Boundary-Value Problems for Systems of Hamilton-Jacobi-Bellman Inclusions with Constraints

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Abstract

We study existence and uniqueness of solutions to boundary-value problems for systems of Hamilton-Jacobi-Bellman first-order partial differential equations subjected to viability constraints in the class of closed set-valued maps. We deduce these results from the fact that the graph of the solution is the viable-capture basin of the graph of the boundary-conditions under an auxiliary system, and then, from their properties. For HJB equations, we recover the existence and uniqueness of lower semicontinuous solutions.

Value Functions In Optimal Control

It is well known that valuation functions of optimal control problems are solutions to Hamilton-Jacobi partial differential equation of the form

$$0 = -\frac{\partial}{\partial t} v(t,x) + \inf_{u \in P(x, v(t,x))}$$

$$\left(\frac{\partial}{\partial x} v(t,x) f(x, v(t,x), u) - g(x, v(t,x), u) \right)$$

with adequate boundary conditions.

The Emergence of Two Inequalities

Observe nevertheless that in this equation, the infimum hides two inequalities:

1. **there exist** $u \in P(x, v(t,x))$ **such that**

$$-\frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(x, v(t,x), u)$$
$$-g(x, v(t,x), u) \leq 0$$

2. **for all** $u \in P(x, v(t,x))$**,**

$$\frac{\partial v(t,x)}{\partial t} - \frac{\partial v(t,x)}{\partial x} f(x, v(t,x), u)$$
$$+g(x, v(t,x), u) \leq 0$$

However, several problems of control theory (reachable maps, etc.) lead to the study of controlled systems **of first-order partial differential equations (or systems of first-order** partial differential inclusions)**: Let** $P : X \times Y \rightsquigarrow \mathcal{U}$ **be a set-valued map associating with any pair** (x, y) **a feasible set** $P(x, y)$ **of controls and** f **and** g **be single-valued maps from** $X \times Y \times \mathcal{U}$ **to finite dimensional vector spaces** X **and** Y **respectively.**

The problem is to find...

... a set-valued map $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ satisfying

1. there exists $u \in P(x, V(t,x))$ such that

$$0 \in -\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x, V(t,x), u) - g(x, V(t,x), u) \tag{1}$$

2. for all $u \in P(x, v(t,x))$,

$$0 \in \frac{\partial V(t,x)}{\partial t} - \frac{\partial V(t,x)}{\partial x} f(x, V(t,x), u) + g(x, V(t,x), u) \tag{2}$$

The Need for Derivatives of Set-Valued Maps

Even in the absence of controls, (as in the case of Burger's equations, for instance) it is well known that such solutions may have shocks — i.e., can be set-valued — and even, when they happen to be single-valued, are not necessarily differentiable in the usual sense.

Derivatives of set-valued maps were introduced in 1981 and used for giving a meaning to generalized lower semicontinuous solutions to HJB equations and prove their existence and uniqueness.

Contingent Cones

We first introduce the contingent cone **to a subset K at a point $x \in K$, introduced in the early thirties independently by Bouligand and Severi, for adapting to any subset the concept of tangent space to manifolds: A direction $v \in X$ belongs to $T_K(x)$ if there exist sequences $h_n > 0$ and $v_n \in X$ converging to 0 and v respectively such that**

$$\forall\, n \geq 0, \;\; x + h_n v_n \;\in\; K$$

This means that the contingent cone is the Painlevé-Kuratowski upper limit of the subsets $\frac{K-x}{h}$ when h converges to 0.

The graph of the set-valued map $DV(t,x,y)$ from $\mathbf{R} \times X$ to Y is equal to the contingent cone to the graph of V at (t,x,y):

$$T_{\mathrm{Graph}(V)}(t,x,y) \; = \; \mathrm{Graph}(DV(t,x,y))$$

This is how Fermat defined in 1637 the derivative of a function as the slope of the tangent to its graph.

