The Substratum of Impulse and Hybrid Control Systems

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Abstract

The behavior of the run of an impulse differential inclusion, and, in particular, of a hybrid control system, is "summarized" by the " initialization map" associating with each initial condition the set of new initialized conditions and more generally, by its "substratum", that is a set-valued map associating with a cadence and a state the next reinitialized state. These maps are characterized in several ways, and in particular, as "set-valued" solutions of a system of Hamilton-Jacobi partial differential inclusions, that play the same role than usual Hamilton-Jacobi-Bellman equations in optimal control.

Keywords: hybrid control, impulse control, differential inclusion, viability, run, execution, periodic, cadenced run, equilibrium, Kakutani Theorem, contingent cone, Marchaud map.

Introduction

Impulse differential inclusions, and in particular, hybrid control systems, are defined by a differential inclusion (or a control system) and a reset map. A run of an impulse differential inclusion is defined by a sequence of cadences, of reinitialized states and of motives describing the evolution along a given cadence between two distinct consecutive impulse times, the value of a motive at the end of a cadence being reset as the next reinitialized state of the next cadence.

A first advantage of introducing impulse differential inclusions is to summarize the usually protracted description of an hybrid system[2] by only two

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[2]See for instance among many papers and books [13, Branicky, Borkar & Mitter], [12, Bensoussan & Menaldi], [17, 18, Matveev & Savkin] and [20, Shaft & Schumacher].

set-valued maps F — the right-hand side of the differential inclusion governing the continuous evolution of a hybrid system — and R, describing the reset map reinitializing the system when required and a constrained set K inside which the evolution of the "run" or "execution" must remain. Hence, for instance, the existence of a run of an hybrid system for every initial set becomes a viability problem of an adequate auxiliary subset under an impulse differential inclusion, that can be characterized elegantly end efficaciously.

The behavior of the run is "summarized" by the " initialization map" $U := U_{(F,R)}$ associating with each initial condition $x_0 \in K$ the set of new initialized conditions $x_1 \in R(x(^{-}t_1))$ when $x(\cdot)$ ranges over the set of solutions to the differential inclusion $x' \in F(x)$ viable in K until they reach $R^{-1}(K)$ at time $t_1 \geq 0$ at $x(^{-}t_1) \in R^{-1}(K)$.

Indeed, the sequence of successive initial conditions x_n of a viable run $x(\cdot)$ of the impulse differential inclusion (F,R) — constituting the "discrete component of the run" — is governed by the discrete system $x_n \in U_{(F,R)}(x_{n-1}) \cap K$ starting at x_0. The knowledge of the sequence of initialized states x_n allows us to reconstitute the "continuous component" of the run by solving the differential inclusion $x' \in F(x)$ starting at each reinitialized state x_n and satisfying the end-point condition $x_{n+1} \in R(x(^{-}t_{n+1}))$, which exists thanks to the definition of the map $U^{K}_{(F,R)}$.

Assume for a while that the impulse differential inclusion is actually an impulse differential equation (f,r) where the maps f and r are single-valued and that the initialization map is single-valued and differentiable. Then we shall prove that the initialization map is a solution to the system of first-order partial differential inclusions

$$\forall\, i = 1, \ldots n, \;\; \sum_{j=1}^{n} \frac{\partial u_i(x)}{\partial x_j} f_j(x) \;=\; 0$$

or, in a more compact form, $0 \;=\; \frac{\partial u}{\partial x} f(x)$, satisfying the "condition"

$$\forall\, x \in K \cap r^{-1}(K), \;\; r(x) \;=\; u(x)$$

Actually, we shall extend this result to general impulse differential inclusions by characterizing the initialization map $U_{(F,R)}$ as a generalized (set-valued) solution — a Frankowska solution — to the system of first-order partial differential inclusions

$$0 \;\in\; \frac{\partial u}{\partial x} F(x)$$

satisfying the "condition"

$$\forall\, x \in K \cap R^{-1}(K), \;\; R(x) \; \subset \; U(x)$$

These are indeed really Dirichlet boundary condition whenever the reset map R is defined only on the boundary ∂K of a closed subset K and maps ∂K into the interior of K. In this case, resetting initial conditions happens only when the continuous evolution of the state governed by the differential inclusion or the control system is about to leave the domain K. Hence the reset map assigns new initialized states in the interior of K.

We shall introduce more generally another set-valued map summarizing the behavior of an impulse differential inclusion, called the substratum, that is the topic of this paper.

