

Boundary-Value Problems for Systems of Hamilton-Jacobi-Bellman Inclusions with Constraints

Jean-Pierre Aubin¹

November 4, 2001

Abstract

We study in this paper boundary-value problems for systems of Hamilton-Jacobi-Bellman first-order partial differential equations and variational inequalities, the solutions of which are constrained to obey viability constraints. They are motivated by some control problems (such as impulse control) and financial mathematics. We shall prove the existence and uniqueness of such solutions in the class of closed set-valued maps, by giving a precise meaning to what a solution means in this case. We shall also provide explicit formulas to this problem. When we deal with Hamilton-Jacobi-Bellman equations, we obtain the existence and uniqueness of Frankowska contingent epi-solutions. We shall deduce these results from the fact that the graph of the solution is the viable-capture basin of the graph of the boundary-conditions under an auxiliary system, and then, from their properties and their characterizations proved in [12, Aubin].

Keywords: partial differential inclusion, systems of Hamilton-Jacobi-Bellman equations, viability, capture basins, method of characteristics, shocks, impulse control, contingent cone, Marchaud map.

Introduction

It is well known that value functions of optimal control problems are solutions to Hamilton-Jacobi partial differential equation of the form

$$-\frac{\partial}{\partial t}v(t, x) + \inf_{u \in P(x, v(t, x))} \left(\frac{\partial}{\partial x}v(t, x)f(x, v(t, x), u) - g(x, v(t, x), u) \right) = 0$$

with adequate boundary conditions.

Observe nevertheless that in this equation, the infimum hides two inequalities:

1. there exist $u \in P(x, v(t, x))$ such that

$$-\frac{\partial}{\partial t}v(t, x) + \frac{\partial}{\partial x}v(t, x)f(x, v(t, x), u) - g(x, v(t, x), u) \leq 0$$

2. for all $u \in P(x, v(t, x))$,

$$\frac{\partial}{\partial t}v(t, x) - \frac{\partial}{\partial x}v(t, x)f(x, v(t, x), u) + g(x, v(t, x), u) \leq 0$$

However, several other problems of control theory lead to the study of controlled systems of first-order partial differential equations (or systems of first-order partial differential inclusions):

¹Réseau de Recherche Viabilité, Jeux, Contrôle
14, rue Domat, F-75005 Paris
Tel/FAX . [33] (0)1-46-33-71-21
e-mail: J.P.Aubin@wanadoo.fr, WEB: <http://viab.dauphine.fr>

Let $P : X \times Y \rightsquigarrow \mathcal{U}$ be a set-valued map associating with any pair (x, y) a feasible set $P(x, y)$ of controls and f and g be single-valued maps from $X \times Y \times \mathcal{U}$ to finite dimensional vector spaces X and Y respectively.

The problem is to find a set-valued map $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ satisfying

1. there exists $u \in P(x, V(t, x))$ such that

$$0 \in \left[-\frac{\partial}{\partial t}V(t, x) + \frac{\partial}{\partial x}V(t, x)f(x, V(t, x), u) \right] - g(x, V(t, x), u) \quad (1)$$

2. for all $u \in P(x, v(t, x))$,

$$0 \in \left[\frac{\partial}{\partial t}V(t, x) - \frac{\partial}{\partial x}V(t, x)f(x, V(t, x), u) \right] + g(x, V(t, x), u) \quad (2)$$

where we shall give a meaning to the derivative

$$\left[\frac{\partial}{\partial t}V(t, x) - \frac{\partial}{\partial x}V(t, x)f(x, V(t, x), u) \right]$$

in Theorem 3.1 below. Indeed, even in the absence of controls, it is well known that such solutions may have shocks — i.e., can be set-valued — and even, when they happen to be single-valued, are not necessarily differentiable in the usual sense.

The definition of solution shall be taken in a generalized sense — Frankowska solution² that we shall define later in the paper.

In order to obtain uniqueness, we have to impose boundary conditions. Furthermore, problems arising in economics, finance and other fields lead to introduce constraints bearing both on the state and on the solution. We shall describe these boundary conditions and constraints by introducing two set-valued maps $\Phi : \mathbf{R}_+ \times X \rightsquigarrow Y$ and $\Psi : \mathbf{R}_+ \times X \rightsquigarrow Y$ such that $\Phi \subset \Psi$, the first one encompassing initial and/or boundary-value conditions, or other conditions as we shall see, the second one viability constraints both on the state variables x — that must remain in the domain of Ψ — and on the solution $V(t, x)$.

We shall prove that **there exists a unique** “solution” $(t, x) \rightsquigarrow V(t, x)$ to this general problem (1,2) satisfying the conditions

$$\forall (t, x) \in \mathbf{R}_+ \times X, \quad \Phi(t, x) \subset V(t, x) \subset \Psi(t, x)$$

in the class of closed set-valued maps (i.e., set-valued maps with closed graph), that depends continuously of the data Φ (in the “graphical sense”, mapping graphical limits to graphical limits, as it is explained later).

Even more, we shall provide an explicit formula when $f(x, u)$ and $P(x)$ do not depend on the variable y and when

$$g(x, y, u) := -M(x, u)y - L(x, u)$$

is affine with respect to y where

²Hélène Frankowska proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is semi-permeable, (i.e., invariant and backward viable) under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant ([43, 44, 46, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions. See also [32, Barron & Jensen] and [31, Bardi & Capuzzo-Dolcetta] for more details. We refer also to [39, 40, Carja & Ursescu] for the study of these equations through the characteristics method using the contingent derivative (and not epi and hypo derivatives). Such concepts have been extended to solutions of systems of first-order partial differential equation without boundary conditions by Hélène Frankowska and the author (see [19, 20, 21, 22, 24, Aubin & Frankowska] and chapter 8 of [2, Aubin]). See also [16, 17, Aubin & Da Prato]. This point of view is used here in the case of boundary value problems.

1. M is a continuous matrix-valued function

$$M : (x, u) \in X \times \mathcal{U} \mapsto M(x, u) \in \mathcal{L}(X, Y)$$

2. L is a continuous “vector-Lagrangian”

$$L : (x, u) \in X \times \mathcal{U} \mapsto L(x, u) \in Y$$

Let us denote by $\mathcal{C} : x \in X \rightsquigarrow \mathcal{C}(x) \subset \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ the set-valued map associating with $x \in X$ the set $\mathcal{C}(x)$ of the pairs $(x(\cdot), u(\cdot))$ solutions to the control system

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in P(x(t)) \end{cases}$$

starting at x at $t = 0$.

In the absence of constraints ($\Psi(t, x) := Y$), we shall prove that setting

$$\begin{cases} J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) \\ := e^{\int_0^t M(x(s), u(s)) ds} \Phi(T - t, x(t)) + \int_0^t e^{\int_0^{\tau} M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{cases}$$

then the set-valued solution V is defined by

$$V(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x)$$

With an adequate choice of the set-valued map Ψ associated with the set-valued map Φ , we find as solution the set-valued map defined by

$$W(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcap_{t \in [0, T]} J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x)$$

We shall find as many formulas as pairs (Ψ, Φ) of set-valued maps (see formula (24) of Theorem 5.1 below).

We can read this type of results the other way around: For instance, the set-valued map $V(\Phi)$ defined by

$$\begin{cases} V(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} \\ \left(e^{\int_0^t M(x(s), u(s)) ds} \Phi(T - t, x(t)) + \int_0^t e^{\int_0^{\tau} M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

is the unique “solution” to the Hamilton-Jacobi partial differential inclusion (1,2) satisfying the initial condition

$$V(0, x) = \Phi(0, x)$$

and

$$\forall t \geq 0, x \in X, \Phi(t, x) \subset V(t, x)$$

They define set-valued analogue of optimal control problems, where the “ \cup operation” replaces the “inf operation” and the “ \cap operation” the “sup operation”. Actually, when

$Y := \mathbf{R}$ and when we associate with two extended functions $\mathbf{c} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R} \cup \{+\infty\}$ and $\mathbf{b} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R} \cup \{+\infty\}$ the set-valued maps

$$\begin{cases} i) & \Phi(t, x) := \mathbf{c}(t, x) + \mathbf{R}_+ \\ ii) & \Psi(t, x) := \mathbf{b}(t, x) + \mathbf{R}_+ \end{cases}$$

we find problems of dynamic valuation and management of portfolios in mathematical finance, used in particular for valuating options, as in [55, Pujal], [30, Aubin, Pujal & Saint-Pierre]) and [56, Pujal & Saint-Pierre]. For instance, we deduce that

$$\left\{ \begin{array}{l} \inf_{y \in V(\Phi)(T, x)} y = \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} \\ \left(e^{\int_0^t M(x(s), u(s)) ds} \mathbf{c}(T - t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \right) \end{array} \right.$$

is the valuation function of a stopping time problem (see section 5 below).

These two explicit formulas are given by the caliber $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ defined in the following way: y belongs to $V(T, x)$ if there exists a control $t \in [0, T] \mapsto u(t)$ and a time $T^* \in [0, T]$ such that the solution $(x(\cdot), u(\cdot), y(\cdot))$ to the control system

$$\begin{cases} i) & x'(t) = f(x(t), y(t), u(t)) \\ ii) & y'(t) = g(x(t), y(t), u(t)) \\ iii) & u(t) \in P(x(t), y(t)) \end{cases}$$

starting at $x(0) = x$, $y(0) = y$ satisfies

$$\begin{cases} i) & \forall t \in [0, T^*], \quad y(t) \in \Psi(T - t, x(t)) \\ ii) & y(T^*) \in \Phi(T - T^*, x(T^*)) \end{cases}$$

Observe that taking $\Phi(t, x) = \emptyset$ whenever $t > 0$ guarantees that $T^* = T$.

We shall prove that this set-valued map is the unique solution to our problem (1,2). Actually, this is a reformulation dictated by problems arising in dynamic economic theory, finance mathematics and control theory of the celebrated “method of characteristics”.

We shall revisit this method using the tools of set-valued analysis and viability theory which go back to the early 80’s³. They find here an unexpected relevance to assert the existence and the uniqueness of the solution to this problem, since such solutions may have shocks — i.e., can be set-valued — and even, when they happen to be single-valued, they are not differentiable in the usual sense. The tools forged by set-valued analysis and viability theory happen to allow us to prove existence and uniqueness in the class of set-valued maps with closed graph only, instead of classes of vector-distributions⁴.

³See [18, Aubin & Frankowska], [2, Aubin], [60, Rockafellar & Wets], etc. for instance.

⁴The strong requirement of pointwise convergence of differential quotients can be weakened in (at least) two ways, each way sacrificing different groups of properties of the usual derivatives:

- **Distributional Derivatives:** Fix the direction v and take the limit of the function $x \mapsto \nabla_h f(x)(v)$ in the weaker sense of distributions. The limit $D_v f$ may then be a distribution, and no longer a single-value map. However, it coincides with the usual limit when f is Gâteaux differentiable. Moreover, one can define difference quotients of distributions, take their limit, and thus, differentiate distributions.

Distributions are no longer functions or maps defined on \mathbf{R}^n , so they loose the pointwise character of functions and maps, but retain the linearity of the operator $f \mapsto D_v f$, mandatory for using the theory of linear operator for solving partial differential equations.

The basic concept useful in our framework is the concept of **viable-capture basin** of a “target” $C \subset K$ viable in a constrained subset $K \subset X$ under a differential inclusion $x' \in F(x)$: It is the subset $\text{Capt}_F^K(C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$, where $\mathcal{S}(x_0)$ denotes⁵ the set of solutions to differential inclusion $x' \in F(x)$ starting at x_0 .

Then we shall prove that *the graph of the solution $(t, x) \rightsquigarrow V(t, x)$ to the above boundary value problem is the viable-capture basin of the graph of the set-valued map Φ viable in the graph of the set-valued map Ψ under the auxiliary differential inclusion*

$$\begin{cases} i) & \tau'(t) = -1 \\ ii) & x'(t) = f(x(t), y(t), u(t)) \\ iii) & y'(t) = g(x(t), y(t), u(t)) \\ iv) & u(t) \in P(x(t), y(t)) \end{cases}$$

and that this solution is **unique** among the solutions with closed graph to this boundary value problem.

In some instances, this viable-capture basin can be computed analytically, and we obtain in this case an explicit formula of the solution to the above boundary value problem.

In all cases, the “Viability/Capturability Algorithm” designed by Patrick Saint-Pierre provides numerically the viable-capture basins, and thus, the solutions to systems of Hamilton-Jacobi-Bellman equations, bypassing finite-difference methods (see [61, Saint-Pierre], [38, Cardaliaguet, Quincampoix & Saint-Pierre], [55, Pujal] and [56, Pujal & Saint-Pierre]) for solutions of Hamilton-Jacobi-Bellman equations derived from mathematical finance.)