Consequently, to say that $g \in Y$ belongs to the contingent derivative $DV(t,x,y)(\pm 1,$ of V at (t,x,y) in the direction $(\pm 1, f) \in \mathbf{R} \times X$ means that

$$\liminf_{h \to 0+, \, f' \to f} \left\| \frac{V(t \pm h, x + hf') - y}{h} - g \right\| \; = \; 0$$

Examples

When $u : \mathbf{R} \times X \mapsto Y$ is single-valued, we set $Du(t,x) := Du(t,x,u(t,x))$.

$Du(t,x)(\pm 1, f) = \pm\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} \cdot f$ **whenever u is differentiable at (t,x). When u is Lipschitz on a neighborhood of (t,x) and when the dimension of X is finite, the domain of $Du(t,x)$ is not empty. Furthermore, the Rademacher Theorem implies that $x \rightsquigarrow Du(t,x)$ is almost everywhere single-valued. Equality $Du(t,x)(-1,-f) = -Du(t,x)(1,f)$ is not true in general.**

Boundary Conditions and Constraints

In order to obtain uniqueness, we have to impose boundary conditions. We introduce constraints **bearing both on the state and on the solution, described by introducing two set-valued maps $\Phi : \mathbf{R}_+ \times X \rightsquigarrow Y$ and $\Psi : \mathbf{R}_+ \times X \rightsquigarrow Y$ such that $\Phi \subset \Psi$. The first one encompassing initial and/or boundary-value conditions, or other conditions as we shall see, the second one viability constraints both on the state variables x — that must remain in the domain of Ψ — and on the solution $V(t,x)$.**

The Boundary-Value Problem

We can prove that there exists a unique "solution" $(t, x) \rightsquigarrow V(t, x)$ to this general problem (1,2) satisfying the conditions :

$\forall\ (t, x) \in \mathbf{R}_+ \times X$,

$$\Phi(t,x)\ \subset\ V(t,x)\ \subset\ \Psi(t,x)$$

in the class of closed set-valued maps (i.e., set-valued maps with closed graph), that depends continuously of the data Φ (in the "graphical sense", mapping graphical limits to graphical limits).

Capture Basins

Let $K \subset X$ be a constrained set and $C \subset K$ be a target. Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ associate with any $x \in X$, $\mathcal{S}(x)$ the set of evolutions $x(\cdot)$ governed by

$$\left\{ \begin{array}{lll} (i) & x'(t) & = & f(x(t),u(t)) \\ (ii) & u(t) & \in & P(x(t)) \end{array} \right. \tag{3}$$

starting from x. The subset $\mathrm{Capt}(K,C) := \mathrm{Capt}_{\mathcal{S}}(K,C)$ of initial states $x_0 \in K$ such that at least one evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the capture basin of C viable in K **under $\mathcal{S}$.**

Characterization of the Solution

Define the set-valued map $V_{(\Psi,\Phi)}$ by

$$\mathrm{Graph}(V_{(\Psi,\Phi)}) := \mathrm{Capt}(\mathrm{Graph}(\Psi), \mathrm{Graph}(\Phi))$$

under the auxiliary differential inclusion

$$\begin{cases} i) & \tau'(t) \ = \ -1 \\ ii) & x'(t) \ = \ f(x(t), y(t), u(t)) \\ iii) & y'(t) \ = \ g(x(t), y(t), u(t)) \\ iv) & u(t) \ \in \ P(x(t), y(t)) \end{cases}$$

Under adequate assumptions, $V_{(\Psi,\Phi)}$ is the unique solution among the solutions with closed graph to this constrained boundary value problem.

The characterizations and properties of capture basins can be applied to study solutions to boundary-value problems to HJB systems of PDI.

