

Systems & Control: Foundations & Applications

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Mutational and Morphological Analysis

Tools for Shape Evolution
and Morphogenesis

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qui est sage comme une image
dont l'évolution garde tout son mystère
et sa régulation échappe à de nombreux contrôles.

Preface

Shapes and images are basically sets, not even smooth.

Therefore, their analysis, their processing, their evolution, their optimization and/or their regulation and control require naturally an intrinsic analysis, for which the tools of *set-valued and morphological analysis* have been designed.

These tools are meant to enrich the panoply of those set out by differential geometry and functional analysis, by relaxing whenever possible the unnecessary restrictions induced in depicting sets by functions. These diverse and ingenious ways of associating with subsets several functions — such as their characteristic functions, indicators, gauges, support functions, distance functions and signed (or oriented) distance functions to quote a few — or regarding a subset as a level-set of a function, benefit from the long familiarity and intimacy of mathematicians with functions and maps and their analysis.

I suggest instead to follow almost systematically the opposite itinerary: Study first the properties of sets, of *power spaces* — by which families of subsets are known — of set-valued or set-defined maps, and second, regard maps through their graphs, in the lost tradition going back to Pierre de Fermat and René Descartes! I hope to convince the reader that by doing so, we can go far enough to solve some problems by this direct approach, bypassing classical analysis.

Paradoxically, as History usually behaves, the investigation of power spaces began at the same time as set theory at the dawn of our almost finished century. Painlevé defined in 1902 the concept of upper and lower limits of sets (bearing the name of Kuratowski limits after its mention in the celebrated Kuratowski book) and Pompeiu, a student of Painlevé, introduced in 1907 the distance on the family of nonempty compact subsets of a metric space (bearing the name of Hausdorff distance after its mention in the renowned Hausdorff book).

However, this set-valued approach, born with the century, was neglected during almost half a century. It has been resurrected under pressure of various problems arising in a manifold of fields after World War II.

Concerning the problems addressed in this book, set-valued analysis, including the contributions of Minkowski and Steiner among others, underlies the original approach proposed under the name of *mathematical morphology*¹ pioneered by Georges Matheron for image processing. Another original approach was called *shape optimization* by Jean C ea and Jean-Paul Zol esio, who introduced the basic concept of shape derivatives of set-defined maps as well as the concept of velocities of tubes in [?, Zol esio] as early as 1976. *Graphical derivatives of set-valued maps* were introduced in the beginning of the 80's, and *mutations* were proposed ten years later by the author for defining *velocities of evolving sets*. One purpose of mutations was to deal either with some aspects of morphogenesis appearing in several biological problems, with “visual control” or target problems in control problems and differential games and other “viability issues.”

The last two decades have witnessed the parallel development of these various approaches making image and shape analysis a scientific Tower of Babel, in which however is buried a deep unity of basic mathematical concepts and tools. Directional derivatives of nonsmooth functions of all kinds and their associated generalized gradients, graphical derivatives and codifferentials of set-valued maps, shape derivatives and mutations, are all linked together. Their appearance met different needs, according to the numerous roles played by differentiability.

Mutational and morphological analysis offers a structure that embraces and inte-

¹The adjective *morphological* was coined by Johann von Goethe (1749-1832), who also worked in biology. He proposed in an essay [?, Goethe] published in 1790 a bold unifying hypothesis stating that most botanical forms evolved from an archetypal plant (Urpflanze). Goethe heralded the founder of morphogenesis, Thompson d’Arcy (see [?, Thompson d’Arcy]). He was quite bitter that his scientific contributions were not taken seriously (except by Isidore Geoffroy Saint-Hilaire) : He complained “The public ... is expected that a person who has distinguished himself on one field ... will not leave his field, much less venture into one entirely unrelated. Should an individual attempt this, no gratitude is shown to him; Indeed, even if he does his task well, he is given no special praise”. This complaint is still of actuality.

grates the underlying frameworks of these competing — yet complementary — concepts. It reveals that their apparent differences reflect actually the differences between the sources of motivations. Divulging the first results on mutational and morphological analysis that I found promising motivated the writing of this monograph and generated the hope that the tools therein will be useful for studying various aspects of “morphogenesis.” Much remains to be done. I expect that many problems opened during this course will soon be closed, if indeed any one problem can ever be said to have been definitively solved.

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Hélène Frankowska did play her usual hidden and efficient role during daily discussions, with, this time, the “collaboration” of Pierre-Cyril, who provided happy perturbations without succeeding to erase the files of this book on the computer.

Alexander Kurzhanski contaminated me with the Eastern European virus of funnel equations during many summers at IIASA (International Institute for Applied Systems Analysis). After a very long period of incubation, the disease finally erupted. This event happened during the spring of 1992 at the Scuola Normale di Pisa, thanks to Giuseppe Da Prato. It was triggered when I became acquainted with the beautiful series of fundamental papers by Michel Delfour and Jean-Paul Zolésio on one hand, and when I received by chance the thesis of Michel Schmitt who introduced me to another way of looking at set-valued analysis motivated by mathematical morphology, pioneered by George Matheron and Jean Serra. Françoise Chatelin, who hired me as scientific adviser at the Laboratoire Central de Recherche of THOMSON-CSF, helped me to develop this collaboration in an industrial environment. Sharing with Terry Rockafellar and Roger Wets for twenty years common perspectives on set-valued analysis and many other topics added a strong influence in this book. With the benefit of information from these various fields, I was then able easily to relate the results underlying them and to let emerge what I shall present under the label of *mutational and morphological analysis*. Thanks also to Marc Quincampoix, Patrick Saint-Pierre and Vladimir Veliov for joining us in this new adventure.