Outline: We begin by giving our definition of impulse differential inclusions. We then recall the characterization of viable subsets under an impulse differential inclusion an derive from it a necessary and sufficient condition for the existence of solutions to hybrid differential inclusions. Then, we devote the next section to the graphical properties of the initialization map U and we derive its properties from the general properties of viable-capture basins of a target by a differential inclusion.

In the last section, we translate the Frankowska characterization of viable-capture basins in terms of kinds of systems of first-order Hamilton-Jacobi partial differential equations characterizing the substratum the solutions of which are the initialization maps and the substratum.

1 Impulse Differential Inclusions

"Hybrid control systems", as they are called by engineers, or "multiple-phase dynamical economies", as they are called by economists (see for instance [16, Day]), or "Integrate and Fire" models in neurobiology (see for instance [14, Brette]) — may be regarded as impulse differential inclusions.

Here, $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ denote finite dimensional vector spaces. Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $P : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\left\{ \begin{array}{ll} i) & x'(t) \; = \; f(x(t), u(t)) \\ ii) & u(t) \; \in \; P(x(t)) \end{array} \right.$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, P(x)) := \{f(x,u)\}_{u \in P(x)}$.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.

We shall say that K *is locally viable under* F if from every $x \in K$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on the nonempty interval $[0, T_x[$ in the sense

$$\forall\, t \in [0, T_x[, \quad x(t) \ \in K$$

and that K is viable if we can take $T_x = +\infty$. It is *locally backward invariant under* F if for every $t_0 \in]0, +\infty[, \ x \in K$, for all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ arriving at x at time t_0, there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, and backward invariant if we can take $s = 0$.

We denote by

$$\mathrm{Graph}(F) := \{(x, y) \in X \in Y \mid y \in F(x)\}$$

the graph of a set-valued map $F : X \rightsquigarrow Y$ and $\mathrm{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its domain.

Let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t-\eta, t[$ where $\eta > 0$, and, for consistency purposes, $x(s) = x(^-t)$ if $s = t$. An impulse differential inclusion (and in particular, an impulse control system) is described by a pair (F, R), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the continuous evolution of the system in K and where R, the reset map, governs the discrete switches to new "initial conditions" when the continuous evolution is doomed to leave K.

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called in the "hybrid system" literature a "run" or an "execution".

Definition 1.1 *Let us consider a finite dimensional vector space* X*, a closed subset* $K \subset X$*, a set-valued map* $F : X \rightsquigarrow X$ *and a set-valued map* $R : X \rightsquigarrow X$*, regarded as a* reset map*. We regard the pair* (F, R) *as the dynamics of an* impulse differential inclusion*.*

A run *of the impulse differential inclusion is a map* $x(\cdot)$ *from* $[0, T]$ *to* X *if* $T < +\infty$ *or from* $[0, +\infty[$ *to* X *if* $T = +\infty$ *which is associated with* *a non decreasing sequence $\mathcal{T}(x(\cdot)) := \{t_n\}_{n\geq 0}$ of* impulse or switching times *$t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ (depending on the run $x(\cdot)$) such that*

1. *$x(t_{n+1}) \in R(x(t_n))$ if $t_{n+1} = t_n$,*
2. *or else, on the interval $[t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ at time t_n until time t_{n+1} at which we take $x(t_{n+1}) \in R(x(^{-}t_{n+1}))$.*

We denote by $\tau_n := t_n - t_{n-1}$ the nth cadence *of the run and by $x_n(\cdot) := x(\cdot + t_n)$ the nth* motive *of the run, a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ on the interval $[0, \tau_n]$ and satisfying the end-point condition $x_n(\tau_n) \in R^{-1}(x_{n+1})$. The sequence of states $x(t_n)$ is called the sequence of* initialized states*.*

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$.

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in R(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

Hybrid systems can be regarded as instances of viable impulse differential inclusions: we refer to [2, Aubin] or [11, Aubin, Lygeros, Quincampoix, Sastry & Seube] for more details on that topic.

2 The Substratum and the Initialization and Impulse Maps

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$ the set of absolutely continuous functions $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t))$$

starting at time 0 at x: $x(0) = x$ and by $\mathcal{S}^K : K \rightsquigarrow \mathcal{C}(0, \infty, K)$ the viable solution map mapping an initial state $x \in K$ to the set $\mathcal{S}^K(x)$ of solutions to the differential inclusion $x' \in F(x)$ starting at $x \in K$ and viable in K.

The set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the solution map associated with F.