This existence and uniqueness result follows from the three following steps:

- **Graphical Derivatives:** Fix the direction x and take the limit of the function $v \mapsto \nabla_h f(x)(v)$ in the weaker sense of “graphical convergence” (the graph of the graphical limit is the Painlevé-Kuratowski upper limit of the graphs). The limit $Df(x)$ may then be a set-valued map, and no longer a single-value map. However, it coincides with the usual limit when f is Gâteaux differentiable. Moreover, one can define difference quotients of set-valued maps, take their limit, and thus, differentiate set-valued maps. These graphical derivatives keep the pointwise character of functions and maps, mandatory for implementing the Fermat Rule, proving inverse function theorems under constraints or using Lyapunov functions, for instance, but lose the linearity of the map $f \mapsto Df(x)$.

In both cases, the approaches are similar: They use (different) convergences *weaker than the pointwise convergence* for increasing the possibility for the difference-quotients to converge. But the price to pay is the loss of some properties by passing to these weaker limits (the pointwise character for distributional derivatives, the linearity of the differential operator for graphical derivatives).

⁵When we are studying the viable-capture basins of targets under differential inclusions, we observe that they are not specific to differential inclusions. They involve only few properties of the solution map \mathcal{S} associating with any initial state x the set $\mathcal{S}(x)$ of solutions $t \mapsto x(t)$ that are solutions to the above differential inclusion starting at x at initial time 0. These properties (translation and concatenation properties, as well as continuity properties) of the solution map $x \rightsquigarrow \mathcal{S}(x)$ are common to other control problems, such as

1. control problems with memory (see the contributions of [50, 51, 52, Haddad], some of them being presented in [2, Aubin]) — before known under the name of functional control problems, the new fashion calling them as “path dependent control systems”,
2. parabolic (diffusion-reaction) type partial differential inclusions (see the contributions of [64, 65, 67, Shi Shuzhong], some of them being presented in [2, Aubin]) — also known as distributed control systems
3. “mutational equations” governing the evolution in metric spaces, including “morphological equations” governing the evolution of sets (see [4, Aubin] for instance).

1. from a characterization proved in [15, Aubin & Catté] stating the viable-capture basin $\text{Capt}^K(C)$ is the **unique** closed subset D between C and K satisfying

$$\begin{cases} i) & \text{Capt}^D(C) = D \\ ii) & \text{Capt}^K(D) = D \end{cases} \quad (3)$$

2. from a characterization stated below of the viable-capture basin of a target C viable in a closed subset K under a differential inclusion $x' \in F(x)$ proved in [12, Aubin]: Let us recall that

- (a) $K \setminus C$ is a repeller means that all solutions $x(\cdot) \in \mathcal{S}(x)$ starting from $x \in K \setminus C$ reach C or leave K in finite time,
- (b) The subset $D \setminus C$ is said **locally viable** under \mathcal{S} if from any initial state $x \in D \setminus C$ starts at least one solution viable in $D \setminus C$ on a nonempty interval,
- (c) A subset $D \subset K$ is **locally backward invariant relatively to K** under \mathcal{S} if for every $x \in D$, all backward solutions starting from x and viable in K on an interval $[0, T]$ are viable in D on $[0, T]$.

Theorem 0.1 *Let us assume that F is Marchaud⁶ and that a closed subset $C \subset K$ satisfies property*

$$K \setminus C \text{ is a repeller under } F \quad (4)$$

*Then the viable-capture basin $\text{Capt}^K(C)$ is the **unique** closed subset D satisfying $C \subset D \subset K$ and*

$$\begin{cases} i) & D \setminus C \text{ is locally viable under } \mathcal{S} \\ ii) & D \text{ is locally backward invariant relatively to } K \text{ under } \mathcal{S} \end{cases} \quad (5)$$

3. from the Viability and Invariance Theorems that translate the necessary and sufficient conditions (5) in terms of tangential conditions. We recall that the **contingent cone** to a subset K at a point $x \in K$, introduced in the early thirties independently by Bouligand and Severi, adapts to any subset the concept of tangent space to manifolds: A direction $v \in X$ belongs to $T_K(x)$ if there exist sequences $h_n > 0$ and $v_n \in X$ converging to 0 and v respectively such that

$$\forall n \geq 0, \quad x + h_n v_n \in K$$

This means that the contingent cone is the Painlevé-Kuratowski upper limit of the subsets $\frac{K-x}{h}$ when h converges to 0 (see for instance [18, Aubin & Frankowska] and [60, Rockafellar & Wets] for more details).

We shall use the following statements of [12, Aubin]:

Theorem 0.2 *Let us assume that F is Marchaud, that $C \subset K$ and K are closed and that $K \setminus C$ is a repeller. Then the viable-capture basin $\text{Capt}_F^K(C)$ is*

⁶ F is a Marchaud map if

$$\begin{cases} i) & \text{the graph and the domain of } F \text{ are nonempty and closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear:} \\ & \exists c > 0 \mid \forall x \in X, \quad \|F(x)\| := \sup_{v \in F(x)} \|v\| \leq c(\|x\| + 1) \end{cases}$$

(a) the **largest** closed subset D satisfying $C \subset D \subset K$ and

$$\forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset \quad (6)$$

(b) if furthermore F is Lipschitz, the **unique** closed subset D satisfying $C \subset D \subset K$ and the Frankowska properties:

$$\begin{cases} i) & \forall x \in D \setminus C, F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall x \in D \cap \text{Int}(K), -F(x) \subset T_D(x) \\ iii) & \forall x \in D \cap \partial K, -F(x) \subset T_D(x) \cup T_{X \setminus K}(x) \end{cases} \quad (7)$$

4. from the property

$$T_{\text{Graph}(V)}(t, x, y) = \text{Graph}(DV(t, x, y))$$

of the contingent derivative $DV(t, x, y)$ of the set-valued map $V : (t, x) \rightsquigarrow V(t, x)$ at the point (t, x, y) of its graph introduced in [1, Aubin]: The graph of the set-valued map $DV(t, x, y)$ from $\mathbf{R} \times X$ to Y is equal to the contingent cone to the graph of V at (t, x, y) (see [1, Aubin]). This is how Fermat defined in 1637 the derivative of a function as the slope of the tangent to its graph. Leibniz and Newton provided the characterization in terms of limits of difference quotients. Here, too, *the contingent derivative $DV(t, x, y)$ is the upper graphical limit of the difference quotients*, the graph of which being by definition the upper limit of the graphs of the difference quotients $\nabla_h V(t, x, y)$ of V at $(t, x, y) \in \text{Graph}(V)$ defined by

$$(\lambda, f) \mapsto \nabla_h V(t, x, y)(\lambda, f) := \frac{V(t + \lambda h, x + hf) - y}{h}$$

Indeed, we observe that

$$\text{Graph}(\nabla_h V(t, x, y)) = \frac{\text{Graph}(V) - (t, x, y)}{h}$$

so that the contingent cone to the graph of V being the upper limit of the graphs of the difference quotients, is equal by definition the graph of the upper graphical limit of the difference quotients.

Consequently, to say that $g \in Y$ belongs to the contingent derivative $DV(t, x, y)(\pm 1, f)$ of V at (t, x, y) in the direction $(\pm 1, f) \in \mathbf{R} \times X$ means that

$$\liminf_{h \rightarrow 0+, f' \rightarrow f} \left\| \frac{V(t \pm h, x + hf') - y}{h} - g \right\| = 0$$

Since the contingent cone is a closed subset, the graph of a contingent derivative is always closed and positively homogenous (this is what remains of the required linearity of the derivative in classical analysis, but, fortunately, we can survive pretty well without linearity).

When $u : \mathbf{R} \times X \mapsto Y$ is single-valued, we set $Du(t, x) := Du(t, x, u(t, x))$. We see at once that $Du(t, x)(\pm 1, f) = \pm \frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} \cdot f$ whenever u is differentiable at (t, x) . When u is Lipschitz on a neighborhood of (t, x) and when the dimension of X is finite, the domain of $Du(t, x)$ is not empty. Furthermore, the Rademacher Theorem stating that a locally Lipschitz single-valued map is almost everywhere differentiable implies that $x \rightsquigarrow Du(t, x)$ is almost everywhere single-valued.

However, in this case, equality $Du(t, x)(-1, -f) = -Du(t, x)(1, f)$ is not true in general. We refer to [18, Aubin & Frankowska], [60, Rockafellar & Wets] for more details.

The above results — which are interesting by themselves for other mathematical models of evolutionary economics, population dynamics, epidemiology — can be applied to many other problems. Dealing with subsets, they can be applied to graphs of single-valued maps as well as set-valued maps, to epigraphs and hypographs of (extended) real-valued functions for solving Hamilton-Jacobi-Bellman equation, to graph of “impulse” maps (which take empty values except in a discrete sets, useful in the study of hybrid systems or inventory management), etc, (see for instance [5, 10, 6, 9, Aubin], [25, 26, 27, 28, Aubin & Haddad], [29, Aubin, Lygeros, Quincampoix, Sastry & Seube], [34, Bensoussan & Menaldi], [62, Saint-Pierre] and their bibliographical references).

Outline — We begin in Section 1 with two non standard motivations arising in macro-economic problems faced by central banks (filtering informations on the economy from past informations and future expectations) and in the study of impulse and hybrid systems, leading to systems of Hamilton-Jacobi-Bellman inclusions. This section (stating the problem without solving them, this task being done in specific articles) can be skipped by true believers in mathematics ... We define in Section 2 the “caliber” of a pair of set-valued map (Φ, Ψ) under a control system, that appears naturally in some control problems and in economic and financial mathematics. We next prove in Section 3 that the caliber is the unique solution to the system of Hamilton-Jacobi-Bellman inclusions satisfying the imposed conditions. In Section 4, we provide a useful stability result, stating roughly that the caliber of graphical limits is the graphical limit of calibers. Section 5 deals with the explicit formula of the caliber when the control system is structured and the exosystem is affine with respect to the second variable. We also prove that in this case, the caliber is the unique solution to a system of “fixed-set equations” and provide interesting “barrier properties” of the boundary of the caliber. We derive in Section 6 the usual characterization theorems of the valuation functions of a large class of control and stopping time problems as Frankowska episolutions to the scalar Hamilton-Jacobi-Bellman equation that justify the usefulness of the results they are derived from, and extend this scalar situation in Section 7 to the case of “dynamical vector optimization”, where we look for intertemporal Pareto minima. \square

1 Motivations

We shall provide two motivations coming from recent issues arising

1. in macroeconomics (filtering informations on the economy from past informations and future expectations),
2. in hybrid systems and impulse control

leading to systems of Hamilton-Jacobi-Bellman partial differential inclusions.

We refer to specific articles for more details, since they use the basic theorems of this present paper to solve the Hamilton-Jacobi-Bellman partial differential inclusions that pop-up in those articles.

Further applications to the value functions of optimal control and stopping time problems are given in Section 6 and to dynamic vector optimization in Section 7.

1.1 Selector through Past Informations and Future Expectations

As a first motivation, we present a problem originating in a research programme under current investigation by Noël Bonneuil, Halim Doss, Georges Haddad, Henri Pages, Dominique Pujal,

Patrick Saint-Pierre and the author on macro-economic problems faced by central banks.

We quote excerpts of the introduction of [68, Svensson & Woodford] “*It is a truism that monetary policy operates under considerable uncertainty about the state of the economy and the size and nature of the disturbances that hit the economy... But in a more realistic case where important variables are forward-looking [and not only backward-looking] variables, the problem of efficient signal-extraction is inherently more complicated ... In the real world, many important indicator variables are forward-looking variables [routinely watched by central banks] ...*”

We suggest to take up this issue by using nonlinear continuous evolutionary models controlled by instruments such as interest rates, keeping the problem of extracting the “real evolution” knowing only backward-looking measurements and forward-looking expectations, that we shall describe by “expectation tubes”.

The use of the Kalman filter for extracting information is replaced by the recent concept of detector introduced in [13, Aubin, Bicchi & Pancanti] and [27, Aubin & Haddad] in the case of “impulse and hybrid control systems”. We adapt this concept of detector in the case of both backward-looking and forward-looking informations and expectations.

For that purpose, we introduce three time variables for describing the evolution of the system: the current (or present) time T , the past time $t \in [0, T]$, and a prediction time or forward-looking time $s \geq T$ where $a := s - T \in \mathbf{R}_+$ is the prediction horizon, used to take into account anticipations, expectations (or make predictions) in the future.

At each past date, the state is measured, or informations on the state are gathered: This is mathematically described by a “detectability tube” (as in [13, Aubin, Bicchi & Pancanti])

$$t \in \mathbf{R}_+ \rightsquigarrow I(t) \subset X := \mathbf{R}^n$$

that provides the limited amount of information about the states at time t . We take $I(t) := X$ when no information is recorded at time t . Hence “discrete” measurements are obtained when $I(t) \neq X$ only for a discrete number of instants t_n .

An example of detectability tubes is given by $I(t) := h^{-1}(y(t))$ where $h : X \mapsto Y$ is an observation map (or measurement map) and where $t \mapsto y(t)$ is the evolution of the observed output:

$$\forall t \in [0, T], y(t) = h(x(t))$$

The same framework houses also the case when the observation map is set-valued: We set $I(t) := H^{-1}(y(t))$ and the above viability condition reads

$$\forall t \in [0, T], y(t) \in H(x(t))$$

so that “tychastic” uncertainties (by opposition to stochastic uncertainties) on the measurements can be incorporated in this framework.