I am also grateful to Jacques Demongeot and James Murray for their keen interest and fruitful discussions which, I hope, only begin, to Simon Bouisset who motivated in part this study with postural dynamics in physiology and to Yves Burnod for helping me to discover biological morphogenesis in neurobiology. I hope that these motivations will provide a new life to the search for mathematical tools allowing us to deal with biological morphogenesis.

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Chris Byrnes is the one who, with the active and friendly complicity of Edwin Beschler, Vice-President of Birkhäuser, took risks in attracting Hélène Frankowska and me to publish *Set-Valued Analysis* in this wonderful and useful series dedicated to *Systems and Control: Foundations and Applications*. This monograph, as *Viability Theory*, a prolongation of *Set-Valued Analysis*, could not find a better shelter than this series and this publisher. Thanks to both of them for trusting us that these approaches will contribute to the development of Systems and Control theories.

Introduction

Various fields demand diverse motivating applications of what can be called loosely “morphological analysis”:

Viability Problems are at the direct origin of mutational and morphological analysis. Viability theory deals with evolution equations, the solutions $t \mapsto x(t)$ of which are viable in “tubes” $t \rightsquigarrow K(t)$ (time-dependent constrained subsets):

$$\forall t \geq 0, x(t) \in K(t)$$

These tubes were first assumed to be given, and the characterization of this viability property led to the introduction of a class of derivatives of set-valued maps, called “graphical derivatives.” Yet, biological and economical considerations led us not only to characterize and regulate viable evolutions of contingent (i.e., multivalued) dynamical systems to given tubes $K(t)$, but also to “endogenize” the evolution of the tube $K(t)$: The issue quickly arose of having the evolution of tubes governed itself by a kind of differential equation, called “morphological equation.” It enabled us to study the necessary and sufficient conditions linking the dynamics governing the evolution of the state $x(t)$ and the dynamics ruling the evolution of the closed subset $K(t)$ in such a way that the above viability condition is satisfied. Unfortunately, the concepts of graphical derivatives of tubes are no longer adequate for defining the velocities of a tube, needed to design “morphological equations” governing the evolution of subsets. This required the construction of another “differential calculus” in the metric space of nonempty compact sets in order to study these morphological equations: The concept of *mutation of a tube* leads us to define tube velocities in an adequate manner.

Beyond the standard viability problems, Anne Gorre investigated problems of a similar nature, requiring for instance the subsets $K(t)$, $L(t)$, $M(t)$ evolving

according to morphological equations to obey at each instant relations such as

$$\forall t \geq 0, \begin{cases} i) & L(t) \cap M(t) \neq \emptyset \text{ (intersectability property)} \\ ii) & K(t) \subset L(t) \cap M(t) \text{ (confinement property)} \end{cases}$$

Image Processing has been approached by Georges Matheron through *mathematical morphology* for designing algorithms of image processing based on Minkowski algebraic operations and set topologies.

Indeed, black-and-white images or shapes are basically subsets (of “pixels”) of two or three dimensional vector spaces \mathbf{R}^n and, in the digital case, “grids” \mathbf{Z}^n . Hence, operations and tools have been designed to be valid in both cases whenever possible, since digital images are the ones which are processed in computers.

Gray-scale and color shapes are maps associating with each pixel gray-scale and three color intensities respectively. By characterizing respectively these maps through their *epigraphs*² and their *graphs* (what is now known under the name of *epigraphical and graphical approach*), gray-scale and color shapes are characterized by subsets of higher dimensional vector spaces. Further, in essence, these maps are not regular, since an image represented by a continuous gray scale function is fuzzy.

Therefore, basically, shapes are subsets, and operations on images — black-and-white, gray or colored — are derived from operations on sets. This is one first reason why set-valued analysis arises in the forefront, with its tools forged since the beginning of this century. Image processing has motivated — and still continues to do so — many problems dealing naturally with subsets and their evolution.

Shape Optimization is concerned with problems of structural mechanics (design and construction of industrial structures) and optimal control of distributed systems. Many such problems can be formulated as the minimization of functionals over a class of subsets under “geometrical” constraints. For implementing the Fermat rule and studying sensitivity analysis, Jean C ea and Jean-Paul Zol esio designed the concept of *shape derivative* of a set-defined function, which is the prototype of the concept of mutation of a map. Instead of using the metric space of nonempty compact subsets³, shape optimization uses the σ -algebras of probability spaces, endowed with the distance associating with any two subsets the measure of their symmetric difference.

²The epigraph of a function is the subset above its graph (see Definition ?? below).

³on which the volume functionals are not continuous, but just upper semicontinuous.

Visual Control, for instance, deals with dynamical problems of states (ranging over vector spaces or power spaces) controlled in the last analysis by shapes or images, i.e., by sets. In such problems, closed loop controls *feed back on images or shapes* for stabilization or viability purposes. Associating vectorial characteristics with these shapes — which allow us to use the tools of control theory — results in information loss that is avoided whenever shapes are just subsets. Taking this direct approach, feedback maps and Lyapunov functions map sets to vectors, and thus, are set-defined maps.

Interval analysis, a domain of *numerical analysis* that studies the evolution of intervals containing the solution of a differential equation, for instance, deals with interval evolution, and so, with set evolution. It thus falls naturally in the realm of mutational and morphological analysis.