We next denote by

$$\vartheta^K(t, x) \;:=\; \bigcup_{x(\cdot) \in \mathcal{S}^K(x)} \{x(t)\} \;\;\&\;\; \vartheta^K(t, C) \;:=\; \bigcup_{x \in C} \vartheta^K(t, x)$$

the K-viable reachable maps (or set-valued flow) of $x \in K$ and $C \subset K$ respectively. We set $\vartheta := \vartheta^X$ when viability constraints are absent.

Definition 2.1 *We associate with the dynamics (F, R) of the impulse differential inclusion its* substratum $\Gamma^K := \Gamma^K_{(F,R)} : \mathbf{R}_+ \times K \rightsquigarrow K$, *that is the set-valued map associating with any $(t, x) \in \mathbf{R}_+ \times K$ the subset*

$$\Gamma^K_{(F,R)}(t,x) \ := \ R(\vartheta^K(t,x)) \cap K$$

of the elements $y \in R(c)$ where $c \in C := K \cap R^{-1}(K)$ through which the solutions to the differential inclusion $x' \in F(x)$ starting at x and viable in K until they reach $R^{-1}(K)$ at time t.

Knowing the substratum $\Gamma^K_{(F,R)}$, we introduce

1. *the* impulse map

$$\mathbf{T}^K_{(F,R)}(x) \ := \ \{t \geq 0 \quad \textit{such that} \ \ \Gamma^K_{(F,R)}(t,x) \ \neq \ \emptyset\}$$

2. *and the* initialization map $U^K_{(F,R)} : K \rightsquigarrow X$

$$U^K_{(F,R)}(x) \ = \ \bigcup_{t \in \mathbf{T}^K_{(F,R)}(x)} \Gamma^K_{(F,R)}(t,x)$$

First, we single out the following property:

Proposition 2.2 *Let (F, R) be an impulse differential inclusion defined on a subset K. Knowing the substratum $\Gamma^K_{(F,R)}$ of (K, F, R), and thus the impulse map $\mathbf{T}^K_{(F,R)}$ and the initialization map $U^K_{(F,R)}$, we can* reconstruct *a viable run of the impulse differential inclusion (F, R) through the following algorithm: Given the cadence τ_n and the initial state x_n, we take*

$$\left\{ \begin{array}{ll} i) & \text{the next cadence } \tau_{n+1} \in \mathbf{T}^K_{(F,R)}(x_n), \\ ii) & \text{the next reinitialized state } x_{n+1} \in \Gamma^K_{(F,R)}(\tau_{n+1}, x_n) \subset U^K_{(F,R)}(x_n), \\ iii) & \text{the next motive } x_n(\cdot) := x(\cdot + t_n)\text{, a solution to } x' \in F(x) \text{ satisfying} \\ & x_n(0) \ = \ x_n \ \ \& \ \ x_n(\tau_{n+1}) \ \in \ R^{-1}(x_{n+1}) \end{array} \right. \tag{1}$$

In other words, in terms of impulse times, given the impulse time t_n and the initial state x_n, we take

$$\left\{\begin{array}{ll} i) & \text{the next impulse time } t_{n+1} \in t_n + \mathbf{T}^K_{(F,R)}(x_n), \\ ii) & \text{the next reinitialized state } x_{n+1} \in \Gamma^K_{(F,R)}(t_{n+1}-t_n, x_n) \subset U^K_{(F,R)}(x_n) \\ iii) & \forall\, t \in [t_n, t_{n+1}], \text{ a solution } x(\cdot) \text{ to the differential inclusion } x' \in F(x) \\ & \text{starting from } x_n \text{ at time } t_n \text{ viable in } K \text{ until it reaches } R^{-1}(x_{n+1}) \\ & \text{at time } t_{n+1}. \end{array}\right. \tag{2}$$

Proof — Take any run $x(\cdot)$ associated with a sequence $\mathcal{T}(x(\cdot)) := \{t_n\}$ of impulse times starting at $x_0 \in K$ and viable in K. Then the sequence $\vec{x} : n \to x(t_n)$ is a solution of the discrete dynamical system $\Gamma^K_{(F,R)}(t_{n+1}-t_n, x_n)$, obviously viable in K.