In order to take into account expectations made at each instant t for future dates $s := t + a$, $a \geq 0$, we describe them mathematically by an **expectation tube** $(t, a) \in \mathbf{R}_+^2 \mapsto I(t, a) \subset X$ where we set $I(t, 0) := I(t)$ for obtaining the detectability tube.

We may assume that if $a_1 \leq a_2$, then $I(t, a_2) \subset I(t, a_1) \subset I(t, 0) =: I(t)$, since the predictions made at time t up to time $s_2 := t + a_2$ are valid up to time $s_1 := t + a_1 \leq s_2$.

Therefore, we associate with any current time T , any horizon $s \geq t$ and any backward-looking time $t \in [0, T]$ the set $I(t, s - t)$ of states measured at time t and that depend upon the duration $a := s - t$ of the expectation interval between t and the horizon s .

Hence, the information/expectation constraint can be summarized by

$$\forall T \geq 0, \forall s \geq T \ \& \ \forall t \in [0, T], x(t) \in I(t, s - t) \tag{8}$$

Let \mathcal{U} be a space of controls, regulees, prices, interest-rate instruments, etc. The dynamics of the state are described by a map $f : (t, a, x, u) \in \mathbf{R}_+^2 \times X \times \mathcal{U} \mapsto X$ and by a set-valued map $P : \mathbf{R}_+ \times X \rightsquigarrow \mathcal{U}$ depicting the state-dependent constraints on the controls $u \in P(a, x)$.

For any current time T and horizon $s \geq T$, we assume that the evolution of the state of the system is governed by the control system

$$\forall t \in [0, T], \begin{cases} i) & x'(t) = f(t, s-t, x(t), u(t)) \\ ii) & u(t) \in P(s-t, x(t)) \end{cases} \quad (9)$$

In other words, at each time $t \in [0, T]$, the velocity $x'(t)$ depends upon time t , the time $s-t$ left to the horizon s and a control $u(t)$ subjected to constraints depending both upon the expected time $s-t$ left to the horizon and the state $x(t)$ at time t .

We also introduce a set-valued map $a \in \mathbf{R}_+ \rightsquigarrow C(a)$ specifying another constraint on the subsets $C(a)$ of initial states that may depends upon the term $a \geq 0$ satisfying

$$\forall a \geq 0, \quad C(a) \subset I(0, a)$$

Definition 1.1 *Let us consider a control system (f, P) , an expectation tube $I : \mathbf{R}_+^2 \rightsquigarrow X$ and a tube $C : \mathbf{R}_+ \rightsquigarrow X$ satisfying $\forall a \geq 0, \quad C(a) \subset I(0, a)$.*

The selector is the set-valued map $\mathbf{S}_{(I,C)} : \mathbf{R}_+^2 \rightsquigarrow X$ that associates with any current time T and any expectation $a := s - T$ the (possibly empty) subset $\mathbf{S}_{(I,C)}(t, a)$ of states $x \in I(t, a)$ such that there exists a solution $x(\cdot) \in \mathcal{S}(t, a, x)$ to the system

$$\begin{cases} \forall t \in [0, T], \quad \forall s := T + a \geq 0, \\ i) & x'(t) = f(t, s-t, x(t), u(t)) \\ ii) & u(t) \in P(s-t, x(t)) \end{cases} \quad (10)$$

such that

$$\begin{cases} i) & x(0) \in C(s) \\ ii) & \forall t \in [0, T], \quad x(t) \in I(t, s-t) \\ iii) & x(T) = x \end{cases}$$

In other words, both the dynamics and the constraints depend upon horizon $s \geq T$ and take into account the informations gathered at any preceding time $t \in [0, T]$ and expectations at time $s-t$ left to the horizon s . The selector is thus a tube associating with any horizon $s \geq T$ the set of states x such that there exists a control $u(\cdot)$ governing the evolution $x(\cdot)$ through control system (9):

$$\forall t \in [0, T], \begin{cases} i) & x'(t) = f(t, s-t, x(t), u(t)) \\ ii) & u(t) \in P(s-t, x(t)) \end{cases}$$

that satisfies for all anterior time $t \in [0, T]$ the expected constraints made at that time t for the future time $s-t$.

We can check easily that

Lemma 1.2 *The graph of the selector $\mathbf{S}_{(I,C)}$ is the capture basin of $\{0\} \times \text{Graph}(C)$ viable in the graph of the tube I under the auxiliary system*

$$\begin{cases} i) & \tau'(t) = -1 \\ ii) & \alpha'(t) = +1 \\ iii) & x'(t) = -f(\tau(t), \alpha(t), x(t)) \end{cases}$$

Proof— Indeed, to say that (T, a, x) belongs to the viable-capture basin of $\{0\} \times \text{Graph}(C)$ viable in $\text{Graph}(I)$ means that there exist an evolution $\hat{x}(\cdot)$ to $\hat{x}'(t) \in -f(T-t, a+t, \hat{x}(t), \tilde{u}(t))$ starting at $\hat{x}(0) := x$ and a time $t^* \geq 0$ such that

$$\begin{cases} i) & \forall t \in [0, t^*], \quad (T-t, a+t, \hat{x}(t)) \in \text{Graph}(I) \\ ii) & (T-t^*, a+t^*, \hat{x}(t^*)) \in \{0\} \times \text{Graph}(C) \end{cases}$$

The second condition means that $t^* = T$ and that $\hat{x}(T)$ belongs to $C(a+T)$. The first one means that for every $t \in [0, T]$, $\hat{x}(t) \in I(T-t, a+t)$. This amounts to saying that the evolution $x(\cdot) := \hat{x}(T-\cdot)$ is a solution to the control system

$$x'(t) = f(t, a+T-t, x(t), u(t))$$

starting at $x(0) := \hat{x}(T) \in C(T+a)$, satisfying $x(T) = x$ and

$$\forall t \in [0, T], \forall a \geq 0, \quad x(t) \in I(t, a+T-t)$$

This means that $x \in \mathbf{S}_{(I,C)}(T, a)$. \square

We shall therefore characterize the selector as a solution to a system of Hamilton-Jacobi-Bellman partial differential inclusions

$$\forall x \in V(t, a), \exists u \in P(a, x) \quad \text{such that} \quad -\frac{\partial V(t, a, x)}{\partial t} + \frac{\partial V(t, a, x)}{\partial a} + f(t, a, x, u) = 0$$

satisfying the initial condition

$$\forall a \geq 0, \quad V(0, a) = C(a)$$

and the viability constraints

$$\forall (t, a) \in \mathbf{R}_+^2, \quad V(t, a) \subset I(t, a)$$

We deduce from the knowledge of the derivatives of the selector the regulation map $R : \mathbf{R}_+^2 \times X \rightsquigarrow \mathcal{U}$ providing the controls (or regulees, prices, interest-rate instruments) that at each time t , for any future date a and any state x answer the detection/prediction problems. The regulation map associates with any triple (t, a, x) the set $R(t, a, x)$ of controls $u \in P(a, x)$ such that the solutions to the new control system

$$\begin{cases} \forall t \in [0, T], \forall s \geq T, \\ i) & x'(t) = f(t, s-t, x(t), u(t)) \\ ii) & u(t) \in R(t, s-t, x(t)) \end{cases} \quad (11)$$

satisfy the constraints (1.1)

$$\begin{cases} i) & x(0) \in C(s) \\ ii) & \forall t \in [0, T], \quad x(t) \in I(t, s-t) \\ iii) & x(T) = x \end{cases}$$

Finally, the Capture Basin Algorithm allows us to compute the selector and the regulation map.

1.2 The Substratum of an Impulse Differential Inclusion

Impulse differential inclusions are described by two set-valued maps F — the right-hand side of the differential inclusion $x' \in F(x)$ governing the continuous evolution of a hybrid system — and Φ , describing the reset map reinitializing the system when required and a constrained set K inside which the evolution of the impulse differential equation must remain. We denote by $\mathcal{S}(x)$ the set of solutions $x(\cdot)$ to the differential inclusion starting at x .

Let us set $x(^-t) := \lim_{\tau \rightarrow t^-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t - \eta, t[$ where $\eta > 0$, and, for consistency purposes, $x(s) = x(^-t)$ if $s = t$.

An evolution of the impulse differential inclusion, called a “run” or an “execution” in the hybrid system community — is a finite or infinite sequence $x(\cdot) := \{(\tau_n, x_n, x_n(\cdot))\}_{n \geq 0}$ made of triples

1. of nonnegative cadences $\tau_n \in [0, +\infty[$,
2. a sequence of reinitialized states x_n ,
3. a sequence of motives $x_n(\cdot) \in \mathcal{S}(x_n)$ satisfying the end-point condition $x_n(\tau_n) \in \Phi^{-1}(x_{n+1})$

defining the sequence of impulse times $t_{n+1} := t_n + \tau_n$ and, on each interval $[t_n, t_{n+1}[$,

$$\forall t \in [t_n, t_{n+1}[, \quad x(t) := x_n(t - t_n) \quad (12)$$

If the sequence is finite and stops at τ_N , we set $\tau_{N+1} := +\infty$ and take $x_N(\cdot) \in \mathcal{S}(x_N)$ (this definition is taken from [25, 26, Aubin & Haddad]).

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$.

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in \Phi(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

We just define the concept of **substratum** of an impulse differential inclusion introduced in [9, Aubin], that summarizes the salient features of a run by considering only its sequences of cadences τ_n and of reinitialized states x_n . Knowing them, we can reconstruct the motives of the run by taking solutions $x_n(\cdot) \in \mathcal{S}(x_n)$ satisfying the end-point condition $x_n(\tau_n) \in \Phi^{-1}(x_{n+1})$. The question that arises is to provide an algorithm that provides these sequences of cadences τ_n and of reinitialized states x_n without solving the impulse differential inclusion, but through an algorithm. The substratum just does that:

Definition 1.3 *We associate with the dynamics (F, Φ) of the impulse differential inclusion its substratum $\Gamma : \mathbf{R}_+ \times K \rightsquigarrow K$, that is the set-valued map associating with any $(T, x) \in \mathbf{R}_+ \times K$ the subset*

$$\Gamma(T, x) := \bigcup_{x(\cdot) \in \mathcal{S}_F^K(x)} \Phi(x(T))$$

of the elements $y \in \Phi(x(T))$ where $x(\cdot) \in \mathcal{S}_F^K(x)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x and viable in K until it reaches $x(T) \in C := K \cap \Phi^{-1}(K)$ at time T .

We associate with the substratum Γ

1. the cadence map

$$\mathbf{C}(x) := \{t \geq 0 \text{ such that } \Gamma(t, x) \neq \emptyset\}$$

2. and the initialization map $\mathbf{I} : K \rightsquigarrow X$

$$\mathbf{I}(x) = \bigcup_{t \in \mathbf{C}(x)} \Gamma(t, x)$$

Knowing the substratum Γ of (K, F, Φ) , and thus the cadence map \mathbf{C} and the initialization map \mathbf{I} , we can reconstruct a viable run of the impulse differential inclusion (F, Φ) through the following algorithm: Given the cadence τ_n and the initial state x_n , we take

$$\left\{ \begin{array}{l} i) \quad \text{the next cadence } \tau_{n+1} \in \mathbf{C}_{(F, \Phi)}(x_n), \\ ii) \quad \text{the next reinitialized state } x_{n+1} \in \Gamma_{(F, \Phi)}(\tau_{n+1}, x_n) \subset \mathbf{I}(x_n), \\ iii) \quad \text{the next motive } x_n(\cdot) := x(\cdot + \tau_n), \text{ a solution to } x' \in F(x) \text{ satisfying} \\ \quad \quad x_n(0) = x_n \ \& \ x_n(\tau_{n+1}) \in \Phi^{-1}(x_{n+1}) \end{array} \right. \quad (13)$$

Assume for a while that the impulse differential inclusion is actually an impulse differential equation (f, φ) where the maps f and φ are single-valued and that the substratum is single-valued and differentiable. We define

$$\Phi(t, x) := \begin{cases} \varphi(x) & \text{if } x \in C := K \cap \varphi^{-1}(K) \\ \emptyset & \text{if } x \notin C \end{cases}$$

and

$$\Psi(t, x) := \begin{cases} K & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

Then we shall prove that the substratum is a “solution” $v(t, x)$ to the system of first-order partial differential inclusions

$$\forall i = 1, \dots, n, \quad -\frac{\partial v_i(t, x)}{\partial t} + \sum_{j=1}^n \frac{\partial v_i(t, x)}{\partial x_j} f_j(x) = 0$$

satisfying the “condition”

$$\forall x \in C := K \cap \varphi^{-1}(K), \quad v(0, x) = \varphi(x)$$

that is either single-valued or takes empty values — and thus, is a set-valued map.

Actually, we shall extend this result to general impulse differential inclusions by characterizing the substratum as a generalized (set-valued) solution — a Frankowska solution — to the system of first-order partial differential inclusions

$$\left\{ \begin{array}{l} i) \quad -\frac{\partial}{\partial t} V(t, x) + \frac{\partial V(t, x)}{\partial x} \cdot u = 0 \\ ii) \quad u \in F(x) \end{array} \right.$$

satisfying the “condition”

$$\forall x \in C := K \cap \Phi^{-1}(K), \quad V(0, x) = \Phi(x)$$

and the constraints $V(t, x) \subset K$.