Marchaud and Lipschitz Maps

F **is a** Marchaud map **if**

$$\left\{ \begin{array}{ll} i) & \textbf{the graph and the domain of } F \\ & \textbf{are nonempty and closed} \\ ii) & \textbf{the values } F(x) \textbf{ of } F \textbf{ are convex} \\ iii) & \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \leq c(\|x\| + 1) \end{array} \right.$$

A set-valued map F **is said to be** λ-Lipschitz **if there exists a constant** $\lambda > 0$ **such that**

$$\forall\, x, y \in X, \;\; F(x) \;\subset\; F(y) + \lambda \|x - y\| B$$

Theorem 1 (Existence)

Assume that the system is Marchaud. Then the set-valued map $V_{(\Psi,\Phi)}$ **is the largest closed set-valued map** $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ **satisfying** $\forall\, (t,x) \in \mathbf{R}_+ \times X$,

$$\Phi(t,x) \ \subset \ V(t,x) \ \subset \ \Psi(t,x) \tag{4}$$

and

$$\left\{ \begin{array}{l} \forall\, y \in V(t,x) \backslash \Phi(t,x), \ \exists\, u \in P(x,y) \\ \text{such that} \\ 0 \ \in \ DV(-1, f(x,y,u)) - g(x,y,u) \end{array} \right. \tag{5}$$

The Regulation Map

Let us set

$$\mathbf{R}(t,x,y) := \{u \in P(x,y) \text{ such that}$$
$$0 \in DV_{(\Psi,\Phi)}(t,x,y)(-1, f(x,y,u))$$
$$-g(x,y,u)\}$$

Knowing $V_{(\Psi,\Phi)}$, any solution satisfying the constraints and reaching the objective in finite time is obtained in the following way:

The Regulation Law

Starting from $y_0 \in V_{(\Psi,\Phi)}(T, x_0) \backslash \Phi(T, x_0)$**, any evolution** $(x(\cdot), y(\cdot), u(\cdot))$ **satisfying**

$$y(t) \ \in \ V_{(\Psi,\Phi)}(T - t, x(t))$$

until the first time $t^\star \in]0, T]$ **when**

$$y(t^\star) \ \in \ \Phi(T - t^\star, x(t^\star))$$

is a solution to the control system: for almost all $t \in [0, T]$**,**

$$\left\{ \begin{array}{lll} i) & x'(t) & = \ f(x(t), y(t), u(t)) \\ ii) & y'(t) & = \ g(x(t), y(t), u(t)) \\ iii) & u(t) & \in \ \mathbf{R}(T - t, x(t), y(t)) \end{array} \right. \quad (6)$$

Theorem 2: Uniqueness

Assume furthermore that the system is Lipschitz. Then $V_{(\Psi,\Phi)}$ **is the unique closed set-valued map** $V : \mathbf{R}_+ \times X \rightsquigarrow Y$ **satisfying (4), (5) and**

$$\begin{cases} \forall\, y \in V(t,x) \cap \Psi^{\circ}(t,x),\ \forall\ \ u \in P(x,y) \\ 0\ \in\ g(x,y,u) + DV(1,-f(x,y,u)) \\ \& \\ \forall\, y \in V(t,x) \cap \Psi^{\partial}(t,x),\ \forall\ \ u \in P(x,y), \\ 0\ \in\ g(x,y,u)+ \\ DV(1,-f(x,y,u)) \cup D\Psi^{c}(+1,-f(x,y,u)) \end{cases} \tag{7}$$

where $\mathrm{Graph}(\Psi^{\circ}) := \mathrm{Int}(\mathrm{Graph}(\Psi))$**,** $\mathrm{Graph}(\Psi^{\partial}) := \partial\mathrm{Graph}(\Psi)$ **and** $\mathrm{Graph}(\Psi^{c}) := X\backslash\mathrm{Graph}(\Psi)$**.**