Dynamic Economic Theory is also a rich source of motivations. Production technologies for instance are often represented in the famous Arrow-Debreu world as sets, which are actually graphs of input-output maps. Keeping this general description of a production process inherited from the static theory, one can tackle the evolution of production sets as a solution of a mutational equation, the above viability requirement expressing that total consumption must belong to the production set. Coalitions of economic agents, diffusion of technologies, immigration and emigration problems provide other avenues of investigation which could use morphological analysis.

Biological Morphogenesis and various other domains of biology⁴ provide an immense reservoir of problems that could be treated by mutational and morphological analysis, whenever the dynamics governing the evolution of biological shapes are not just dependent on the individual points that belong to the shape, but should also depend upon the shape itself.

In physiology, for instance, one can provide a mathematical metaphor of “postural (dynamical) equilibria” when $K(t)$ describes the convex hull of the feet at time t and $x(t)$ the barycenter of the body, which has to belong to this set at every moment.

Environmental problems and various domains of ecology, dealing with the evolution of populations and their interactions, also open new areas of investigation of set evolution.

⁴I have chosen to postpone the biological applications to a forthcoming monograph.

Front Propagation problems, more directly related to physics, are also set evolution problems. Some of them, e.g. the celebrated “eikonal” equation, can be treated in the framework of mutational equations. A vast literature deals with such questions representing the evolution of sets $K(t) \subset X$ as level sets of functions:

$$K(t) := \{x \in X \mid u(t, x) = 0\}$$

When u is smooth, the evolution of the set is described by the evolution of the function $u(t, \cdot)$ governed by a partial differential equation. Whenever the subsets $K(t)$ are solutions to a mutational equation, no regularity conditions are asked.

This list of motivating applications is far from being exhaustive.

Why did it take so long to resurrect set-valued analysis from the purgatory to which it has been sent ... by Bourbaki among others ?

One of the many reasons may be due to the fact that “power spaces”, spaces of subsets of a basic space, inherit only few of the properties of the basic space. The various families of subsets of a vector space inherit only certain (poor) algebraic structures, such as lattices, dioids, max-plus algebras and the like, allowing however morphological analysts, for instance, to solve many problems of image processing.

Topological properties of the basic space are neither canonically nor easily transferred to the various power spaces, with the exception of the space of nonempty compact sets.

Also, the “pointwise approach” consisting in regarding set-valued maps from X to Y as single-valued maps from X to the power space $\mathcal{P}(Y)$ became prevalent in analysis, to the detriment of the “old” graphical approach that mostly regarded maps as graphs, a view going back to Pierre de Fermat and René Descartes. Hence, functional analysis became for very good reasons the successful standard, above all when differentiability questions arose. The loss of the differentiability paradise was at first glance the punishment inflicted on those exploring the set-valued purgatory, depriving the sinner of the grace of differential and integral calculus. Actually, many techniques, such as representing a subset as a level-set, tend to use sly ways for circumventing this obstacle. I do not recommend the deprivation of these efficient tools of functional analysis, but let me just advocate addition to the tool-box of other instruments designed by set-valued analysis. *Du passé, je ne souhaite surtout pas faire table rase.*

Graphical derivatives⁵ of set-valued maps, while efficient for dealing with viability

⁵introduced in [?, ?, Aubin]. See for instance [?, Aubin & Frankowska] and [?, Rockafellar & Wets] for some historical informations.

issues and adapting inverse function theorems to set-valued maps, did not offer convenient ways for defining the *velocity of tubes* needed for the design of analogues of differential equations governing the evolution of subsets. Some of us were also reluctant to introduce still new concepts and definitions of differentiability to an already rich ménagerie of derivatives, codifferentials and other subdifferentials.

Still, the pressure for defining velocities of tubes remained. They led Jean-Paul Zolésio to describe as early as 1976 the velocities of tubes as vector fields, which becomes now a particular case of the concept of “mutation” of a set-valued map. Much to my relief, it is closely related to the graphical derivative by a formula that reestablishes unity in a field crowded with so many definitions, which makes a “multiverse” the universe inherited from Pierre de Fermat, Isaac Newton and Gottfried Leibniz.

Oddly enough, the concept of “mutations” of set-valued maps was already there, discreetly hidden in the concept of “shape derivatives” of “set-defined maps” introduced by Jean C'éa and Jean-Paul Zolésio: Since set-defined maps go from $\mathcal{P}(X)$ to Y whereas set-valued maps go from X to $\mathcal{P}(Y)$, it happened to be sufficient to “invert” the definition of shape derivatives to obtain “mutations” (kind of derivatives) of set-valued maps.

Allow me to elaborate this simple idea in more detail.

Consider a “set-defined map” V mapping subsets K of a vector space X to vectors $V(K) \in Y$ in a finite dimensional vector space Y . The underlying idea behind the concept of shape derivative was to replace the usual differential quotients $\frac{U(x + hv) - U(x)}{h}$ measuring the variation of a map $U : X \mapsto Y$ on *half-lines* $x + hv$ by differential quotients $\frac{V(\vartheta_\varphi(h, K)) - V(K)}{h}$ where $\varphi : X \mapsto X$ is a Lipschitz map, $\vartheta_\varphi(h, x)$ denotes the value at time h of the solution to the differential equation $z' = \varphi(z)$ starting at $z(0) = x$ at time 0 and $\vartheta_\varphi(h, K) := \{\vartheta_\varphi(h, x)\}_{x \in K}$ the *reachable set* from K at time h of φ . These “curves” $h \mapsto \vartheta_\varphi(h, K)$, associated with Lipschitz maps φ , play the role of directions when one defines directional derivatives of usual functions. Hence, if the limit

$$\overset{\circ}{V}(K)\varphi := \lim_{h \rightarrow 0^+} \frac{V(\vartheta_\varphi(h, K)) - V(K)}{h}$$

exists, it is called the *directional shape derivative of V at K* in the “direction” φ .