Indeed, the substratum is a particular case of a caliber with $f(x, y, u) := u$, $g(x, y, u) := 0$, $P(x, y) := F(x)$, $\Phi(0, x) := \Phi(x)$ if $x \in K$, $\Phi(0, x) := \emptyset$ if $x \notin K$, $\Phi(t, x) := \emptyset$ if $t > 0$, $\Psi(t, x) := K$ if $x \in K$ and $\Psi(t, x) := \emptyset$ otherwise.

2 The Caliber of Dynamical Constraints and Objective

The purpose of this section is to show how “viability techniques” may be efficient for solving systems of first-order partial differential inclusions arising in different fields of control theory and hybrid systems.

We denote by $L^1(0, \infty; \mathcal{U})$ the set of measurable functions from $[0, +\infty[$ to a vector space \mathcal{U} , the control space.

We consider a control system of the form

$$\begin{cases} i) & x'(t) = f(x(t), y(t), u(t)) \\ ii) & y'(t) = g(x(t), y(t), u(t)) \\ iii) & u(t) \in P(x(t), y(t)) \end{cases}$$

We denote by $\mathcal{B}(x, y)$ the set of solutions $(x(\cdot), y(\cdot), u(\cdot)) \in \mathcal{C}(0, \infty; X \times Y) \times L^1(0, \infty; \mathcal{U})$ to the above system starting at (x, y) at time 0.

Definition 2.1 *We say that the control system is*

1. *Marchaud if the set-valued map $P : X \times Y \rightsquigarrow \mathcal{U}$ is Marchaud, if $f : X \times Y \times \mathcal{U} \mapsto X$ and $g : X \times Y \times \mathcal{U} \mapsto Y$ are continuous and affine⁷,*
2. *Lipschitz if the set-valued map $P : X \rightsquigarrow \mathcal{U}$ is Lipschitz and if $f : X \times Y \times \mathcal{U} \mapsto X$ and $g : X \times Y \times \mathcal{U} \mapsto Y$ are Lipschitz.*

We associate the set-valued map $G : \mathbf{R}_+ \times X \times Y \rightsquigarrow \mathbf{R}_+ \times X \times Y$ defined by

$$G(T, x, y) := \{-1\} \times \{f(x, y, u)\} \times \{g(x, y, u)\}_{u \in P(x, y)} \quad (14)$$

and we denote by \mathcal{R} the set-valued map defined the formula

$$\mathcal{R}(T, x, y) = \{(T - \cdot, x(\cdot), y(\cdot), u(\cdot))\}_{(x(\cdot), y(\cdot), u(\cdot)) \in \mathcal{B}(x, y)}$$

We infer that *the set-valued map G is a Marchaud (resp. Lipschitz) map whenever the control system is Marchaud (resp. Lipschitz).*

We introduce now dynamical constraints and objectives defined by

1. a set-valued map $\Phi : \mathbf{R}_+ \times X \rightsquigarrow Y$ defining an **objective**, regarded as an obstacle in problems of unilateral mechanics, for instance.
2. a set-valued map $\Psi : \mathbf{R}_+ \times X \rightsquigarrow Y$ defining dynamical constraints. State constraints are involved in the domain

$$\text{Dom}(\Psi) := \{(t, x) \in \mathbf{R}_+ \times X \mid \Psi(t, x) \neq \emptyset\}$$

of the set-valued map Ψ .

We shall assume that

$$\begin{cases} i) & \forall (t, x) \in \mathbf{R}_+ \times X, \quad \Phi(t, x) \subset \Psi(t, x) \\ ii) & \forall t < 0, \forall x \in X, \quad \Psi(t, x) := \emptyset \end{cases}$$

⁷We actually need only that the values $\{f(x, y, u), g(x, y, u)\}_{u \in P(x, y)}$ are closed and convex.

Definition 2.2 *The two constraint and objective set-valued maps being given, the caliber $(T, x) \rightsquigarrow V_\Psi(\Phi)(T, x)$ of the pair (Ψ, Φ) under the controlled system is the set-valued map associating with the pair (T, x) made of the horizon T and the initial state x the set of initial observations y such that there exist a control $t \in [0, T] \mapsto u(t)$ and a time $T^* \in [0, T]$ such that a solution $(x(\cdot), u(\cdot), y(\cdot)) \in \mathcal{B}(x, y)$ starting at $x(0) = x$, $y(0) = y$ satisfies*

$$\begin{cases} i) & \forall t \in [0, T^*], \quad y(t) \in \Psi(T - t, x(t)) \\ ii) & y(T^*) \in \Phi(T - T^*, x(T^*)) \end{cases} \quad (15)$$

We observe at once the following property: *The caliber satisfies the initial condition*

$$\forall x \in X, \quad V_\Psi(\Phi)(0, x) = \Phi(0, x)$$

Indeed, condition (15)ii) with $T = 0$ means that $y \in V_\Psi(\Phi)(0, x)$, implying that $T^* = 0$ and $(0, x, y) \in \text{Graph}(\Phi)$, i.e., $y \in \Phi(0, x)$. Hence $V_\Psi(\Phi)(0, x) \subset \Phi(0, x) \subset V_\Psi(\Phi)(0, x)$. \square

What is the connection between this problem and the basic viability theorems? The answer is simple: The graph of the caliber is the capture basin of the graph of the set-valued map Φ viable in the graph of Ψ under the auxiliary control system G .

Proposition 2.3 *The graph of the caliber $V_\Psi(\Phi)$ is equal to the viable-capture basin of $\text{Graph}(\Phi)$ viable in $\text{Graph}(\Psi)$ under the auxiliary system \mathcal{R} :*

$$\text{Graph}(V_\Psi(\Phi)) = \text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi)}(\text{Graph}(\Phi))$$

Proof — It is enough to translate conditions (15) in the form

$$\begin{cases} i) & \forall t \in [0, T^*], \quad (T - t, x(t), y(t)) \in \text{Graph}(\Psi) \\ ii) & (T - T^*, x(T^*), y(T^*)) \in \text{Graph}(\Phi) \end{cases} \quad (16)$$

to recognize that

$$\text{Graph}(V_\Psi(\Phi)) = \text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi)}(\text{Graph}(\Phi)) \quad \square$$

This being checked, it will be sufficient to translate the properties of capture basins in terms of caliber.

3 Set-Valued Solutions to Systems of Hamilton-Jacobi-Bellman Inclusions

In the case of controlled systems, we shall relate the caliber with the set-valued solution to the controlled Hamilton-Jacobi-Bellman partial differential equations

1. there exists $u \in P(x, V(t, x))$ such that

$$0 \in \left[-\frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(x, V(t, x), u) - g(x, V(t, x), u) \right]$$

2. for all $u \in P(x, v(t, x))$,

$$0 \in \left\langle \frac{\partial}{\partial t} V(t, x) - \frac{\partial}{\partial x} V(t, x) f(x, V(t, x), u) \right\rangle + g(x, V(t, x), u)$$

satisfying the initial condition

$$V(0, x) = \Phi(0, x)$$

and the constraints

$$\forall t \geq 0, x \in X, \Phi(t, x) \subset V(t, x) \subset \Psi(t, x)$$

in a sense that we make precise in the following Theorem.

Theorem 3.1 *Let us assume that the system is Marchaud.*

1. Then the caliber $V_\Psi(\Phi)$ of (Ψ, Φ) is the **largest** closed set-valued map $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ satisfying

$$\forall (t, x) \in \mathbf{R}_+ \times X, \Phi(t, x) \subset V(t, x) \subset \Psi(t, x) \quad (17)$$

and

$$\left\{ \begin{array}{l} \forall y \in V(t, x) \setminus \Phi(t, x), \exists u \in P(x, y) \\ \text{such that} \\ 0 \in DV(-1, f(x, y, u)) - g(x, y, u) \end{array} \right. \quad (18)$$

2. Let us set

$$\mathbf{R}(t, x, y) := \{u \in P(x, y) \mid 0 \in DV_\Psi(\Phi)(t, x, y)(-1, f(x, y, u)) - g(x, y, u)\}$$

Knowing the caliber, any solution satisfying the constraints (15)i) and reaching the objective (15)ii) in finite time is obtained in the following way: Starting from (x_0, y_0) such that $y_0 \in V_\Psi(\Phi)(T, x_0) \setminus \Phi(T, x_0)$, any solution $(x(\cdot), y(\cdot), u(\cdot))$ to the control system: for almost all $t \in [0, T]$,

$$\left\{ \begin{array}{l} i) \quad x'(t) = f(x(t), y(t), u(t)) \\ ii) \quad y'(t) = g(x(t), y(t), u(t)) \\ iii) \quad u(t) \in \mathbf{R}(T-t, x(t), y(t)) \end{array} \right. \quad (19)$$

starting at (x, y) is a solution satisfying

$$y(t) \in V_\Psi(\Phi)(T-t, x(t))$$

until the first time $t^* \in]0, T]$ when

$$y(t^*) \in \Phi(T-t^*, x(t^*))$$

3. If we assume furthermore that the system is Lipschitz, then the caliber $V_\Psi(\Phi)$ of (Ψ, Φ) is the **unique** closed set-valued map $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ satisfying (17), (18) and

$$\left\{ \begin{array}{l} \forall y \in V(t, x) \cap \Psi^\circ(t, x), \forall u \in P(x, y) \\ 0 \in DV(1, -f(x, y, u)) + g(x, y, u) \\ \& \\ \forall y \in V(t, x) \cap \Psi^\partial(t, x), \forall u \in P(x, y) \\ 0 \in DV(1, -f(x, y, u)) \cup D\Psi^c(+1, -f(x, y, u)) + g(x, y, u) \end{array} \right. \quad (20)$$

where $\text{Graph}(\Psi^\circ) := \text{Int}(\text{Graph}(\Psi))$, $\text{Graph}(\Psi^\partial) := \partial\text{Graph}(\Psi)$ and $\text{Graph}(\Psi^c) := X \setminus \text{Graph}(\Psi)$.

Proof

1. The graph of the caliber $V_\Psi(\Phi)$ being equal to the capture basin $\text{Capt}_G^{\text{Graph}(\Psi)}(\text{Graph}(\Phi))$ by Proposition 2.3, we can apply the first part of Theorem 0.1 since $\text{Graph}(\Psi) \subset \mathbf{R}_+ \times X \times Y$ is a repeller under G . Hence it is the largest graph of a set-valued map $V : X \rightsquigarrow Y$ between $\text{Graph}(\Phi)$ and $\text{Graph}(\Psi)$ such that, for any $(t, x, y) \in \text{Graph}(V) \setminus \text{Graph}(\Phi)$ — i.e., whenever $y \in V(t, x) \setminus \Phi(t, x)$ — there exists a control $u \in P(x)$ such that

$$(-1, f(x, y, u), -g(x, y, u)) \in T_{\text{Graph}(V)}(t, x, y) \quad (21)$$

In other words, it is the graph of the largest closed set-valued map V satisfying

$$\Phi(t, x) \subset V(t, x) \subset \Psi(t, x)$$

and, whenever $y \in V(t, x) \setminus \Phi(t, x)$, there exists $u \in P(x, y)$ such that

$$0 \in DV(-1, f(x, y, u)) - g(x, y, u)$$

2. The solutions $(T - \cdot, x(\cdot), y(\cdot))$ viable in the graph of Ψ until they reach the graph of Ψ satisfy: for almost all $t \geq 0$.

$$\begin{cases} u(t) \in P(x(t), y(t)) \\ \& \\ (-1, f(x(t), y(t), u(t)), g(x(t), y(t), u(t))) \in T_{\text{Graph}(V)}(T - t, x(t), y(t)) \end{cases}$$

This condition can be rewritten

$$\text{for almost all } t \geq 0, \quad u(t) \in R(T - t, x(t), y(t))$$

3. Under the Lipschitz conditions, Theorem 0.2 states that the graph of $V_\Psi(\Phi)$ is the unique closed subset $\text{Graph}(V)$ satisfying the Frankowska properties:

$$\begin{cases} i) & \forall (t, x, y) \in \text{Graph}(V) \setminus \text{Graph}(\Phi), \\ & \exists u \in P(x, y) \mid (-1, f(x, y, u), g(x, y, u)) \in T_{\text{Graph}(V)}(t, x, y) \\ ii) & \forall (t, x, y) \in \text{Graph}(V) \cap \text{Int}(\text{Graph}(\Psi)), \forall u \in P(x, y) \\ & (1, -f(x, y, u), -g(x, y, u)) \in T_{\text{Graph}(V)}(t, x, y) \\ iii) & \forall (t, x, y) \in \text{Graph}(V) \cap \partial(\text{Graph}(\Psi)), \forall u \in P(x, y) \\ & (1, -f(x, y, u), -g(x, y, u)) \in T_{\text{Graph}(V)}(t, x, y) \cup T_{X \setminus \text{Graph}(\Psi)}(t, x, y) \end{cases}$$

Using the fact that $\text{Graph}(DV)(t, x, y) = T_{\text{Graph}(V)}(t, x, y)$, we infer the third part of the Theorem. \square

4 Stability Properties

We state the following Theorem of [12, Aubin]:

Theorem 4.1 *Let us consider a sequence of closed subsets C_n satisfying $\text{Viab}(K) \subset C_n \subset K$ and*

$$\text{Lim}_{n \rightarrow +\infty} C_n := \text{Limsup}_{n \rightarrow +\infty} C_n = \text{Liminf}_{n \rightarrow +\infty} C_n$$

If the set-valued map F is Marchaud and Lipschitz and if K is closed and backward invariant under F , then

$$\text{Lim}_{n \rightarrow +\infty} \text{Capt}^K(C_n) = \text{Capt}^K(\text{Lim}_{n \rightarrow +\infty} C_n) \quad (22)$$

Theorem 4.1 implies “continuity properties” of the caliber regarded as a map $\Phi \mapsto V_\Psi(\Phi)$. For that purpose, let us recall the following definitions of graphical convergence (see [18, Aubin & Frankowska] and/or [60, Rockafellar & Wets] for instance):

1. the upper limit of the graphs

$$\text{Limsup}_{n \rightarrow +\infty} \text{Graph}(\Phi_n) =: \text{Graph}(\text{Lim}^\sharp_{n \rightarrow +\infty} \Phi_n)$$

is the graph of the graphical upper limit $\text{Lim}^\sharp_{n \rightarrow +\infty} \Phi_n$ defined by

$$(\text{Lim}^\sharp_{n \rightarrow +\infty} \Phi_n)(T, x) = \liminf_{n \rightarrow +\infty, s \rightarrow t, y \rightarrow x} V(\Phi_n)(s, y)$$

2. the lower limit of the graphs

$$\text{Liminf}_{n \rightarrow +\infty} \text{Graph}(\Phi_n) =: \text{Graph}(\text{Lim}^b_{n \rightarrow +\infty} \Phi_n)$$

is the graph of the graphical lower limit.