Conversely, assume that the substratum $\Gamma^K_{(F,R)}$ is known. The above algorithm (2) starting at time 0 and state $x_0 \in K$ provides a run $x(\cdot)$ associated with the sequence $\mathcal{T}(x(\cdot)) := \{t_n\}$ of impulse times of the impulse differential inclusion (F, R) viable in K. □

Actually, if we are interested only in the sequence of reinitialized states and not necessarily in knowledge of the sequence of impulse times, the knowledge of the initialization map $U^K_{(F,R)}$ is sufficient:

Proposition 2.3 *A subset K is viable under the impulse differential inclusion (F, R) if and only if the domain of the* initialization map $U^K_{(F,R)}$ *is equal to K.*

Proof — Assume that K is viable under (F, R) and prove that for every $x \in K$, $U^K_{(F,R)}(x) \neq \emptyset$. Take any $x_0 \in K$. By definition, there exists a run $x(\cdot)$ associated with a sequence $\mathcal{T}(x(\cdot)) := \{t_n\}$ of impulse times viable in K. Then the sequence $\vec{x} : n \to x(t_n)$ is a solution of the discrete dynamical system $U^K_{(F,R)}$, obviously viable in K.

Conversely, assume that K is viable under the discrete system $U_{(F,R)}$, i.e., that for every $x \in K$, $U^K_{(F,R)}(x) \neq \emptyset$. We shall prove that K is viable under the impulse differential inclusion (F, R). Let x_0 given in K and a solution $\vec{x} : n \to x_n \in U^K_{(F,R)}(x_{n-1}) \cap K$ to the discrete dynamical system $U^K_{(F,R)}$. By definition of the initialization map $U^K_{(F,R)}$, we can associate with

$$x_n \;\in\; U^K_{(F,R)}(x_{n-1}) \;:=\; \bigcup_{t \in \mathbf{T}^K_{(F,R)}(x_{n-1})} \Gamma^K_{(F,R)}(t, x_{n-1})$$

some $\tau_{n-1} \in \mathbf{T}^K_{(F,R)}(x_{n-1})$ such that

$$x_n \ := \ x_n(\tau_{n-1}) \ \in \ R(\vartheta^K(\tau_{n-1}, x_{n-1}))$$

where $x_n(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at time 0 from x_{n-1}. Setting $t_n := t_{n-1} + \tau_{n-1}$ and $x(t) := x_n(t + t_{n-1})$ if $t \in [t_{n-1}, t_n]$, we have checked that $x(\cdot)$ is a run to the impulse differential inclusion (F, R) associated with the sequence $\{t_n\}_{n\geq 0}$ of impulse times t_n starting from x_0 and viable in K. □

3 Some Prerequisite from Viability Theory

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 3.1 (Marchaud Map) *We shall say that F is a* Marchaud map *if*

$$\begin{cases} i) & \textit{the graph of } F \textit{ is closed} \\ ii) & \textit{the values } F(x) \textit{ of } F \textit{ are convex} \\ iii) & \textit{the growth of } F \textit{ is linear: } \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \ \leq \ c(\|x\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(x, u) \mapsto f(x, u)$ is continuous, affine with respect to the controls u and with linear growth and when P is Marchaud.

We recall the following version of the important Theorem 3.5.2 of **Viability Theory**, [1, Aubin]:

Theorem 3.2 *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}$ is upper semicompact with nonempty values: This means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.*

Our purpose is to characterize the viability of a subset K under an impulse differential inclusion:

Definition 3.3 *We shall say that a subset K is viable under an impulse differential inclusion (F, R) if from any initial state x of K starts at least one run viable in K.*

The Viability Theorem[3] and its consequences imply the following

Theorem 3.4 *Let (F, R) be an impulse differential inclusion and $K \subset X$ be a closed subset. Assume that F is Marchaud and that $R^{-1}(K)$ is closed. Then the following statements are equivalent*

1. *K is viable under (F, R),*
2. *The subset[4] $K \backslash R^{-1}(K)$ is locally viable under F,*
3. *K, F and R are linked through the tangential condition[5]*

$$\forall\, x \in K \backslash R^{-1}(K), \quad F(x) \cap T_K(x) \neq \emptyset$$

(see [2, Aubin] or [11, Aubin, Lygeros, Quincampoix, Sastry & Seube] for a proof.)