In other words, the “half curve” $h \mapsto \vartheta_\varphi(h, K)$ plays the role of half line $h \mapsto x + hu$ in vector spaces. Both can be regarded as *transitions* mapping each element to a neighboring one in a given “direction”, depicted by a vector in the case of vector spaces and by a Lipschitz map in the case of subsets. We can thus interpret the shape

derivative $\overset{\circ}{V}(K)\varphi \in Y$ as a direction such that *the transition $V(K) + h \overset{\circ}{V}(K)\varphi$ of the image $V(K)$ and the image $V(\vartheta_\varphi(h, K))$ of the transition $\vartheta_\varphi(h, K)$ are equivalent in the sense that*

$$\lim_{h \rightarrow 0^+} \frac{d(V(\vartheta_\varphi(h, K)), V(K) + h \overset{\circ}{V}(K)\varphi)}{h} = 0$$

Since the set $\mathcal{K}(X)$ of nonempty compact subsets of X is only a metric space, without linear structure, replacing half-lines by curves to measure variations is indeed a very reasonable strategy. For this special metric space, these “curves” ϑ_φ are examples of “*transitions*” with which one can define the analogue of a differential calculus of set-defined maps. The above reformulation of the concept of a directional shape derivative⁶ *uses only the notions of transitions in the metric spaces $\mathcal{K}(X)$ and the vector space Y .* The only privilege of being a vector space is to have transitions of the form $x + hu$ associated with directions belonging to the same vector space.

Since this strategy worked well for shape maps, it should work as well for set-valued maps $F : X \mapsto \mathcal{K}(Y)$, and indeed, it does for solving certain classes of problems. For that purpose, and in a similar way, we supply the vector space X with transitions $h \mapsto x + hu$ and the metric space $\mathcal{K}(Y)$ (supplied with the Pompeiu-Hausdorff distance⁷) with the transitions $h \mapsto \vartheta_\varphi(h, K)$ associated with Lipschitz maps $\varphi \in \text{Lip}(Y, Y)$, as we did when defining shape derivatives. *We thus compare the transition $\vartheta_\varphi(h, F(x))$ of the image $F(x)$ and the image $F(x + hu)$ under F of the transition $x + hu$. It is then tempting to say that either $\varphi \in \text{Lip}(Y, Y)$ or the associated transition ϑ_φ belongs to the mutation of F at x in the direction u if*

$$\lim_{h \rightarrow 0^+} \frac{d(F(x + hu), \vartheta_\varphi(h, F(x)))}{h} = 0$$

and to denote the subset $\overset{\circ}{F}(x)(u)$ of such transitions φ as the *mutation of F at x in the direction u* . The Lipschitz map $\varphi \in \text{Lip}(X, X)$, plays the role of a “directional derivative of the set-valued map F at x in the direction u .” A new name had to be coined, because many concepts of derivatives of a set-valued map — *graphical derivatives*⁸,

⁶With such a concept, an inverse function theorem allowing us to invert locally a shape map V whenever its shape derivative $\text{Lip}(X, X) \mapsto Y$ is surjective is proved in [?, Doyen], and many applications to shape optimization under constraints are derived in Doyen’s paper.

⁷defined by

$$d(L, M) := \max \left(\sup_{y \in L} \inf_{z \in M} d(y, z), \sup_{z \in M} \inf_{y \in L} d(y, z) \right)$$

⁸according to a term coined by R.T. Rockafellar. See [?, Aubin & Frankowska], [?, Aubin] and [?, Rockafellar & Wets] among other authors for an exposition of their properties.

such as contingent, adjacent and circatangent derivatives, as well as other *pointwise concepts* — have been used extensively. Naturally, several Lipschitz maps may belong to the same mutation of a function at a given point in a given transition, but they are all equivalent in an appropriate sense to be clarified later.

For instance, in the case of tubes $t \rightsquigarrow K(t)$ with nonempty compact values, we define a *velocity of the tube* $K(t)$ as the mutation $\overset{\circ}{K}(t) := \overset{\circ}{K}(x)(1)$ of K at t in the forward direction 1:

$$\lim_{h \rightarrow 0^+} \frac{d(\vartheta_\varphi(h, K(t)), K(t+h))}{h} = 0$$

Whenever f maps compact subsets K to Lipschitz maps $x \mapsto f(K)(x)$, this concept of velocity introduced in [?, Zolésio] allows us to give a meaning to the so-called *morphological equations*

$$\overset{\circ}{K}(t) \ni f(K(t))$$

governing the evolutions of sets $K(t)$, as differential equations $x'(t) = g(x(t))$ govern the evolutions of vectors $x(t)$. This goes a step farther than the evolution of tubes driven by set-independent dynamics, the so-called “funnel equations”, that were extensively studied by Soviet mathematicians in the 89’s, in particular by Filippova, Kurzhanski, Panasyuk, Tolstogonov and their collaborators.

When the tube $t \rightsquigarrow K(t) := \{x \in X \mid u(t, x) = 0\}$ is defined by the level sets of a smooth nondegenerate function u , the Lipschitz map $\varphi(t)$ defined by

$$x \mapsto \varphi(t)(x) := -\frac{u'_t(t, x)}{\|u'_x(t, x)\|} \frac{u'_x(t, x)}{\|u'_x(t, x)\|}$$

belongs to the mutation $\overset{\circ}{K}(t)$ of $K(t)$. Its restriction to $K(t)$

$$\forall x \in K(t), \quad \varphi(t)(x) = -\frac{u'_t(t, x)}{\|u'_x(t, x)\|} \quad (\text{unit normal to } K(t) \text{ at } x)$$

is the *normal velocity* to $K(t)$ of differential geometry, so that the level set approach, when valid, relates naturally to the mutational approach.