Then we derive the following “continuity” properties of the calibers:

Theorem 4.2 *Let us consider a sequence of nontrivial set-valued maps $\Psi_n : \mathbf{R}_+ \times X \rightsquigarrow Y \cup \{+\infty\}$ and $\Phi_n : \mathbf{R}_+ \times X \rightsquigarrow Y \cup \{+\infty\}$ such that*

$$\forall (T, x) \in \mathbf{R}_+ \times X, \quad \Phi_n(T, x) \subset \Psi_n(T, x)$$

The calibers satisfy also the following properties:

1. *Let us assume that the auxiliary control system F is Marchaud. Then*

$$\lim^\sharp_{n \rightarrow +\infty} V_{\Psi_n}(\Phi_n)(T, x) \subset V_{\lim^\sharp_{n \rightarrow +\infty} \Psi_n} \left(\lim^\sharp_{n \rightarrow +\infty} \Phi_n \right) (T, x)$$

2. *Let us assume that the auxiliary control system F is Marchaud and Lipschitz. Then*

$$\begin{cases} V \left(\lim^b_{n \rightarrow +\infty} \Phi_n \right) (T, x) \subset \lim^b_{n \rightarrow +\infty} V(\Phi_n)(T, x) \\ \subset \lim^\sharp_{n \rightarrow +\infty} V(\Phi_n)(T, x) \subset V \left(\lim^\sharp_{n \rightarrow +\infty} \Phi_n \right) (T, x) \end{cases}$$

Therefore, if the sequence of set-valued maps Φ_n converges graphically to Φ , the caliber of the graphical limit is the graphical limit of the calibers.

Proof — If the system is Marchaud, Theorem 4.1 implies that

$$\text{Limsup}_{n \rightarrow +\infty} \text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi_n)}(\text{Graph}(\Phi_n)) \subset \text{Capt}_{\mathcal{R}}^{\text{Limsup}_{n \rightarrow +\infty} \text{Graph}(\Psi_n)}(\text{Limsup}_{n \rightarrow +\infty} \text{Graph}(\Phi_n))$$

Hence we deduce from Proposition 2.3 characterizing the graph of the caliber as a viable-capture basin and from the definitions of graphical limits that

$$\lim^\sharp_{n \rightarrow +\infty} V_{\Psi_n}(\Phi_n)(T, x) \subset V_{\lim^\sharp_{n \rightarrow +\infty} \Psi_n} \left(\lim^\sharp_{n \rightarrow +\infty} \Phi_n \right) (T, x)$$

Under Lipschitz conditions of G , Theorem 4.1 implies that

$$\text{Capt}_{\mathcal{R}}(\text{Liminf}_{n \rightarrow +\infty} \text{Graph}(\Phi_n)) \subset \text{Liminf}_{n \rightarrow +\infty} \text{Capt}_{\mathcal{R}}(\text{Graph}(\Phi_n))$$

Hence we deduce that

$$V \left(\lim^b_{n \rightarrow +\infty} \Phi_n \right) (T, x) \subset \lim^b_{n \rightarrow +\infty} V(\Phi_n)(T, x)$$

This completes the proof. \square

5 Caliber of Structured Problems with Linear Exosystems

Of special interest is the particular case when the first differential equation does not depend upon the variable y and when the set-valued map $P : X \rightsquigarrow \mathcal{U}$ does not depend on the observation variable y : We thus obtain a **structured system** — as age structured systems in demography, when x plays the role of the age variable — of the form

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & y'(t) = g(x(t), y(t), u(t)) \\ iii) & u(t) \in P(x(t)) \end{cases}$$

where $y(\cdot)$ is often regarded as an **observation** of the state (see for instance [14, Aubin, Bonneuil & Maurin]). In control theory, the second controlled equation is called the **exosystem**.

We denote by $\mathcal{C} : x \in X \rightsquigarrow \mathcal{C}(x) \in \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ the set-valued map associating with $x \in X$ the set $\mathcal{C}(x)$ of the pairs $(x(\cdot), u(\cdot))$ solutions to the control system

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in P(x(t)) \end{cases}$$

starting at x at $t = 0$.

We shall also set $g(x, y, u) := -M(x, u)y - L(x, u)$ where

1. M is a bounded continuous matrix-valued function

$$M : (x, u) \in X \times \mathcal{U} \mapsto M(x, u) \in \mathcal{L}(X, Y)$$

2. L is a continuous⁸ “vector-Lagrangian”

$$L : (x, u) \in X \times \mathcal{U} \mapsto L(x, u) \in Y$$

with linear growth.

defining the map $(y; (x(\cdot), u(\cdot))) \rightsquigarrow \mathcal{S}(y; (x(\cdot), u(\cdot)))$ associating the subset of evolutions

$$y(t) = e^{-\int_0^t M(x(s), u(s))ds} \left(y - \int_0^t e^{\int_0^\tau M(x(s), u(s))ds} L(x(\tau), u(\tau))d\tau \right) \quad (23)$$

to the linear dynamical system

$$y'(t) = -M(x(t), u(t))y(t) - L(x(t), u(t))$$

starting at $y \in Y$.

We already know that the caliber is the unique Frankowska solution to

1. there exists $u \in P(x, V(t, x))$ such that

$$0 \in -\frac{\partial}{\partial t}V(t, x) + \frac{\partial}{\partial x}V(t, x)f(x, V(t, x), u)'' - M(x, u)V(t, x) - L(x, u)$$

⁸We could take L to be set-valued map, but we restrict to the single-valued case for simplicity.

2. for all $u \in P(x, v(t, x))$,

$$0 \in \left[\frac{\partial}{\partial t} V(t, x) - \frac{\partial}{\partial x} V(t, x) f(x, V(t, x), u) \right] + M(x, u) V(t, x) + L(x, u)$$

satisfying the viability constraints

$$\forall t \geq 0, x \in X, \Phi(t, x) \subset V(t, x) \subset \Psi(t, x)$$

and the initial condition

$$V(0, x) = \Phi(0, x)$$

In summary, we now deal with a **structured problem** where the exosystem is linear with respect to the observations. In this case, we shall be able to provide an explicit formula of the caliber V .

For that purpose, we introduce the subset

$$\left\{ \begin{array}{l} J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) \\ := e^{\int_0^t M(x(s), u(s)) ds} \Phi(T - t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{array} \right.$$

(where t ranges over $[0, T]$). The controls — most often prices or other regulees in economics, portfolio in finance — appear **both** in the matrix M and in the Lagrangian L .

We associate with Ψ the set-valued map J_{Ψ} defined by

$$\left\{ \begin{array}{l} J_{\Psi}(t; (x(\cdot), u(\cdot)))(T, x) \\ := e^{\int_0^t M(x(s), u(s)) ds} \Psi(T - t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{array} \right.$$

and

$$K_{\Psi}(t, x; (x(\cdot), u(\cdot))) := \bigcap_{s \in [0, t]} J_{\Psi}(s, x; (x(\cdot), u(\cdot)))$$

We next introduce

$$L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) := K_{\Psi}(t, x; (x(\cdot), u(\cdot))) \cap J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x)$$

5.1 Explicit Formula of the Caliber

We shall prove now that the caliber V of the pair (Ψ, Φ) under \mathcal{B} is equal to the set-valued map $V_{\Psi}(\Phi)$ defined by

$$V_{\Psi}(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) \quad (24)$$

Note that if $\Psi_1 \subset \Psi_2$ and $\Phi_1 \subset \Phi_2$, then $V_{\Psi_1}(\Phi_1) \subset V_{\Psi_2}(\Phi_2)$ and that

$$\forall t \geq 0, \forall x \in X, \Phi(t, x) \subset V_{\Psi}(\Phi)(t, x) \subset \Psi(t, x)$$

We shall use the fact that its graph is the viable-capture basin of the graph of the cost function Φ viable under the graph of Ψ under the auxiliary system \mathcal{R} .

Theorem 5.1 *Let us assume that the set-valued maps Ψ and Φ are nontrivial. Then the caliber of the pair (Ψ, Φ) is equal to the set-valued map $V_\Psi(\Phi)$ defined by (24).*

*Furthermore, the caliber is the **unique solution** V to the two following “fixed set equations”*

:

$$\begin{cases} V_V(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} L_V^\Phi(t, x; (x(\cdot), u(\cdot))) \\ = V(T, x) \\ = V_\Psi(V)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} L_V^\Psi(t, x; (x(\cdot), u(\cdot))) \end{cases} \quad (25)$$

Moreover, when F is Marchaud and the set-valued maps Φ and Ψ are closed, the graph of the caliber $V_\Psi(\Phi)$ is closed.

5.2 Examples

1. We see that the set-valued map defined by

$$V(\Phi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} J_\Phi(t; (x(\cdot), u(\cdot)))(T, x) = V_{\mathbf{Y}}(\Phi)$$

is the caliber of (Ψ, Φ) where the set-valued map $\Psi = \mathbf{Y}$ is defined by

$$\mathbf{Y}(t, x) := \begin{cases} Y & \text{if } t \geq 0 \\ \emptyset & \text{if } t < 0 \end{cases}$$

Indeed, we observe that taking $\Psi = \mathbf{Y}$, then

$$K_{\mathbf{Y}}(t, x; (x(\cdot), u(\cdot))) = J_{\mathbf{Y}}(t, x; (x(\cdot), u(\cdot))) = Y$$

so that, for any set-valued map Φ , we have $L_{\mathbf{Y}}^\Phi = J_\Phi$ and thus, $V(\Phi) = V_{\mathbf{Y}}(\Phi)$.

2. We also observe that the set-valued map

$$W(\Psi)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcap_{t \in [0, T]} J_\Psi(t; (x(\cdot), u(\cdot)))(T, x) = V_\Psi(\Psi_\emptyset)$$

is the caliber of (Ψ, Φ) where we take $\Phi := \Psi_\emptyset : \mathbf{R}_+ \times X \rightsquigarrow Y$ defined by

$$\Psi_\emptyset(t, x) := \begin{cases} \Psi(0, x) & \text{if } t = 0 \\ \emptyset & \text{if } t \neq 0 \end{cases}$$

Indeed, we see that $J_{\Psi_\emptyset}(t, x; (x(\cdot), u(\cdot))) = \emptyset$ if $t < T$ and $J_{\Psi_\emptyset}(T, x; (x(\cdot), u(\cdot))) = J_\Psi(T, x; (x(\cdot), u(\cdot)))$. Therefore

$$L_{\Psi_\emptyset}^\Psi(t, x; (x(\cdot), u(\cdot))) := \begin{cases} K_\Psi(T, x; (x(\cdot), u(\cdot))) & \text{if } t = T \\ \emptyset & \text{if } t < T \end{cases}$$

and thus, $W(\Psi) = V_\Psi(\Psi_\emptyset)$.

3. Let us introduce a time-independent set-valued map $U : X \rightsquigarrow X$. We shall associate with three pairs (Ψ, Φ) of set-valued maps associated with U the three following calibers:

(a) Taking $\Phi := U_\emptyset$ and $\Psi = \mathbf{Y}$, we obtain

$$\bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \left(e^{\int_0^T M(x(s), u(s)) ds} U(x(T)) + \int_0^T e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \right)$$

When $M = 0$, the above problem boils down to the set-valued equivalent of the “Bolza map”

$$\bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \left(U(x(T)) + \int_0^T L(x(\tau), u(\tau)) d\tau \right)$$

and the “Mayer map”

$$\bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} U(x(T))$$

when furthermore $L = 0$. This is the case of the substratum of an impulse differential inclusion defined above (see also [9, Aubin]).