We shall also need some other prerequisites from Viability Theory:

Definition 3.5 *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set. The subset $\mathrm{Capt}^K(C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K. A subset K is a* repeller *under F if all solutions starting from K leave K in finite time. A subset D is* locally backward invariant relatively to *K if all backward solutions starting from D viable in K are actually viable in K.*

We shall use the following characterization of capture basin (see [6, Aubin]):

Theorem 3.6 *Let us assume that F is Marchaud and that the subsets $C \subset K$ and K are closed. If $K \backslash C$ is a repeller (this is the case when K itself is a repeller), then the viable-capture basin $\mathrm{Capt}^K(C)$ of the target C under $\mathcal{S}$ is the* **unique** *closed subset satisfying $C \subset D \subset K$ and*

$$\left\{ \begin{array}{ll} i) & D \backslash C \text{ is locally viable under } \mathcal{S} \\ ii) & D \textit{ is locally backward invariant relatively to } K \end{array} \right. \tag{3}$$

[3] See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [1, Aubin].

[4] The subset $K \backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C.

[5] The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [8, Aubin & Frankowska]) or [19, Rockafellar & Wets] for more details).

4 The Graph of the Substratum

We begin by characterizing the graph of the substratum $\Gamma^K_{(F,R)}$:

Theorem 4.1 *Let us assume that F is Marchaud, that $C \subset R$ is closed and that the graph of $R : C \rightsquigarrow X$ is closed.*

Then the substratum $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ is the **unique** *set-valued map with closed graph satisfying*

$$\forall\, x \in K, \;\; \Gamma^K_{(F,R)}(0,x) \;:=\; R(x) \cap K$$

and, for any $T > 0$

1. *for any $y \in \Gamma^K_{(F,R)}(T,x)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on $[0,T]$ such that*

$$\forall\, t \in [0,T], \;\; y \in \Gamma^K_{(F,R)}(T-t, x(t)) \tag{4}$$

2. *for any $y \in K \backslash \Gamma^K_{(F,R)}(T,x)$, for every solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on $[0,T]$, then*

$$\forall\, t \in [0,T], \;\; y \in K \backslash \Gamma^K_{(F,R)}(T-t, x(t))$$

As a consequence[6]*, for any $T > 0$ and for any $y \in \partial_K \Gamma^K_{(F,R)}(T,x)$, for every solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ satisfying (4), then*

$$\forall\, t \in [0,T], \;\; y \in \partial_K \Gamma^K_{(F,R)}(T-t, x(t))$$

For proving Theorem 4.1, we shall first observe that the graph of the substratum of (K,F,R) is a viable-capture basin and next, deduce the above results from the characterization of viable-capture basins. Let us recall that we denoted by $R^{|K}_{|K}$ the graphical restriction of R to $K \times K$ defined by

$$R^{|K}_{|K}(x) \;:=\; \begin{cases} R(x) \cap K & \text{if} \quad x \in K \\ \emptyset & \text{if} \quad x \notin K \end{cases}$$

We observe that $C := \mathrm{Dom}(R^{|K}_{|K}) = K \cap R^{-1}(K)$, that $\mathrm{Im}(R^{|K}_{|K})$ and that $\mathrm{Graph}(R^{|K}_{|K}) = \mathrm{Graph}(R) \cap (K \times K)$.

[6]The relative boundary $\partial_K D$ to K of a subset $D \subset K$ is equal to $\overline{D} \cap \overline{K \backslash X}$.

Lemma 4.2 *The graph of the substratum* $\Gamma_{(F,R)}^K$ *of* (K, F, R) *is the viable-capture basin of* $\{0\} \times \mathrm{Graph}(R_{|K}^{|K})$ *under the set-valued map* $\{-1\} \times F \times \{0\}$*:*

$$\mathrm{Graph}(\Gamma_{(F,R)}^K) \;=\; \mathrm{Capt}_{\{-1\}\times F\times\{0\}}^{\mathbf{R}_+\times K\times K}\left(\{0\} \times \mathrm{Graph}(R_{|K}^{|K})\right)$$

and $\forall\, x \in C := K \cap R^{-1}(K)$, $\Gamma_{(F,R)}^K(0, x) = R(x) \cap K$.

Proof — Indeed, to say (T, x, y) belongs to the viable-capture basin

$$\mathrm{Capt}_{\{-1\}\times F\times\{0\}}^{\mathbf{R}_+\times K\times K}\left(\{0\} \times \mathrm{Graph}(R_{|K}^{|K})\right)$$

means that there exists a solution $x(\cdot) \in \mathcal{S}(x)$ and $\bar{t} \in [0, T]$ such that

$$\begin{cases} i) \quad \forall\, t \in [0, \bar{t}], \;\; (T - t, x(t), y) \in \mathrm{Capt}_{\{-1\}\times F\times\{0\}}^{\mathbf{R}_+\times K\times K}(\{0\} \times \mathrm{Graph}(R_{|K}^{|K}) \\ ii) \quad (T - \bar{t}, y, x(\bar{t})) \;\in\; \{0\} \times \mathrm{Graph}(R_{|K}^{|K}) \end{cases}$$

i.e., if and only if $\bar{t} = T$ and

$$\begin{cases} i) \quad \forall\, t \in [0, T[, \;\; x(t) \;\in\; K \\ ii) \quad y \;\in\; R(x(T)) \cap K \end{cases}$$

This is equivalent to say that $y \in \Gamma_{(F,R)}^K(T, x) \cap K$.