Without the detour through shape derivatives of set-defined maps, taking a Lipschitz map as a natural candidate to be a directional derivative, could be felt as counter-intuitive. By the way, we shall enrich this structure by using not only Lipschitz single-valued maps, but also Lipschitz or maximal monotone set-valued maps as candidates to be mutations of a set-valued map! Enlarging the class of transitions increases the chances of a set-valued map F to have mutations.

Once these examples were understood, it was observed that the proofs of the theorems did not involve the explicit properties of the Pompeiu-Hausdorff distance on the space of nonempty compact subsets of a vector space. Actually, using this specific metric space was a burden. It happens that *linearity is not really indispensable for designing a differential calculus*, although it simplifies the definitions of derivatives of maps and their study. It is enough to replace the linear structure of vector spaces by a *mutational structure on a metric space*: It is described by a space of *transitions* $(h, x) \mapsto \vartheta(h, x)$ satisfying a small number of axioms. Therefore, if two metric spaces E and F are mutational spaces and $f : E \mapsto F$ is a map from E to F , we say that a transition $\tau \in \overset{\circ}{f}(x)(\vartheta)$ is a mutation of f at x in the direction of the transition ϑ if *the transition $\tau(h, f(x))$ of the image $f(x)$ and the image $f(\vartheta(h, x))$ under f of the transition $\vartheta(h, x)$ are equivalent* in the sense that

$$\lim_{h \rightarrow 0^+} \frac{d(f(\vartheta(h, x)), \tau(h, f(x)))}{h} = 0$$

This simple structure allows us to adapt in the framework of mutational spaces a large number of the important results of differential calculus and in particular, the analogues of differential equations, which we call “mutational equations.” We can also introduce the concept of *tangent transition* to a given subset and adapt to metric spaces some useful geometrical concepts.

However, in the former Soviet Union and present Bielorussia, Panasyuk did also extend in 1985 the concept of differential equations in metric spaces in an abstract way under the name of “quasidifferential equations” along similar ideas.

Fortunately, one can transfer most of the basic results of differential calculus and differential equations in vector spaces to mutational calculus and mutational equations in any mutational space, including naturally the mutational space of nonempty compact subsets of a vector space as the main example treated in this book. But, in the process, we opened more problems than we were able to close.

This book joins a “club” devoted to the analysis, the optimization, the evolution and the regulation of shapes, images or the “propagation of fronts.” Without duplicating them, I made however an effort to make this book self contained, as much as possible. Finding reasonable compromises is never easy. This book supplements the results of set-valued analysis as they are presented in [?, Aubin & Frankowska] and [?, Rockafellar & Wets] with minimum overlap. It refers to the book of [?, Beer] for an exhaustive study of topologies on power spaces and to [?, Moore] for an introduction to interval

analysis.

It does not address the important issue of geometric measure as presented in [?, ?, Ambrosio], [?, Federer], [?, Morgan] and many other monographs on this topic, nor the issues of nonsmooth curvature measured by second-order tangent sets, stochastic tangent sets and quadrics. The situation of this latter topic is moving too fast to be frozen in a book exposition at this time. This text does not cover either shape optimization as presented in the monographs [?, Sokolowski & Zolesio], [?, ?, Delfour & Zolesio], or the Grenander-Mumford-Shah approach for which I refer to [?, Morel & Solimini]. The intersection with the monographs [?, Matheron], [?, ?, Serra], [?, Giardina & Dougherty], [?, Schmitt & Mattioli], [?, Schmitt & Vincent] and [?, Heijmans] devoted to mathematical morphology is also reduced to a minimum. The level-set approach is presented in [?, Sethian].

Outline of the Book

This book is divided into four parts:

- The first part is devoted to *mutational analysis*. It contains two chapters, the first one concentrating on mutational equations and providing the abstract tools for studying set evolution, the second one addressing other issues of mutational analysis that can be bypassed on a first reading by those only interested in morphological equations.
- The second part deals with *morphological and set-valued analysis*. It contains three chapters. The first one describes several classes of transitions on the space of nonempty compact subsets, while the second presents morphological equations governing the evolution of nonempty compact subsets and the various issues dealing with intersectability and confinement of set evolutions. The third chapter links morphological analysis with some results of *set-valued analysis*, where in particular graphical derivatives of a set-valued map are related to their mutations.
- The third part presents *geometrical morphology* in a first chapter and *algebraic morphology* in a second one. In the first chapter, the concepts of normals and proximal normals for deriving dual formulations of set evolution is presented, while the second one connects *algebraic techniques* characterizing mathematical morphology with the general morphological concepts arising in set evolution.
- The fourth part is an appendix that provides a summary of the statements of the basic theorems concerning differential inclusions needed in the book.

We describe now in more detail a guide for the reader's journey through this book.

1. Mutational Equations

The first chapter deals with mutational spaces and mutations of maps. The need to extend concepts of derivatives in metric spaces is not new and has known

many attempts since T. Ważewski introduced, as early as 1946, the concept of *allongements contingentiels supérieur et inférieur*. But we follow here another track motivated by the evolution and control of tubes, shape analysis and mathematical morphology, as explained in the introduction. Hence, Chapter 1 starts with the definitions of *transitions* on a metric space, the *structure of mutational spaces*, the concepts of *mutations* (i.e., “derivatives”) of maps from one metric space to another and of *tangent transition* to a closed subset by adapting the concept of contingent directions introduced by Georges Bouligand in 1930.