(b) Taking $\Phi := U_\emptyset$ and $\Psi := U$, we obtain

$$\bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcap_{t \in [0, T]} \left(e^{\int_0^t M(x(s), u(s)) ds} U(x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \right)$$

(c) Taking $\Phi := U$ and $\Psi := \mathbf{Y}$, we obtain

$$\bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} \left(e^{\int_0^t M(x(s), u(s)) ds} U(x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \right)$$

When $M = 0$, we find

$$V(U)(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} \left(U(x(t)) + \int_0^t L(x(\tau), u(\tau)) d\tau \right)$$

This map — the set-valued analogue of the valuation function of “obstacle problems” — involves “max-plus” operations and is the set-valued equivalent of the **mathematical fear – or faith** with respect to a Maslov measure introduced in dynamical optimization by Pierre Bernhard (see [35, 35, 37, Bernhard]).

Regarding the caliber as a transform $\Phi \mapsto V(\Phi)$ mapping closed set-valued maps to closed set-valued map, we observe that the caliber satisfies

$$V \left(\bigcup_{i=1, \dots, n} \Phi_i \right) = \bigcup_{i=1, \dots, n} V(\Phi_i)$$

the extensivity property: $\Phi \subset V(\Phi)$ and the monotonicity property: If $\Phi_1 \subset \Phi_2$, then $V(\Phi_1) \subset V(\Phi_2)$.

We recall that Theorem 4.2 implies that the value transform is also “upper continuous” in the sense that

$$\text{Limsup}_{n \rightarrow +\infty, s \rightarrow t, y \rightarrow x} V(\Phi_n)(s, y) \subset V \left(\text{Limsup}_{n \rightarrow +\infty, s \rightarrow t, y \rightarrow x} \Phi_n(s, y) \right) (t, x)$$

and actually, under stronger assumptions, “continuous” in the sense that the caliber of a graphical limit is the graphical limit of calibers.

5.3 Proof of the Explicit Formula

First, to say that a pair (T, x, y) belongs to the viable-capture basin

$$\text{Graph}(V) := \text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi)}(\text{Graph}(\Phi) \subset \text{Graph}(\Psi))$$

means that there exist a solution $(T - \cdot, \tilde{x}(\cdot), \tilde{u}(\cdot), y(\cdot)) \in \mathcal{R}(T, x, y)$ to the auxiliary control problem and some $t^* \geq 0$ such that

$$\begin{cases} i) & \forall t \in [0, t^*], (T - t, \tilde{x}(t), y(t)) \in \text{Graph}(\Psi) \\ ii) & (T - t^*, \tilde{x}(t^*), y(t^*)) \in \text{Graph}(\Phi) \end{cases}$$

or, equivalently, such that

$$\begin{cases} i) & \forall t \in [0, t^*], y(t) \in \Psi(T - t, \tilde{x}(t)) \\ ii) & y(t^*) \in \Psi(T - t^*, \tilde{x}(t^*)) \end{cases}$$

By the very definition of $\mathcal{R}(T, x, y)$ and by the definition (23) of the auxiliary system that its component $y(\cdot)$ satisfies

$$y(t) = e^{-\int_0^t M(\tilde{x}(s), \tilde{u}(s)) ds} \left(y - \int_0^t e^{\int_0^\tau M(\tilde{x}(s), \tilde{u}(s)) ds} L(\tilde{x}(\tau), \tilde{u}(\tau)) d\tau \right)$$

this implies that (T, x, y) belongs to $\text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi)}(\text{Graph}(\Phi))$ if and only if there exists a solution $(\tilde{x}(\cdot), \tilde{u}(\cdot)) \in \mathcal{C}(x)$ satisfying for almost all $t \in [0, \bar{t}]$,

$$\begin{cases} i) & y \in e^{\int_0^t M(\tilde{x}(s), \tilde{u}(s)) ds} \Psi(T - t, \tilde{x}(t), \tilde{u}(t)) + \int_0^t e^{\int_0^\tau M(\tilde{x}(s), \tilde{u}(s)) ds} L(\tilde{x}(\tau), \tilde{u}(\tau)) d\tau \\ ii) & y \in e^{\int_0^{t^*} M(\tilde{x}(s), \tilde{u}(s)) ds} \Phi(T - t^*, \tilde{x}(t^*)) + \int_0^{t^*} e^{\int_0^\tau M(\tilde{x}(s), \tilde{u}(s)) ds} L(\tilde{x}(\tau), \tilde{u}(\tau)) d\tau \end{cases} \quad (26)$$

Since we set

$$K_{\Psi}(t, x; (\tilde{x}(\cdot), \tilde{u}(\cdot))) := \bigcap_{s \in [0, t]} J_{\Psi}(s, x; (\tilde{x}(\cdot), \tilde{u}(\cdot)))$$

and

$$L_{\Psi}^{\Phi}(t, x; (\tilde{x}(\cdot), \tilde{u}(\cdot))) := K_{\Psi}(t, x; (\tilde{x}(\cdot), \tilde{u}(\cdot))) \cap J_{\Phi}(t, x; (\tilde{x}(\cdot), \tilde{u}(\cdot)))$$

this is equivalent to state that y belongs to $V_{\Psi}(\Phi)(T, x)$.

Formula (3) of [15, Aubin & Catté] and the first part of this theorem implies that the caliber is the unique solution to the system (25).

Finally, the closedness of the graph of the caliber is an immediate consequence of Theorem 0.1 because $\text{Graph}(\Psi) \subset \mathbf{R}_+ \times X \times Y$ being a repeller under G , which is Marchaud, the viable-capture basin $\text{Graph}(V_{\Psi})(\Phi) := \text{Capt}_{\mathcal{R}}^{\text{Graph}(\Psi)}(\text{Graph}(\Phi))$ is closed. \square

5.4 The Barrier Property of the Caliber

We begin with a first form of the barrier property:

Proposition 5.2 *Let us consider $y_T \in \partial_{\Psi(T,x)}V_{\Psi}(\Phi)(T,x)$. Then, any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from x satisfying inclusion: for every $t \in [0, t^*]$*

$$y_T \in L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) \quad (27)$$

until the first time t^* when

$$y_T \in \Phi(T - t^*, x(t^*))$$

actually satisfies: for every $t \in [0, t^*]$

$$\forall y_T \in \partial L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x)$$

Proof — Since y_T belongs to $V_{\Psi}(\Phi)(T, x)$, there exists a solution $(x^*(\cdot), u^*(\cdot))$ and a time $t^* \in [0, T]$ such that

$$y_T \in L_{\Psi}^{\Phi}(t^*, x; (x^*(\cdot), u^*(\cdot)))$$

Since y_T belongs to $\partial_{\Psi(T,x)}V_{\Psi}(\Phi)(T, x)$, it can be approximated by elements $y_n \in \Psi(T, x) \setminus V_{\Psi}(\Phi)(T, x)$. We know that for every t and any solution $(x(\cdot), u(\cdot))$, either y_n does not belongs to $K_{\Psi}(t, x; (x(\cdot), u(\cdot)))$ or else, it belongs to $K_{\Psi}(t, x; (x(\cdot), u(\cdot))) \setminus V(T, x)$.

We claim that

$$\left\{ \begin{array}{l} \forall (x(\cdot), u(\cdot)) \in \mathcal{C}(x), \forall t \in [0, T], \\ K_{\Psi}(t, x; (x(\cdot), u(\cdot))) \setminus V(T, x) \subset X \setminus J_V(t, x; (x(\cdot), u(\cdot))) \end{array} \right. \quad (28)$$

Indeed, take y in $K_{\Psi}(t, x; (x(\cdot), u(\cdot))) \setminus V(T, x)$. This means that for every $s \in [0, t]$, y belongs to $J_{\Psi}(s, x; (x(\cdot), u(\cdot)))$, or, equivalently, that

$$\forall s \in [0, t], (T - s, x(s), y(s)) \in \text{Graph}(\Psi)$$

where

$$y(t) := e^{-\int_0^t M(x(s), u(s))ds} \left(y - \int_0^t e^{\int_0^{\tau} M(x(s), u(s))ds} L(x(\tau), u(\tau))d\tau \right)$$

Since (T, x, y) does not belong to the viable capture basin, we infer that every solution in $\mathcal{R}(T, x, y)$ starting at (T, x, y) is viable in the graph $\text{Graph}(\Psi)$ of Ψ before hitting the graph $\text{Graph}(V)$ of V : Therefore, if

$$\forall s \in [0, t], y(s) \in \Psi(T - s, x(s), u(s))$$

we have

$$y(t) \in X \setminus V(T - t, x(t)) \subset X \setminus \Phi(T - t, x(t))$$

that can be written in the form

$$y \in X \setminus J_V(t, x; (x(\cdot), u(\cdot))) \subset X \setminus J_{\Phi}(t, x; (x(\cdot), u(\cdot)))$$

Therefore, inclusion (28) holds true and thus, y_n does not belong to $J_V(t, x; (x(\cdot), u(\cdot)))$. Consequently, for any $t \in [0, T]$, for any $n \geq 0$,

$$y_n \in X \setminus L_{\Psi}^V(t, x; (x(\cdot), u(\cdot))) \subset X \setminus L_{\Psi}^{\Phi}(t, x; (x(\cdot), u(\cdot)))$$

This is in particular true for the solution $(x^*(\cdot), u^*(\cdot))$ and for any $t \in [0, t^*]$. Hence, letting y_n converge to y_T , we deduce the conclusion of the proposition. \square

We can strengthen this result and prove the following “barrier property”. Let us recall that $J_{V_\Psi(\Phi)}(0, x; (x(\cdot), u(\cdot))) = V_\Psi(\Phi)(T, x)$. We shall prove that the boundary condition

$$y_T \in \partial V_\Psi(\Phi)(T, x) = \partial J_{V_\Psi(\Phi)}(0, x; (x(\cdot), u(\cdot)))$$

propagates as long as y_T remains in the interior of $K_\Psi(t, x; (x(\cdot), u(\cdot)))$:

Theorem 5.3 *Let us consider $y_T \in \partial V_\Psi(\Phi)(T, x) \cap \text{Int}(\Psi(T, x))$. Then, any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from x satisfying the inclusion*

$$y_T \in J_{V_\Psi(\Phi)}(t, x; (x(\cdot), u(\cdot))) \quad (29)$$

until the first time \bar{t} when

$$y_T \notin \text{Int}(K_\Psi(\bar{t}, x; (x(\cdot), u(\cdot))))$$

actually satisfies

$$\forall t \in [0, \bar{t}], \quad y \in \partial J_{V_\Psi(\Phi)}(t; (x(\cdot), u(\cdot)))(T, x) \quad (30)$$

Proof — Since y_T belongs to $\partial V_\Psi(\Phi)(T, x)$, it can be approximated by elements $y_n \in \Psi(T, x) \setminus V_\Psi(\Phi)(T, x)$. By assumption, for any $t < \bar{t}$, there exist δ_t such that

$$B(y_T, \delta_t) \subset K_\Psi(t; (x(\cdot), u(\cdot)))(T, x)$$

and thus, that there exists some N_t such that, for any $n \geq N_t$, y_n belongs to $B(y_T, \delta_t)$, and thus

$$y_n \in K_\Psi(t; (x(\cdot), u(\cdot)))(T, x) \setminus V_\Psi(\Phi)(T, x)$$

But by (28), we know that in this case, y_n does not belong to $J_{V_\Psi(\Phi)}(t; (x(\cdot), u(\cdot)))(T, x)$. \square

6 Frankowska contingent epi-solutions to Hamilton-Jacobi-Bellman equations

When $Y := \mathbf{R}$, we can associate with two extended functions $\mathbf{c} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R} \cup \{+\infty\}$ and $\mathbf{b} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R} \cup \{+\infty\}$ such that

$$\forall (t, x) \in \mathbf{R}_+ \times X, \quad 0 \leq \mathbf{b}(t, x) \leq \mathbf{c}(t, x)$$

the set-valued maps Φ and Ψ defined by

$$\begin{cases} i) & \Phi(t, x) := \mathbf{c}(t, x) + \mathbf{R}_+ \\ ii) & \Psi(t, x) := \mathbf{b}(t, x) + \mathbf{R}_+ \end{cases}$$

by setting $\Phi(t, x) := \emptyset$ whenever $\mathbf{c}(t, x) = +\infty$. We observe that $\text{Graph}(\Phi) = \mathcal{E}p(\mathbf{c})$ and that

$$D\Phi(t, x, \mathbf{c}(t, x))(\pm 1, v) = D\uparrow \mathbf{c}(t, x)(\pm 1, v) + \mathbf{R}_+$$

where $D\uparrow \mathbf{c}(t, x)$ is the contingent epiderivative of \mathbf{c} at (t, x) in the direction $(\pm 1, v)$, defined by

$$D\uparrow \mathbf{c}(t, x)(\pm 1, v) = \liminf_{h \rightarrow 0+, v' \rightarrow v} \frac{\mathbf{c}(t \pm h, x + hv') - \mathbf{c}(t, x)}{h}$$