Explicit Formulas

We can provide an explicit formula when $f(x,u)$ and $P(x)$ do not depend on the variable y and when

$$g(x,y,u) \ := \ -M(x,u)y - L(x,u)$$

is affine with respect to y where

1. **M is a continuous matrix-valued function**

$$M : (x,u) \in X \times \mathcal{U} \mapsto M(x,u) \in \mathcal{L}(X,Y)$$

2. **L is a continuous "vector-Lagrangian"**

$$L : (x,u) \in X \times \mathcal{U} \mapsto L(x,u) \in Y$$

The Control System

Let us denote by $\mathcal{C} : x \in X \rightsquigarrow \mathcal{C}(x) \subset \mathcal{C}(0,\infty;X) \times L^1(0,\infty;\mathcal{U})$ **the set-valued map associating with** $x \in X$ **the set** $\mathcal{C}(x)$ **of the pairs** $(x(\cdot),u(\cdot))$ **solutions to the control system**

$$\left\{ \begin{array}{lll} i) & x'(t) & = & f(x(t),u(t)) \\ ii) & u(t) & \in & P(x(t)) \end{array} \right.$$

starting at x **at** $t = 0$**.**

Explicit Formula

Set $(\Psi(t,x) := Y)$ **and**

$$\begin{cases} J_\Phi(t;(x(\cdot),u(\cdot)))(T,x) \\ := \; e^{\int_0^t M(x(s),u(s))ds}\Phi(T-t,x(t)) \\ + \displaystyle\int_0^t e^{\int_0^\tau M(x(s),u(s))ds} L(x(\tau),u(\tau))d\tau \end{cases}$$

If $\forall\, t>0,\ \Phi(t,x)=\emptyset,\ M=0$**, then**

$$\begin{cases} J_\Phi(t;(x(\cdot),u(\cdot)))(T,x) \\ := \; \Phi(0,x(T)) + \displaystyle\int_0^t L(x(\tau),u(\tau))d\tau \end{cases}$$

(Set-Valued Bolza Problem)

Explicit Formula 1

The set-valued solution V defined by

$$V_{(Y,\Phi)}(T,x) \ := \ \bigcup_{(x(\cdot),u(\cdot))\in\mathcal{C}(x)} \ \bigcup_{t\in[0,T]}$$

$$J_{\Phi}(t;(x(\cdot),u(\cdot)))(T,x)$$

is the unique "solution" to the Hamilton-Jacobi partial differential inclusion (1,2) satisfying the initial condition $V(0,x) = \Phi(0,x)$ and

$$\forall\, t \geq 0,\ x \in X,\ \ \Phi(t,x) \ \subset \ V(t,x)$$

Explicit Formula 2

Set $\Phi_{\emptyset}(0,x) := \Phi(0,x)$ **and** $\forall\ t > 0,\ \Phi_{\emptyset}(t,x) := \emptyset$**. The set-valued map defined by**

$$V_{(\Phi,\Phi_{\emptyset})}(T,x) := \bigcup_{(x(\cdot),u(\cdot))\in\mathcal{C}(x)} \bigcap_{t\in[0,T]}$$

$$J_{\Phi}(t;(x(\cdot),u(\cdot)))(T,x)$$

is the solution satisfying $V(t,x) \subset \Phi(t,x)$**.**

We find as many formulas as pairs (Ψ,Φ) **of set-valued maps.**

Vector Optimization Problems

When $Y := \mathbf{R}^n$ is supplied with the natural order relation $\leq$ associated with the positive orthant $\mathbf{R}^n_+$, we can associate with two maps $\vec{\mathbf{c}} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R}^n_+$ and $\vec{\mathbf{b}} : \mathbf{R}_+ \times X \rightsquigarrow \mathbf{R}^n_+$ such that

$$\forall \, (t,x) \in \mathbf{R}_+ \times X, \;\; 0 \; \leq \; \vec{\mathbf{b}}(t,x) \; \leq \; \vec{\mathbf{c}}(t,x)$$

the set-valued maps Φ and Ψ defined by

$$\begin{cases} i) & \Phi(t,x) \; := \; \vec{\mathbf{c}}(t,x) + \mathbf{R}^n_+ \\ ii) & \Psi(t,x) \; := \; \vec{\mathbf{b}}(t,x) + \mathbf{R}^n_+ \end{cases}$$