With these tools at hand, this chapter then proceeds to adapt to the case of “mutational equations” (i.e., “differential equations” in metric spaces) many basic theorems regarding differential equations — such as the Cauchy-Lipschitz, Nagumo theorems, the basic concepts of viability kernel and capture basins. We shall also define and characterize Lyapunov functions, and even, using the concept of viability kernel, construct optimal lower semicontinuous exponential Lyapunov functions larger than or equal to a given function.

2. Mutational Analysis

Logic wants this second chapter to be placed here, but it can be skipped as long as other issues than set evolutions are not concerned. It is mainly concerned with the Invariant Manifold Theorem and with the adaptation to the case of metric spaces of inverse function theorems, the Fermat Rule for characterizing the minimizer of a function defined on metric spaces, and other results of nonlinear analysis. It concludes with the presentation of the concept of integral of functions with values in metric spaces introduced in 1949 by Shafik Doss.

3. Morphological Spaces

The third chapter is devoted to the design of mutational structures on *power spaces* — families of subsets of a basic space — and mutations of *power maps*⁹, which map a power space to another one, thus including set-valued and set-defined maps. It deals essentially with the metric space $\mathcal{K}(X)$ of nonempty compact subsets of a finite dimensional vector space X equipped with the Pompeiu-Hausdorff distance.

We present several examples of transitions, starting with “structuring transitions” underlying mathematical morphology: In this case, the transitions are the “curves” $\vartheta_B(h, K) := K + hB$, where B is a convex compact subset regarded as a *structuring element* to which one compares other subsets. We then present

⁹among which we find pertinent concepts of image processing.

the “shape transitions” inherited from shape analysis, which are the flows associated with a Lipschitz single-valued map. Next, we put these examples on the same footing by regarding these two classes of transitions as particular cases of “morphological transitions” defined as flows associated with a Lipschitz set-valued map with convex compact images. We naturally characterize morphological transitions that are equivalent at a given compact subset. This leads us to the concept of a subset *semi-permeable* under a set-valued map, which is a set both invariant and backward viable under this set-valued map. We shall prove that a morphological transition is equivalent to the neutral transition at a compact subset K if and only if K is semi-permeable under this morphological transition. Hence, equilibria of morphological equations are semi-permeable. This property happens to be also shared by the graphs of the reachable maps of time-dependent differential inclusions, by invariant manifolds of systems of differential equations or inclusions, by the epigraphs of value functions in optimal control as it was observed by H el ene Frankowska in a series of papers [?, ?, ?, Frankoska], and by viability kernels, pointing out the hidden but crucial role played by this concept. One could also use maximal monotone maps for defining transitions. This also allows us to enrich the panoply of basic morphological operations (dilations, erosions and their combinations) based on the Minkowski operations on subsets. We thus are able to endow the space $\mathcal{K}(X)$ with various mutational structures, making them what are called *morphological spaces*. Therefore, we can define mutations of power maps, and in particular, of set-valued and set-defined maps. This chapter ends with Christian Hess’ results implying that the Aumann integral of compact-valued maps coincides with the Doss integral for metric-valued maps, regarded as maps taking their values in the space $\mathcal{K}(X)$.

4. Morphological Dynamics

This chapter deals with the principal motivational topic of this book, which is the study of evolution of viable, intersectable and confined tubes. *Tubes* are nicknames for set-valued maps $K : t \in [0, T] \rightsquigarrow E$. Since we know how to define their velocities, thanks to the concept of mutation $\overset{\circ}{K}(t)$ of K at t in the forward direction 1, we can define *morphological equations*

$$\overset{\circ}{K}(t) \ni f(K(t))$$

governing the evolution of a tube $t \rightsquigarrow K(t)$, where the dynamics f associates with any compact subset K the Lipschitz set-valued map $x \rightsquigarrow f(K)(x)$ defining a morphological transition on $\mathcal{K}(X)$.

When $f : X \mapsto X$ is a single valued map or, more generally, when $F : X \rightsquigarrow X$ is a set-valued map, the evolution of tubes has been studied by Russian and Bielorussian mathematicians either as “viability tubes”, or as reachable tubes of a differential inclusion regarded as solutions to “funnel differential equations or inclusions.”

But “interval equations” arising in interval analysis and Steiner morphological equations are examples of set-dependent equations that do not fall within the framework of funnel equations : there are equations of the form

$$\overset{\circ}{K}(t) \ni f(V(K(t)))$$

where the set-defined map V maps subsets $K \subset X$ of the state space X to state vectors and where $f : X \mapsto X$. Visual control problems and other control problems that feed back on shapes or images fall also in this category of equations, which are instances of morphological equations.

For exposing the main results dealing with morphological equations, we naturally start with the concept of primitives of time-dependent — but state independent — transitions, which are solutions to funnel equations. We then extend successively the Cauchy-Lipschitz Theorem and the Nagumo Theorem. In particular, the Morphological Nagumo Theorem allows us to characterize the dynamics on the states and the sets that yield evolutions obeying “geometrical constraints.”