We set

$$\begin{cases} J_{\mathbf{c}}(t; (x(\cdot), u(\cdot)))(T, x) \\ := e^{\int_0^t M(x(s), u(s)) ds} \mathbf{c}(T-t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{cases}$$

and then,

$$K_{\mathbf{b}}(t, x; (x(\cdot), u(\cdot))) := \sup_{s \in [0, t]} J_{\mathbf{b}}(s, x; (x(\cdot), u(\cdot)))$$

We next integrate this cumulated cost together with the former cost $J_{\mathbf{c}}(t, x; (x(s), u(s)))$ by introducing the new cost function

$$L_{\mathbf{b}}^{\mathbf{c}}(t; (x(\cdot), u(\cdot)))(T, x) := \max(K_{\mathbf{b}}(t, x; (x(\cdot), u(\cdot))), J_{\mathbf{c}}(t; (x(\cdot), u(\cdot)))(T, x))$$

We shall deduce from Theorem 5.1 the following consequence :

Theorem 6.1 *Let us assume that the extended functions \mathbf{b} and \mathbf{c} are nontrivial and non negative. The viable-capture basin $\text{Capt}_{\mathcal{R}}^{\mathcal{E}p(\mathbf{b})}(\mathcal{E}p(\mathbf{c}))$ of $\mathcal{E}p(\mathbf{c})$ under \mathcal{R} is the epigraph of the valuation function $V_{\mathbf{b}}(\mathbf{c})$ defined by*

$$V_{\mathbf{b}}(\mathbf{c})(T, x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{b}}^{\mathbf{c}}(t; (x(\cdot), u(\cdot)))(T, x)$$

Furthermore, any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from $x \in X$ satisfying the inequality: for every $t \in [0, t^*]$

$$\begin{cases} V_{\mathbf{b}}(\mathbf{c})(T, x) \\ \geq e^{\int_0^t M(x(s), u(s)) ds} V_{\mathbf{b}}(\mathbf{c})(T-t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{cases} \quad (31)$$

until the first time t^* when

$$V_{\mathbf{b}}(\mathbf{c})(T-t^*, x(t^*)) = \mathbf{c}(T-t^*, x(t^*))$$

is an optimal solution for the optimal time t^* .

Finally, the valuation function is a solution \mathbf{v} to the two following functional equations stating that the functions $L_{\mathbf{v}}^{\mathbf{b}}$ and $L_{\mathbf{v}}^{\mathbf{c}}$ have the same infimum than $L_{\mathbf{b}}^{\mathbf{c}}$:

$$\begin{cases} \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{v}}^{\mathbf{c}}(t, x; (x(\cdot), u(\cdot))) \\ = \mathbf{v}(T, x) \\ = \inf_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \inf_{t \in [0, T]} L_{\mathbf{b}}^{\mathbf{v}}(t, x; (x(\cdot), u(\cdot))) \end{cases} \quad (32)$$

Proof — It is enough — and easy — to check that

$$L_{\mathbf{v}}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) = L_{\mathbf{b}}^{\mathbf{c}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+$$

and thus, that

$$\inf_{y \in V_{\mathbf{v}}^{\Phi}(t, x)} y = V_{\mathbf{b}}^{\mathbf{c}}(t, x)$$

since $\inf(\bigcup_{i \in I} [a_i, \infty]) = \inf_{i \in I} a_i$. \square

Theorem 5.3 implies the following form of the optimality principle:

Theorem 6.2 *Let us assume that $V_{\mathbf{b}(\mathbf{c})}(T, x) > \mathbf{b}(T, x)$. Then any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from $x \in \text{Dom}(V_{\mathbf{b}(\mathbf{c})})$ satisfying inequalities*

$$\begin{cases} V_{\mathbf{b}(\mathbf{c})}(T, x) \\ \geq e^{\int_0^t M(x(s), u(s)) ds} V_{\mathbf{b}(\mathbf{c})}(T-t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{cases} \quad (33)$$

until the first time $t^* \in [0, T]$ when

$$V_{\mathbf{b}(\mathbf{c})}(T-t^*, x(t^*)) = \mathbf{b}(T-t^*, x(t^*))$$

actually satisfies equality

$$\begin{cases} \forall t \in [0, t^*], V_{\mathbf{b}(\mathbf{c})}(T, x) \\ = e^{\int_0^t M(x(s), u(s)) ds} V_{\mathbf{b}(\mathbf{c})}(T-t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{cases} \quad (34)$$

The first statement of Theorem 3.1 implies that the valuation function is a Frankowska contingent episolution to Hamilton-Jacobi-Bellman equations:

Theorem 6.3 (Frankowska) *Let us assume that the control system (P, f, L, M) is Marchaud and that the functions \mathbf{b} and \mathbf{c} are nontrivial, non negative and lower semicontinuous.*

Then the valuation function $V_{\mathbf{b}(\mathbf{c})}$ is characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ satisfying for every $(t, x) \in]0, \infty[\times X$

$$\begin{cases} i) & \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x) \\ ii) & \text{if } (t, x) \in \Omega(\mathbf{v}), \\ & \inf_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(-1, f(x, u)) + L(x, u) + M(x, u) \mathbf{v}(t, x)) \leq 0 \end{cases}$$

Let us set

$$\mathbf{R}(t, x) := \{u \in P(x) \mid D_{\uparrow} V_{\mathbf{b}(\mathbf{c})}(t, x)(-1, f(x, u)) + L(x, u) + M(x, u) V_{\mathbf{b}(\mathbf{c})}(t, x) \leq 0\}$$

Knowing the valuation function, an optimal solution is obtained in the following way: Starting from x_0 such that $V_{\mathbf{b}(\mathbf{c})}(T, x_0) < \mathbf{c}(T, x_0)$, any solution $(x(\cdot), u(\cdot))$ to the control system

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in \mathbf{R}(T-t, x(t)) \end{cases} \quad (35)$$

is an optimal solution, and the first time $t^ \geq 0$ when*

$$V_{\mathbf{b}(\mathbf{c})}(T-t^*, x(t^*)) = \mathbf{c}(T-t^*, x(t^*))$$

is the optimal time.

The second part of Theorem 3.1 implies the following existence and uniqueness result:

Theorem 6.4 (Frankowska) *Let us assume that the control system (P, f, L, M) is Marchaud and Lipschitz and that \mathbf{b} and \mathbf{c} are nontrivial, non negative and lower semicontinuous.*

Then the valuation function $V_{\mathbf{b}}(\mathbf{c})$ is the unique lower semicontinuous episolution \mathbf{v} to the system of differential inequalities: for every $(t, x) \in \text{Dom}(\mathbf{v})$,

$$\left\{ \begin{array}{l} i) \quad \mathbf{b}(t, x) \leq \mathbf{v}(t, x) \leq \mathbf{c}(t, x) \\ ii) \quad \text{if } \mathbf{v}(t, x) < \mathbf{c}(t, x), \\ \quad \inf_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(-1, f(x, u)) + L(x, u) + M(x, u)\mathbf{v}(t, x)) \leq 0 \\ iii) \quad \text{if } \mathbf{v}(t, x) > \mathbf{b}(t, x), \\ \quad \sup_{u \in P(x)} (D_{\uparrow} \mathbf{v}(t, x)(1, -f(x, u)) - L(x, u) - M(x, u)\mathbf{v}(t, x)) \leq 0 \\ iv) \quad \text{if } \mathbf{v}(t, x) = \mathbf{b}(t, x), \\ \quad \sup_{u \in P(x)} [\min(D_{\uparrow} \mathbf{v}(t, x)(1, -f(x, u)), D_{\downarrow} \mathbf{b}(t, x)(1, -f(x, u))) \\ \quad - L(x, u) - M(x, u)\mathbf{v}(t, x)] \leq 0 \end{array} \right. \quad (36)$$

Remark — Condition (36)iv) is automatically satisfied whenever

$$\sup_{u \in P(x)} (D_{\downarrow} \mathbf{b}(t, x)(1, -f(x, u)) - L(x, u) - M(x, u)\mathbf{v}(t, x)) \leq 0$$

We refer to the papers [43, 44, 46, Frankowska] for other differential properties of the value function obtained using the tools of the epigraphical approach, and in particular, by duality, the links with viscosity solutions and lower semicontinuous bilateral solutions introduced also in [32, Barron & Jensen] by PDE's methods.

7 Vector Optimal Control Problems

When $Y := \mathbf{R}^n$ is supplied with the natural order relation \leq associated with the positive orthant \mathbf{R}_+^n , we can associate with two maps $\vec{\mathbf{c}} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R}_+^n$ and $\vec{\mathbf{b}} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R}_+^n$ such that

$$\forall (t, x) \in \mathbf{R}_+ \times X, \quad 0 \leq \vec{\mathbf{b}}(t, x) \leq \vec{\mathbf{c}}(t, x)$$

the set-valued maps Φ and Ψ defined by

$$\left\{ \begin{array}{l} i) \quad \Phi(t, x) := \vec{\mathbf{c}}(t, x) + \mathbf{R}_+^n \\ ii) \quad \Psi(t, x) := \vec{\mathbf{b}}(t, x) + \mathbf{R}_+^n \end{array} \right.$$

In the following formulas, the supremums and the infimums are taken component by component. We set

$$\left\{ \begin{array}{l} J_{\vec{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) \\ := e^{\int_0^t M(x(s), u(s)) ds} \vec{\mathbf{c}}(T - t, x(t)) + \int_0^t e^{\int_0^\tau M(x(s), u(s)) ds} L(x(\tau), u(\tau)) d\tau \end{array} \right.$$

and then,

$$K_{\vec{\mathbf{b}}}(t, x; (x(\cdot), u(\cdot))) := \sup_{s \in [0, t]} J_{\vec{\mathbf{b}}}(s, x; (x(\cdot), u(\cdot)))$$

We next integrate this cumulated cost together with the former cost $J_{\vec{\mathbf{c}}}(t, x; (x(s), u(s)))$ by introducing the new cost function

$$L_{\vec{\mathbf{b}}}^{\vec{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) := \max(K_{\vec{\mathbf{b}}}(t, x; (x(\cdot), u(\cdot))), J_{\vec{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x))$$

and the subset

$$V_{\bar{\mathbf{b}}}(\bar{\mathbf{c}})(T, x) := \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} L_{\bar{\mathbf{b}}}^{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x)$$

We shall deduce from Theorem 5.1 the following consequence :

Theorem 7.1 *Let us assume that for any $(x, u) \in X \times \mathcal{U}$ and for any $y \in \mathbf{R}_+^n$, whenever $y_i = 0$, then $(M(x, u)y)_i = 0$.*

Then

$$\forall T \geq 0, x \in X, \quad V_{\Psi}(\Phi)(T, x) = V_{\bar{\mathbf{b}}}(\bar{\mathbf{c}})(T, x) + \mathbf{R}_+^n$$

Proof — It is enough to check that

$$L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) = L_{\bar{\mathbf{b}}}^{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+^n$$

By assumption, the cone \mathbf{R}_+^n is forward and backward invariant under the differential equation $y'(t) = M(x(t), u(t))y(t)$, so that

$$\forall y \in \mathbf{R}_+^n, \quad e^{\pm \int_0^t M(x(\tau), u(\tau)) d\tau} y \in \mathbf{R}_+^n$$

Therefore, since $\bar{\mathbf{c}}(t, x) \in \mathbf{R}_+^n$, we infer that

$$J_{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) = J_{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+^n$$

Next, we observe that if $a_i \in \mathbf{R}^n$,

$$\bigcap_{i \in I} (a_i + \mathbf{R}_+^n) = \sup_{i \in I} a_i + \mathbf{R}_+^n$$

and thus, that

$$K_{\Psi}(t; (x(\cdot), u(\cdot)))(T, x) = L_{\bar{\mathbf{b}}}^{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+^n$$

and that

$$L_{\Psi}^{\Phi}(t; (x(\cdot), u(\cdot)))(T, x) = L_{\bar{\mathbf{b}}}^{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+^n$$

Finally, we note that

$$\begin{cases} V_{\Psi}(\Phi)(T, x) \\ = \bigcup_{(x(\cdot), u(\cdot)) \in \mathcal{C}(x)} \bigcup_{t \in [0, T]} \left[L_{\bar{\mathbf{b}}}^{\bar{\mathbf{c}}}(t; (x(\cdot), u(\cdot)))(T, x) + \mathbf{R}_+^n \right] \\ = V_{\bar{\mathbf{b}}}(\bar{\mathbf{c}})(T, x) + \mathbf{R}_+^n \end{cases}$$

□

Recall that for a closed subset $A \subset \mathbf{R}^n$ satisfying $A = A + \mathbf{R}_+^n$, the interior of A is equal to

$$\text{Int}(A) = A + \overset{\circ}{\mathbf{R}_+^n}$$

and thus, that the boundary of A is equal to the set of (weak) Pareto optima of A : Indeed, $y \in \partial A$ if and only if for any $z \in A$, there exists at least $i \in \{1, \dots, n\}$ such that $y_i \leq z_i$. We say that $z \gg y$ if for any $i \in \{1, \dots, n\}$, $z_i > y_i$.

