discoveries that helped me to revise some of the perspectives I formed earlier.

However, several pedagogical choices remain invariant: Convey the feeling of the variety of applications. With an obvious consequence: For keeping the length of the exposition within reasonable limits — about 120 teaching hours — restrict the initiation to functional analysis to the linear framework, remain in the simple Hilbertian structure, present distributions as elements of Sobolev spaces. I shall thus be able to cast a rapid glance on boundary-value problems for elliptic and parabolic partial differential equations. I added a short introduction to set-valued analysis and presented the Nagumo Theorem on the viability of closed subsets under differential equations. It is not only interesting by itself, but allows us to forge efficient tools for solving rapidly and easily other problems, such as boundary-value problems for systems of first-order partial differential equations or minimal and stopping time problems or building Lyapunov functions, and more. I took away the chapter on nonlinear analysis of the first edition, and, here and there, sections or paragraphs which are no longer essential.

For illustrating as soon as possible the abstract exposition, I chose applications derived from numerical analysis, systems theory, calculus of variations, control theory, optimization of allocations of scarce resources, demography (McKendrick boundary-value problems), convex and nonsmooth analysis, set-valued analysis. This selection will not be to everyone's taste: It is partial. For allocating time and space for a short presentation of these examples, I had to go as far as to sacrifice the use of weak topologies and to deprive the reader of the grace of the weak compactness of the unit ball of the dual of a Banach space. However, as long as the linear theory is concerned, one can survive without it. This allows us to provide a *larger number of results in a simplest way*, at the price, of course, of generality.

I hope that by doing so, I may persuade the readers of the advantages of an abstract approach to theories motivated by concrete problems, and to attract them to applied and motivated mathematics.

Naturally, the nature and the deep meaning of mathematical concepts and statements evolve with time. This was the case during the course of the century of the views on differential calculus, inherited from Pierre de Fermat, Isaac Newton and Gottfried Leibniz three centuries ago, and formalized when Augustin-Louis Cauchy a little more than one century ago defined rigorously the concept of limit. The consensus on the for-

malization of derivatives as limits of difference quotients for the pointwise convergence was so strong that the concept of derivative became a permanent reality, protected from any dissident view. This could have been the case in this kind of paradise in which one is free to choose the assumptions and the rules of the game. The overwhelming curiosity and the concern for interpreting the environment with the help of mathematical metaphors was Eve's apple. Are all problems arising outside pure mathematics "well-posed" in the Hadamard sense? Should the nondifferentiable functions popping up in so many fields be deprived forever of the benefits of some properties of the derivatives?

Since then, the history of derivatives of functions and maps is a kind of *mathematical striptease*, the modern version of what Parmenides and the pre-Socratic Greeks called *a-letheia*, the dis-covering, un-veiling of the world that surrounds us. This is nothing else than the drive to "abstraction", isolating, in a given perspective, the relevant information in each concept and investigate the interplay between them.

Indeed, one by one, and very shyly, the required properties of the derivative of a function or of a functional were taken away. We shall go quite far to leave the derivatives with the bare minimum.

This is quite natural, though, because each problem demands its own amount of properties that the derivative should enjoy, i.e., its own degree of regularity. Without going too far by always requiring minimal assumptions, some problems could not be solved by sticking to the richest structure. The right balance between generality and readability is naturally a subjective choice.

The concepts of derivatives of functionals go back to Volterra in 1887. Then Gâteaux, in a note written in 1913 and published in 1919 after his death during the First World War, introduced the concept of *first variation*: If  $f : \mathbf{R}^n \mapsto \mathbf{R}^m$  denotes a map from one finite dimensional vector space to another and

$$\nabla_h f(x)(v) := \frac{f(x + hv) - f(x)}{h}$$

denotes its differential quotients, the first variation of  $f$  at  $x$  in the direction  $v$  is the limit  $Df(x)(v)$  of these differential quotients when it exists. In defining the *Gâteaux derivative*  $Df(x)$ , Fréchet added the requirement that the map  $v \mapsto Df(x)(v)$  is linear and continuous! He proposed his own concept of derivative (with the mandatory linearity) for a function as early as in 1912 in the case of functions and in 1925 for maps from a normed space to another one. Mathematicians of this period still asked many properties for the derivatives of functionals and were not ready to give away linearity.

These definitions were too restrictive, so that they were weakened in several ways, and led to a *ménagerie* of concepts: strong or weak Fréchet and Gâteaux derivatives,

Hadamard, bounded (Suchomlinov), locally uniform (Vainberg) derivatives, Dini directional semiderivatives or derivatives from the right, to quote a few.

However, this was not enough, the topologies used to define the limits of difference quotients were still too strong for allowing more maps to retain some kind of differentiability. But weakening the topologies allows us to get more limits at the price of obtaining these limits outside the set of single-valued maps. This was even worse than losing the linearity of the directional derivatives.

However, Serge Sobolev and Laurent Scwhartz did dare to introduce weak derivatives and *distributions* in the forties for obtaining solutions to partial differential equations, Jean-Jacques Moreau and Terry Rockafellar to define *set-valued subdifferential* of convex functions to implement the Fermat rule in optimization at the beginning of the sixties, the eighties witnessing the eclosion of *graphical derivatives* of set-valued maps and set-valued analysis for dealing, for instance, with control systems and differential games, the nineties the appearance of *mutations of set-valued maps* for grasping kind of differential equations — called mutational equations — governing the evolution of sets. This process of differentiating “less and less differentiable maps”, so to speak, continues its random course to unknown shores.

Allow me to elaborate briefly this point at his early stage of the exposition: The strong requirement of pointwise convergence of differential quotients can be weakened in (at least) two ways, each way sacrificing different groups of properties of the usual derivatives:

- Fix the direction  $v$  and take the limit of the function  $x \mapsto \nabla_h f(x)(v)$  in the weaker sense of distributions to be defined later in this book. The limit  $D_v f$  may then be a distribution, and no longer a single-value map. However, it coincides with the usual limit when  $f$  is Gâteaux differentiable. Moreover, one can define difference quotients of distributions, take their limit, and thus, differentiate distributions.

Distributions, as we shall see, are no longer functions or maps defined on  $\mathbf{R}^n$ , so they lose the pointwise character of functions and maps, but retain the linearity of the operator  $f \mapsto D_v f$ , mandatory for using the theory of linear operator for solving partial differential equations.

- Fix the direction  $x$  and take the limit of the function  $v \mapsto \nabla_h f(x)(v)$  in the weaker sense of “graphical convergence” to be defined later in this book. The limit  $Df(x)$  may then be a set-valued map, and no longer a single-value map. However, it coincides with the usual limit when  $f$  is Gâteaux differentiable. Moreover, one can define difference quotients of set-valued maps, take their limit, and thus,

differentiate set-valued maps. These graphical derivatives keep the pointwise character of functions and maps, mandatory for implementing the Fermat Rule, proving inverse function theorems under constraints or using Lyapunov functions, for instance, but lose the linearity of the map  $f \mapsto Df(x)$ .

In both cases, the approaches are similar: They use (different) convergences *weaker than the pointwise convergence* for increasing the possibility for the difference-quotients to converge. But the price to pay is the loss of some properties by passing to these weaker limits (the pointwise character for distributional derivatives, the linearity of the differential operator for graphical derivatives).

We shall use both of them for studying boundary value-problems for partial differential equations, the second approach being for instance involved in the definition of *viscosity solutions* to Hamilton-Jacobi variational equations and inequalities.

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*Paris, France, 30th January 1999*