An extension of the Invariant Manifold Theorem to morphological equations is also provided, leading to partial mutational inclusions, the solutions of which are set-defined maps $K \mapsto u(K)$ allowing us to *track* the solutions $t \mapsto K(t)$ of a given morphological equation by the solution $t \mapsto x(t) = u(K(t))$ of a given differential equation. *Such a map $K \mapsto u(K)$ provides a way of associating vector characteristics of sets compatible with the two given dynamics.*

In the same spirit, the Morphological Nagumo Theorem implies that the system

$$\begin{cases} i) & x'(t) = f(x(t), K(t)) \\ ii) & \overset{\circ}{K}(t) \ni g(x(t), K(t)) \end{cases}$$

provides *viable evolutions*, i.e., evolutions $t \mapsto x(t)$ and $t \rightsquigarrow K(t)$ satisfying

$$\forall t \geq 0, \quad x(t) \in K(t)$$

if and only if the dynamics f and g satisfy

$$\forall K \in \mathcal{K}(x), \quad \forall x \in K, \quad f(x, K) \in g(x, K)(x) + T_K(x)$$

where $T_K(x)$ denotes the *Bouligand contingent cone* to K at $x \in K$, which is the set of directions $v \in X$ such that there exist sequences $h_n \rightarrow 0+$ and $v_n \rightarrow v$ such that $x + h_nv_n \in K$ for all $n \geq 0$.

Anne Gorre proved that a system of mutational equations

$$\begin{cases} i) & \mathring{L}(t) \ni g(L(t), M(t)) \\ ii) & \mathring{M}(t) \ni h(L(t), M(t)) \end{cases}$$

governs *intersectable evolutions*, i.e., evolutions satisfying

$$\forall t \geq 0, L(t) \cap M(t) \neq \emptyset$$

if and only if for every intersecting pair (L, M) of nonempty compact subsets,

$$\exists x \in L \cap M, \text{ such that } (f(L, M)(x, x) - g(L, M)(x, x)) \cap P_M^L(x) \neq \emptyset$$

where $P_M^L(x)$ denotes the *Bouligand paratingent cone* to (L, M) at $x \in L \cap M$, which is the set of directions $v \in X$ such that there exist sequences $h_n \rightarrow 0+$, $x_n \in L$ converging to x and $v_n \rightarrow v$ such that $x_n + h_nv_n \in M$ for all $n \geq 0$. Hence, the two cones introduced almost sixty years ago by Georges Bouligand found profound implications.

In the same way, she characterized systems of mutational equations

$$\begin{cases} i) & \mathring{K}(t) \ni f(K(t), L(t), M(t)) \\ i) & \mathring{L}(t) \ni g(K(t), L(t), M(t)) \\ ii) & \mathring{M}(t) \ni h(K(t), L(t), M(t)) \end{cases}$$

providing *confined evolutions*, i.e., evolutions satisfying

$$\forall t \geq 0, K(t) \subset L(t) \cap M(t)$$

The theory of Lyapunov functions can also be adapted to the case of morphological equations whenever one can compute the mutations of some set-defined functions, such as the marginal function of an optimization problem. We then use set-defined functions as Lyapunov functions for studying asymptotic stability of a set and asymptotic targeting of tubes solutions to morphological equations. We end this chapter with an illustration by problems by means of visual control

5. Set-Valued Analysis

It is time now to relate the notions of *mutations* with the concept of *graphical derivatives* of set-valued maps, and, for that purpose, to recall basic facts of set-valued analysis.

We begin by recalling that not only black and white images are naturally subsets, but also that gray-scale shapes and color shapes are functions and maps that are represented by their epigraphs or hypographs¹⁰ and graphs, which again are subsets. “Morphological operations”, such as “dilations” and “erosions” defined on subsets can then be adapted to gray-scale images when regarded as epigraphs, and give rise to “inf-convolutions” of gray-scale images.

Since Painlevé, we know two ways of defining limits of a sequence of (closed) sets (including the empty set). Either taking the set of limits of sequences of elements of these sets, and this is the *lower limit*, or taking the set of cluster points of such sequences, and this the *upper limit*. In other words, lower limits are “thick limits” and upper limits are “thick cluster points”, so to speak. When dealing with nonempty compact subsets, the limit of a (uniformly bounded) sequence for the Pompeiu-Hausdorff distance coincides with both the upper and the lower Painlevé-Kuratowski limit.

Once equipped with these concepts of Painlevé-Kuratowski limits and their basic properties, we proceed by introducing continuity concepts of set-valued maps and power maps.

By regarding set-valued maps as graphs, it is natural to introduce the concept of *graphical convergence*, where the graph of the upper or lower graphical limit of a sequence of maps is the upper or lower limit of their graphs. Equipped with this graphical convergence, one can always define the upper graphical limit of difference quotients of a set-valued map and obtain in this way the concept of *contingent derivative* of a set-valued map, the graph of which is the contingent cone to the graph. In the same way, for gray-scale images, we can use the epigraphical approach for naturally obtaining the concepts of *epigraphical convergence* of a sequence of extended functions. Equipped with this epigraphical

¹⁰The morphological tradition uses the concept of hypograph, also called sub-graph or shade. Unfortunately, the literature on convex analysis and mathematical programming characterizes functions through their epigraphs — a viewpoint known under the name of *epigraphical approach*, because it allows us to speak only of convex sets and convex functions. Since many mathematical results are presented in this framework, we have chosen epigraphs despite the mathematical morphological tradition.

convergence, one can also define the lower epigraphical limit of difference quotients of a function and obtain in this way the concept of *contingent epiderivative* of a function, the epigraph of which is the contingent cone to the epigraph.