Hence, we deduce the following consequence of Theorem 5.3:

Theorem 7.2 *Let us consider $y_T \gg \vec{\mathbf{b}}(T, x)$ be a Pareto minimum of the set $J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}(T, x)$. Consider any solution $(x(\cdot), u(\cdot)) \in \mathcal{C}(x)$ starting from $x \in \text{Dom}(V_{\mathbf{b}}(\mathbf{c}))$ satisfying*

$$y_T \geq J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}(t, x; (x(\cdot), u(\cdot))) \quad (37)$$

until the first time t^ when, for at least one component $i = 1, \dots, n$,*

$$y_{T_i} \leq K_{\vec{\mathbf{b}}_i}(t^*, x; (x(\cdot), u(\cdot)))$$

Then y_T actually remains a Pareto minimum of the sets $J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}((t; (x(\cdot), u(\cdot))))(T, x)$ whenever $t \in [0, t^]$.*

References

- [1] AUBIN J.-P. (1981) *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, Advances in Mathematics, Supplementary studies, Ed. Nachbin L., 160-232
- [2] AUBIN J.-P. (1991) **Viability Theory** Birkhäuser, Boston, Basel
- [3] AUBIN J.-P. (1997) **Dynamic Economic Theory: A Viability Approach**, Springer-Verlag Birkhäuser, Boston, Basel
- [4] AUBIN J.-P. (1999) **Mutational and morphological analysis: tools for shape regulation and morphogenesis**, Birkhäuser
- [5] AUBIN J.-P. (2001) **Impulse Differential Inclusions and Hybrid Systems: A Viability Approach**, Lecture Notes, Université Paris-Dauphine
- [6] AUBIN J.-P. (2000) *Optimal Impulse Control Problems and Quasi-Variational Inequalities Thirty Years Later: a Viability Approach*, in **Contrôle optimal et EDP: Innovations et Applications**, IOS Press
- [7] AUBIN J.-P. (2000) *Boundary-Value Problems for Systems of First-Order Partial Differential Inclusions*, NoDEA, 7, 67-90
- [8] AUBIN J.-P. (2000) *Lyapunov Functions for Impulse and Hybrid Control Systems*, Proceedings of the CDC 2000 Conference
- [9] AUBIN J.-P. (2001) *The Caliber of Impulse and Hybrid Control Systems*, Proceedings of "Evolution 2000", Birkhäuser
- [10] AUBIN J.-P. (2001) *The Substratum of Impulse and Hybrid Control Systems*, in **Hybrid Systems: Computation and Control**, 105-118, Di Benedetto & Sangiovanni-Vincentelli Eds, Proceedings of the HSCC 2001 Conference, LNCS 2034, Springer-Verlag
- [11] AUBIN J.-P. (2002) *Dynamic Core of Fuzzy Dynamical Cooperative Games*, Annals of Dynamic Games, Ninth International Symposium on Dynamical Games and Applications, Adelaide, 2000
- [12] AUBIN J.-P. (2001) *Viability Kernels and Capture Basins of Sets under Differential Inclusions*, SIAM J. Control, 40, 853-881
- [13] AUBIN J.-P., BICCHI A. & PANCANTI S. (In preparation) *Detectability of Evolutions by Tubes*,

- [14] AUBIN J.-P., BONNEUIL N. & MAURIN F. (200) *Non-linear Structured Population Dynamics with Co-Variables*, Math. Popul. Studiesm, 91, 1-31
- [15] AUBIN J.-P. & CATTE F. (submitted) *Fixed-Point and Algebraic Properties of Viability Kernels and Capture Basins of Sets*,
- [16] AUBIN J.-P. & DA PRATO G. (1990) *Solutions contingentes de l'équation de la variété centrale*, Comptes-Rendus de l'Académie des Sciences, Paris, 311, 295-300
- [17] AUBIN J.-P. & DA PRATO G. (1992) *Contingent Solutions to the Center Manifold Equation*, Annales de l'Institut Henri-Poincaré, Analyse Non Linéaire, 9, 13-28
- [18] AUBIN J.-P. & FRANKOWSKA H. (1990) **Set-Valued Analysis**, Birkhäuser, Boston, Basel
- [19] AUBIN J.-P. & FRANKOWSKA H. (1990) *Inclusions aux dérivées partielles gouvernant des contrôles de rétroaction*, Comptes-Rendus de l'Académie des Sciences, Paris, 311, 851-856
- [20] AUBIN J.-P. & FRANKOWSKA H. (1991) *Systèmes hyperboliques d'inclusions aux dérivées partielles*, Comptes-Rendus de l'Académie des Sciences, Paris, 312, 271-276
- [21] AUBIN J.-P. & FRANKOWSKA H. (1992) *Hyperbolic systems of partial differential inclusions*, Annali Scuola Normale di Pisa, 18, 541-562
- [22] AUBIN J.-P. & FRANKOWSKA H. (1995) *Partial differential inclusions governing feedback controls*, J. Convex Analysis, 2, 19-40
- [23] AUBIN J.-P. & FRANKOWSKA H. (1996) *The Viability Kernel Algorithm for Computing Value Functions of Infinite Horizon Optimal Control Problems*, J.Math. Anal. Appl., 201, 555-576
- [24] AUBIN J.-P. & FRANKOWSKA H. (1997) *Set-valued Solutions to the Cauchy Problem for Hyperbolic Systems of Partial Differential Inclusions*, NoDEA, 4, 149-168
- [25] AUBIN J.-P. & HADDAD G. (2001) *Cadenced runs of impulse and hybrid control systems*, International Journal Robust and Nonlinear Control
- [26] AUBIN J.-P. & HADDAD G. (2001) *Path-Dependent Impulse and Hybrid Systems*, in **Hybrid Systems: Computation and Control**, 119-132, Di Benedetto & Sangiovanni-Vincentelli Eds, Proceedings of the HSCC 2001 Conference, LNCS 2034, Springer-Verlag
- [27] AUBIN J.-P. & HADDAD G. (2001) *Detectability under impulse differential inclusions*, Proceedings of the ECC 2001 Conference
- [28] AUBIN J.-P. & HADDAD G. (submitted) *Detectability through Measurements under Impulse Differential Inclusions*,
- [29] AUBIN J.-P., LYGEROS J., QUINCAMPOIX. M., SASTRY S. & SEUBE N. (2001) *Impulse Differential Inclusions: A Viability Approach to Hybrid Systems*, IEEE Transactions on Automatic Control
- [30] AUBIN J.-P., PUJAL D. & SAINT-PIERRE P. (2001) *Dynamic Management of Portfolios with Transaction Costs under Contingent Uncertainty*, preprint
- [31] BARDI M. & CAPUZZO DOLCETTA I. (1998) **Optimal control and viscosity solutions to Hamilton-Jacobi-Bellman equations**, Birkhäuser

- [32] BARRON E.N. & JENSEN R. (1990) *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations, 15, 1713-1742
- [33] BARRON E.N. & JENSEN R. (1991) *Optimal control and semicontinuous viscosity solutions*, Proceedings of the AMS, 113, 393-402
- [34] BENSOUSSAN A. & MENALDI (1997) *Hybrid Control and Dynamic Programming*, Dynamics of Continuous, Discrete and Impulse Systems, 3, 395-442
- [35] BERNHARD P. (1995) *Expected Values, feared values and partial information optimal control*, in **New trends in dynamic games and applications**, Annals of the ISDG, 3, 3-24, Birkhauser
- [36] BERNHARD P. (1997) *On the performance index of feared value control*, ISDG symposium, Sils-Maria, Switzerland
- [37] BERNHARD P. (2000) *Max-Plus algebra and mathematical fear in dynamic optimization*, J. Set-Valued Analysis,
- [38] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (1999) *Set-valued numerical methods for optimal control and differential games*, In **Stochastic and differential games. Theory and numerical methods**, Annals of the International Society of Dynamical Games, 177-247, Birkhäuser
- [39] CARJA O. & URSESCU C. (1993) *The characteristics method for a first-order partial differential equation*, An. Stiin. Univ. Al. I. Cuza, Sect. 1 a Mat., 39, 367-396
- [40] CARJA O. & URSESCU C. (1994) *Viscosity solutions and partial differential inequalities*, in **Evolution equations, control theory and biomathematics**, 39-44, Dekker
- [41] FRANKOWSKA H. (1987) *L'équation d'Hamilton-Jacobi contingente*, Comptes-Rendus de l'Académie des Sciences, PARIS, Série 1, 304, 295-298
- [42] FRANKOWSKA H. (1987) *Optimal trajectories associated to a solution of contingent Hamilton-Jacobi equations*, IEEE, 26th, CDC Conference, Los Angeles, December 9-11
- [43] FRANKOWSKA H. (1989) *Optimal trajectories associated to a solution of contingent Hamilton-Jacobi equations*, Applied Mathematics and Optimization, 19, 291-311
- [44] FRANKOWSKA H. (1989) *Hamilton-Jacobi equation: viscosity solutions and generalized gradients*, J. of Math. Analysis and Appl. 141, 21-26
- [45] FRANKOWSKA H. (1991) *Lower semicontinuous solutions to Hamilton-Jacobi-Bellman equations*, Proceedings of 30th CDC Conference, IEEE, Brighton, December 11-13
- [46] FRANKOWSKA H. (1993) *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation*, SIAM J. on Control and Optimization,
- [47] FRANKOWSKA H. (to appear) **Control of Nonlinear Systems and Differential Inclusions**, Birkhäuser
- [48] FRANKOWSKA H., PLASKACZ S. & RZEZUCHOWSKI T. (1995) *Measurable Viability Theorems and Hamilton-Jacobi-Bellman Equation*, Journal of Differential Equations, 116, 265-305
- [49] FRANKOWSKA H., PLASKACZ S. & RZEZUCHOWSKI T. (1995) *Théorèmes de Viabilité Mesurable et l'équation d'Hamilton-Jacobi-Bellman*, Comptes-Rendus de l'Académie des Sciences, Série 1, Paris, 131-134

- [50] HADDAD G. (1981) *Monotone trajectories of differential inclusions with memory*, Isr. J. Math., 39, 83-100
- [51] HADDAD G. (1981) *Monotone viable trajectories for functional differential inclusions*, J. Diff. Eq., 42, 1-24
- [52] HADDAD G. (1981) *Topological properties of the set of solutions for functional differential differential inclusions*, Nonlinear Anal. Theory, Meth. Appl., 5, 1349-1366
- [53] MATVEEV A.S., SAVKIN A.V. (2000) **Qualitative Theory of Hybrid Dynamical Systems**, Birkhäuser
- [54] MATVEEV A.S., SAVKIN A.V. (2001) **Hybrid dynamical systems: Controller and sensor switching problems**, Birkhäuser
- [55] PUJAL D. (2000) *Valuation et gestion dynamiques de portefeuilles*, Thèse de l'Université de Paris-Dauphine
- [56] PUJAL D. & SAINT-PIERRE P. (2001) *L'algorithme du bassin de capture appliqué pour évaluer des options européennes, américaines ou exotiques*, preprint
- [57] QUINCAMPOIX M. (1990) *Frontières de domaines d'invariance et de viabilité pour des inclusions différentielles avec contraintes*, Comptes-Rendus de l'Académie des Sciences, Paris, 311, 411-416
- [58] QUINCAMPOIX M. (1992) *Enveloppes d'invariance pour des inclusions différentielles Lipschitziennes : applications aux problèmes de cibles*, Comptes-Rendus de l'Académie des Sciences, Paris, 314, 343-347
- [59] QUINCAMPOIX M. & VELIOV V. (1998) *Viability with a target: theory and applications*, in **Applications of mathematics in engineering**, 47-54, Heron Press
- [60] ROCKAFELLAR R.T. & WETS R. (1997) **Variational Analysis**, Springer-Verlag
- [61] SAINT-PIERRE P. (1994) *Approximation of the viability kernel*, Applied Mathematics & Optimisation, 29, 187-209
- [62] SAINT-PIERRE P. (to appear) *Approximation of capture basins for hybrid systems*, Proceedings of the HSCC 2001 Conference
- [63] SHAFT (van der) A. & SCHUMACHER H. (1999) **An introduction to hybrid dynamical systems**, Springer-Verlag, Lecture Notes in Control, 251
- [64] SHI SHUZHONG (1986) *Théorèmes de viabilité pour les inclusions aux dérivées partielles*, C. R. Acad. Sc. Paris, 303, Série I, 11-14
- [65] SHI SHUZHONG (1988) *Nagumo type condition for partial differential inclusions*, Nonlinear Analysis, T.M.A., 12, 951-967
- [66] SHI SHUZHONG (1988) *Optimal control of strongly monotone variational inequalities*, SIAM J. Control & Optimization, 26, 274-290
- [67] SHI SHUZHONG (1989) *Viability theorems for a class of differential-operator inclusions*, J. of Diff. Eq., 79, 232-257
- [68] SVENSSON L.E.O. & WOODFORD M. (2000) *Indicator variables for optimal policy*, NBRR WP 7953