Consequently, to say that y belongs to $\Gamma_{(F,R)}^K(0, x)$ means that $y \in R(x) \cap K$. $\square$

Proof of Theorem 4.1 — We observe first that the map $\{-1\} \times F \times \{0\} : \mathbf{R} \times X \times X \rightsquigarrow \mathbf{R} \times X \times X$ is Marchaud and that $\mathbf{R}_+ \times K \times K$ is a repeller under this map since any solution $(T - t, x(t), y)$ starting at (T, x, y) leaves $\mathbf{R}_+ \times K \times K$ at time T. Theorem 3.6 states that the viable-capture basin

$$\mathrm{Graph}(\Gamma_{(F,R)}^K) \;=\; \mathrm{Capt}_{\{-1\}\times F\times\{0\}}^{\mathbf{R}_+\times K\times K}\left(\{0\} \times \mathrm{Graph}(R_{|K}^{|K})\right)$$

is the unique closed subset $\mathcal{V} \subset \mathbf{R} \times K \times K$ containing $\{0\} \times \mathrm{Graph}(R_{|K}^{|K})$ satisfying

1. $\mathcal{V} \backslash (\{0\} \times \mathrm{Graph}(R_{|K}^{|K}))$ is locally viable under $\{-1\} \times F \times \{0\}$
2. and
$$\mathrm{Capt}_{\{-1\}\times F\times\{0\}}^{\mathbf{R}_+\times K\times K}(\mathcal{V}) \;=\; \mathcal{V}$$
   This states that whenever $(T, x, y) \in (\mathbf{R}_+ \times K \times K) \backslash \mathcal{V}$, all solutions to the differential inclusion $(t', x', y') \in \{-1\} \times F(x) \times \{0\}$ leave $(\mathbf{R}_+ \times K \times K)$ before possibly reaching the target $\{0\} \times \mathrm{Graph}(R_{|K}^{|K})$.

The first statement means that whenever (T,x,y) belongs to $\mathcal{V}$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ such that $(T-t,x(t),y)$ belongs to $\mathcal{V}$ until it reaches $\{0\}\times \mathrm{Graph}(R_{|K}^{|K})$. This is equivalent to saying that

$$\forall\, t\in[0,T],\ \ y\in \Gamma_{(F,R)}^{K}(T-t,x(t))$$

The second statement means that whenever (T,x,y) does not belong to $\mathcal{V}$, all solutions $x(\cdot)$ to the differential inclusion $x'\in F(x)$ are such that $(T-t,x(t),y)$ do not belong to $\mathcal{V}$ whenever $(T-t,x(t),y)\in \mathbf{R}_+\times K\times K$, i.e., whenever $x(\cdot)$ is viable in K on the interval $[0,T]$. This is equivalent to saying that for all solutions to $x'\in F(x)$ viable in K on the interval $[0,T]$,

$$\forall\, t\in[0,T],\ \ y\in K\backslash\Gamma_{(F,R)}^{K}(T-t,x(t))$$

Let us consider now $y\in\partial\Gamma_{(F,R)}^{K}(T,x)$ where $T>0$. This means that there exists a sequence $y_n\in K$ such that $y_n\in K\backslash\Gamma_{(F,R)}^{K}(T,x)$ converges to y. Hence (T,x,y_n) does not belong to the capture basin of $\{0\}\times\mathrm{Graph}(R_{|K}^{|K})$ viable in $\mathbf{R}_+\times K\times K$. Therefore we know that for any solution $x(\cdot)\in\mathcal{S}(x)$ viable in K on $[0,T]$, for any $t\in[0,T]$, $y_n\in K\backslash\Gamma_{(F,R)}^{K}(T-t,x(t))$ and, in particular, that $y_n\in K\backslash\Gamma_{(F,R)}^{K}(0,x(T))=R(x(T))$. Taking any solution $x(\cdot)\in\mathcal{S}(x)$ satisfying (4) and the limit when $n\to+\infty$, we infer that

$$\forall\, t\in[0,T],\ \ y\in\partial_K\Gamma_{(F,R)}^{K}(T-t,x(t))$$

and that

$$y\in\partial_K R(x(T))$$

5 Hamilton-Jacobi Characterization of the Substratum

Before stating the general result characterizing the substratum as a solution to a system of first-order partial differential inclusions, let us consider the following particular case:

Proposition 5.1 *Let us assume that $f:X\mapsto X$ is Lipschitz, $r:X\mapsto X$ is single-valued and continuous, that $\varpi_{(K,C)}^{\flat}$ is continuous, that K is viable under (f,r) and $\Gamma_{(f,r)}^{K}$ is differentiable. Then it is the* **unique** *solution to the system of first-order partial differential equations*

$$\forall\, x\in K\backslash C,\ \forall\, j=1,\ldots,n,\ -\frac{\partial u_j(t,x)}{\partial t}+\sum_{i=1}^{n}\frac{\partial u_j(t,x)}{\partial x_i}f_i(x)\ =\ 0$$

or, in a more compact form,

$$\forall\, x \in K \backslash C, \;\; -\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} f(x) \; = \; 0$$

satisfying the condition

$$\forall\, x \in C, \;\; u(0,x) \; = \; r(x)$$

We shall deduce from Theorem 5.3 below. Indeed, thanks to the concepts of contingent derivative, we shall show that the substratum $\Gamma^K_{(F,R)}$ is the unique (set-valued) solution in the "Frankowska sense" to the "Hamilton-Jacobi inclusion"

$$0 \; \in \; -\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} \cdot F(x) \tag{5}$$

satisfying the condition

$$\forall\, x \in C, \;\; V(0,x) \; = \; R(x) \cap K$$

We refer to [5, 7, Aubin], [9, Aubin & Frankowska] and their references for set-valued solutions to systems of Hamilton-Jacobi inclusions. For that purpose, we recall that the (graphical contingent) derivative of a set-valued map $V : \mathbf{R}_+ \times K \rightsquigarrow K$ may be defined by the relation

$$\mathrm{Graph}(DV(T,x,y)) \; := \; T_{\mathrm{Graph}(V)}(T,x,y)$$

Definition 5.2 *We shall say that a set-valued map $V : \mathbf{R}_+ \times K \rightsquigarrow K$ is a* Frankowska solution *to the Hamilton-Jacobi system of first-order partial differential inclusions (5) satisfying the initial condition $V(0,x) = R(x)$ if its graph is closed, if*

$$\forall\, t > 0, \; \forall\, y \in V(t,x), \; \exists\, v \in F(x) \;\;\; \textit{such that} \;\; 0 \in DV(t,x,y)(-1,v)$$

and if for every $v \in F(x)$

$$\forall\, t \geq 0, \; \forall\, y \in V(t,x), \;\;\; 0 \in DV(t,x,y)(1,-v)$$

or

$$\left\{ \begin{array}{ll} i) & -v \in T_{X \backslash K}(x) \;\; \text{if} \;\; x \in \partial K \\ ii) & -v \in T_K(x) \;\; \text{if} \;\; y \in \partial K \end{array} \right.$$

Theorem 5.3 *Let us assume that F is Marchaud, that $C := K \cap R^{-1}(K)$ is closed and that the graph of $R : C \rightsquigarrow K$ is closed.*

1. *The substratum $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ is the* **largest** *set-valued map $V : \mathbf{R}_+ \times K \rightsquigarrow K$ with closed graph contained in $K \times K$ satisfying*

   $$\forall\, t > 0,\ y \in V(t,x),\ \exists\, v \in F(x) \quad \textit{such that}\ \ 0 \in DV(t,x,y)(-1,v)$$

   and the condition $V(0,x) = R(x) \cap K$,

2. *If furthermore, F is assumed to be Lipschitz, the substratum $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ is the* **unique** *Frankowska solution $V : \mathbf{R}_+ \times K \rightsquigarrow K$ to the Hamilton-Jacobi system of first-order differential inclusions (5) satisfying the initial condition $V(0,x) = R(x)$.*