Therefore, we can distinguish the “*graphical derivatives*” from ‘*mutations*.’ The “*graphical derivatives*” are *local*: they are defined at each point of the *graph* of a set-valued map, since their graphs are “tangent cones” of the graph of the set-valued map. Graphical derivatives are set-valued maps. “*Mutations*” are *global*, in the sense that they are defined at every point of the domain of the set-valued map. The mutations are Lipschitz set-valued maps with compact convex images. We then present a formula linking these two radically different concepts. This formula can be used in many applications: We illustrate it by computing the contingent derivatives of level sets of functions, with derivatives of morphological tubes and transport of functions and maps.

After characterizing the morphological primitive, we then compute the mutational epiderivatives of marginal functions — as well as mutational epiderivatives of inf-convolutions — and derive the Mattioli formula which shows that the dilations of lower semicontinuous functions are generalized solutions (called Frankowska’s solutions) to nonlinear Hamilton-Jacobi equations studied independently in [?, Brocket & Maragos] and [?, ?, ?, Alvarez, Guichard, Lions P.-L. & Morel]). These equations, among which one can find the *eikonal equation*, can be regarded as equations involving the infinitesimal generator of a general *Koopman process* defined on set-defined functions.

After introducing so many derivatives, this chapter ends with a comparison of derivatives of distributions and contingent derivatives of set-valued maps. One way to place all these concepts on the same footing is to associate set-valued maps with distributions, allowing them to recover their lost pointwise character.

6. Morphological Geometry

Although the concept of tangent directions and transitions was already used several times, we devote this chapter to more specific geometric considerations, which use in particular the several concepts of normals to subsets, dual to tangential notions.

The basic operation here is the *projector* Π_K on a closed subset K , which associates with a given point the set of elements of K achieving its distance to K . It is an upper semicontinuous maximal monotone operator, single-valued outside a subset called the *skeleton* of the closure of its complement, and locally Lipschitz when it is single-valued on a neighborhood of the set K . We associate with the

projector the geometrical concepts of proximal normals and normals for any kind of subsets. We then compute the derivatives of the distance functions and, when it is differentiable, to the projector onto a convex subset.

Next, we characterize through proximal normals the concepts of *viability*, *invariance* and, above all, of *semi-permeability* under a set-valued map. These are examples of general concepts of *discriminating domains and kernels* of a Hamiltonian introduced by Pierre Cardaliaguet for formulating dual characterizations of these concepts and for providing a geometrical definition of a tube being a “Cardaliaguet solution” to a first-order front propagation problem. This is illustrated by showing that a morphological primitive is the Cardaliaguet solution of the associated Hamiltonian. We then provide dual formulations of solutions to graphical equations and to Frankowska’s (lower semicontinuous) solutions first-order Hamilton-Jacobi equation, whenever the primitive can be described epigraph of lower semicontinuous function (which coincide with the viscosity solution when the solution is continuous). This section singles out the links between these concepts in the framework of morphological primitives.

Projections on a subset K can be regarded as the points of K reached from $x \in X$ with minimal time under the differential inclusion $x'(t) \in B$, where B is the unit ball. Replacing the unit ball by a Lipschitz convex-valued map $\Phi : X \rightsquigarrow X$, the *chronector* on K , which associates with any point the elements of K reached with minimal time under the differential inclusion $x'(t) \in \Phi(x)$, plays the role of the projectors, and thus, is worth much further study.

Applied mathematical morphology deals with geometric and algebraic operations not only of subsets of vector spaces, but subsets of digital grids that can be programmed in computer software. In particular, the gauges of the “structuring elements” will allow us to define distances, and in particular, “homomorphic” distances defined on digital grids and vector spaces. Since structuring elements that are common in the digital and continuous worlds are not smooth (for instance, they are cubes or polytopes), nonsmooth analysis will be solicited: The smooth Euclidian distance, associated with the smooth Euclidian unit ball, has sometimes to be replaced by distances associated with cubes or polyhedra, which are no longer differentiable. Proximal normals, which can be defined for subsets of digital grids, are essential for defining the basic “morphological concept of skeleton of a set.” We thus devote a short section to morphological analysis on grids.

7. Morphological Algebra

This chapter is devoted to algebraic properties of morphological tubes and other algebraic issues in the tradition of mathematical morphology. We single-out the dioid (or “max-plus”) structure of the family of sets, where the first operation is the union (or closed union) and the second operation is the sum (or the closed sum). Luc Doyen, Juliette Mattioli and Laurent Najman studied several classes of power maps associated with a differential inclusion : They proved that morphological dilations and erosions are openings, the accessibility map is a closing, the viability and invariance kernels are openings, as well as the algebraic properties of the various basins of attraction one can define.

We then present the Galois (or residual) transform of maps from a dioid to another and their numerous properties, and show that the Galois transform of the invariance kernel is the accessibility map.

We conclude by leaving the realm of power spaces to adjoin the confines of logic : we shall use the concepts of closing and of Galois transform for defining the implication of “nonconsistent logics” for which there exist elements satisfying both a property and its nonconsistent negation. For instance, the accessibility map associated with a Lipschitz single-valued map can be regarded as temporal logic and the accessibility map associated with a Lipschitz set-valued map a vicarious temporal logic.

8. Differential Inclusions: A Tool-Box

We summarize in the appendix the results on differential inclusions and viability theory that we need. We begin by stating the Filippov Theorem on the existence and localization of one solution to a differential inclusion with Lipschitz set-valued dynamics and define the invariance kernel.

We then proceed with the Viability Theorem for Marchaud differential inclusions (i.e., with convex-valued closed graph dynamics enjoying linear growth).

We conclude by presenting the third class of differential inclusions associated with maximal monotone dynamics.