Vector Functionals

In the following formulas, the supremums and the infimums are taken component by component. We set

$$\begin{cases} J_{\vec{\mathbf{c}}}(t;(x(\cdot),u(\cdot)))(T,x) \\ := \ e^{\int_0^t M(x(s),u(s))ds}\vec{\mathbf{c}}(T-t,x(t)) \\ + \int_0^t e^{\int_0^\tau M(x(s),u(s))ds} L(x(\tau),u(\tau))d\tau \end{cases}$$

and then,

$$K_{\vec{\mathbf{b}}}(t,x;(x(\cdot),u(\cdot))) \ := \ \sup_{s\in[0,t]} J_{\vec{\mathbf{b}}}(s,x;(x(\cdot),u(\cdot)))$$

We next integrate this cumulated cost together with the former cost $J_{\vec{\mathbf{c}}}(t,x;(x(s),u(s))$ **by introducing the new cost function**

$$L_{(\vec{\mathbf{b}},\vec{\mathbf{c}})}(t;(x(\cdot),u(\cdot)))(T,x) \ :=$$

$$\max(K_{\vec{\mathbf{b}}}(t,x;(x(\cdot),u(\cdot))),J_{\vec{\mathbf{c}}}(t;(x(\cdot),u(\cdot)))(T,x)$$

and the subset $V_{(\vec{\mathbf{b}},\vec{\mathbf{c}})}(T,x) :=$

$$\bigcup_{(x(\cdot),u(\cdot))\in\mathcal{C}(x)} \bigcup_{t\in[0,T]} L_{(\vec{\mathbf{b}},\vec{\mathbf{c}})}(t;(x(\cdot),u(\cdot)))(T,x)$$

If for any $(x,u)\in X\times\mathcal{U}$ **and for any** $y\in\mathbf{R}^n_+$**, whenever** $y_i=0$**, then** $(M(x,u)y)_i = 0$**, then** $\forall\, T\geq 0, x\in X,$

$$V_{(\Psi,\Phi)}(T,x) \;=\; V_{(\vec{\mathbf{b}},\vec{\mathbf{c}})}(T,x)+\mathbf{R}^n_+$$

Pareto Minima

Recall that for a closed subset $A \subset \mathbf{R}^n$ satisfying $A = A + \mathbf{R}^n_+$, the interior of A is equal to

$$\mathrm{Int}(A) \;=\; A + \overset{\circ}{\mathbf{R}}{}^n_+$$

and thus, that the boundary of A is equal to the set of (weak) Pareto optima of A : $y \in \partial A$ if and only if for any $z \in A$, there exists at least $i \in \{1, \ldots, n\}$ such that $y_i \leq z_i$.

We say that $z >> y$ if for any $i \in \{1, \ldots, n\}$, $z_i > y_i$.

Let us consider $y_T >> \vec{\mathbf{b}}(T,x))$ be a Pareto minimum of the set $J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}(T,x)$. Consider any solution $(x(\cdot),u(\cdot)) \in \mathcal{C}(x)$ starting from $x \in \mathrm{Dom}(V_{\mathbf{b}}(\mathbf{c}))$ satisfying

$$y_T \ \geq \ J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}(t,x;(x(\cdot),u(\cdot))) \tag{8}$$

until the first time t^* when, for at least one component $i = 1,\ldots,n$,

$$y_{T_i} \ \leq \ K_{\vec{\mathbf{b}}_i}(t^*,x;(x(\cdot),u(\cdot))))$$

Then y_T actually remains a Pareto minimum of the sets $J_{V_{\vec{\mathbf{b}}}(\vec{\mathbf{c}})}((t;(x(\cdot),u(\cdot)))(T,x)$ whenever $t \in [0,t^*]$.