Proof — When F is Marchaud, to say that $\mathrm{Graph}(V)\backslash(\{0\}\times\mathrm{Graph}\left(R^{|K}_{|K}\right))$ is locally viable under $\{-1\} \times F \times \{0\}$ means that

$$\forall\ (t,x,y) \in \mathrm{Graph}(V)\backslash(\{0\}\times\mathrm{Graph}\left(R^{|K}_{|K}\right)),\ \ \{-1\}\times F(x)\times\{0\}\cap T_{\mathrm{Graph}(V)}(t,x,y) \neq \emptyset$$

We observe that $(t,x,y) \in \mathrm{Graph}(V)\backslash(\{0\} \times \mathrm{Graph}(R^{|K}_{|K}))$ whenever $t > 0$ and we recall that

$$T_{\mathrm{Graph}(V)}(t,x,y) \ = \ \mathrm{Graph}(DV(t,x,y))$$

so that the above condition reads

$$\forall\, t > 0, \forall\, y \in \Gamma^K_{(F,R)}(t,x),\ \exists\, v \in F(x) \quad \text{such that}\ \ 0 \ \in \ DV(t,x,y)(-1,v)$$

When F is assumed to be Lipschitz, to say that

$$\mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}(\mathrm{Graph}(V)) \ = \ \mathrm{Graph}(V)$$

means that

1. for any $(t,x,y) \in \mathrm{Graph}(V)) \cap \mathrm{Int}(\mathbf{R}_+ \times K \times K)$,

   $$(\{1\} \times -F(x) \times \{0\}) \ \subset \ T_{\mathrm{Graph}(V)}(t,x,y) \ = \ \mathrm{Graph}(DV(t,x,y))$$

   This is equivalent to say that for every $v \in F(x)$,

   $$\forall\, t > 0,\ x \in \mathrm{Int}(K),\ y \in V(t,x) \cap \mathrm{Int}(K),\ \ 0 \ \in \ DV(t,x,y)(1,-v) \tag{6}$$

2. and otherwise, for any $(t,x,y) \in \text{Graph}(V)) \cap \partial(\mathbf{R}_+ \times K \times K)$,

$$(\{1\}\times -F(x)\times\{0\}) \ \subset \ T_{\text{Graph}(V)}(t,x,y) \cup T_{(\mathbf{R}\times X\times X)\backslash(\mathbf{R}_+\times K\times K)}(t,x,y)$$

This means that for every $v \in F(x)$,

$$\begin{cases} i) & 0 \ \in \ DV(t,x,y)(1,-v) \text{ if } t=0, y \in R(x) \\ ii) & 0 \ \in \ D(t,x,y)(1,-v) \text{ or } \ -v \in T_{X\backslash K} \text{ if } t \geq 0, x \in \partial K, \ y \in R(x) \\ iii) & 0 \ \in \ D(t,x,y)(1,-v) \text{ or } \ -v \in T_K \text{ if } t \geq 0, \ y \in R(x) \cap \partial K \end{cases}$$

Indeed,

$$\overline{(\mathbf{R}\times X\times X)\backslash(\mathbf{R}_+\times K\times K)} = (\mathbf{R}_-\times K\times K)\cup(\mathbf{R}_+\times(X\backslash K)\times K)\cup(\mathbf{R}_+\times K\times(X\backslash K))$$

Therefore, condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times K \times K$ at $(0,x,y)$ is impossible, condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times (X\backslash K) \times K$ at (t,x,y) when $x \in \partial K$ means that $-v$ belongs to $T_{X\backslash K}(x)$ and condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times K \times (X\backslash K)$ at (t,x,y) when $y \in \partial K$ means that $-v$ belongs to $T_K(x)$. $\square$

For the initialization map, we obtain the following Hamilton-Jacobi inclusion :

Theorem 5.4 *Let us assume that F is Marchaud, that $C := K \cap R^{-1}(K)$ is closed and that the graph of $R : C \leadsto K$ is closed.*

1. *The initialization map $U^K_{(F,R)} : K \leadsto K$ is the* **largest** *set-valued map $V : \mathbf{R}_+ \times K \leadsto K$ with closed graph contained in $K \times K$ satisfying*

$$\forall\, y \in V(x),\ \exists\, v \in F(x) \quad \textit{such that} \ \ 0 \in DV(x,y)(v)$$

2. *If furthermore, F is assumed to be Lipschitz, the initialization map $U^K_{(F,R)} : K \leadsto K$ is the* **unique** *Frankowska solution $V : \mathbf{R}_+ \times K \leadsto K$ to the Hamilton-Jacobi system of first-order differential inclusions (5) satisfying the condition $\forall\, x \in C,\ V(x) = R(x)$.*

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