Stochastic Invariance for Differential Inclusions

Jean-Pierre Aubin, Giuseppe Da Prato & Hélène Frankowska

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1 Introduction

The first aim of this report is to combine two ways for representing uncertainty through stochastic differential inclusions: a "stochastic uncertainty", driven by a Wiener process, and a "contingent uncertainty", driven by a set-valued map [1] as well as to consider stochastic control problems with continuous dynamic and state dependent controls.

The second objective is to extend to stochastic differential inclusions the Invariance Theorem for nonstochastic differential inclusions (see [5, Aubin] for an exhaustive presentation, historical comments and a bibliography) with Lipschitz right-hand side, using the concepts of stochastic tangent sets to given set-valued rnadom variable or stochastic tubes. A characterization of invariance has been given in [20, Bardi & Goatin] in terms of "second-order normal cones", using techniques of viscosity solutions for second-order Hamilton-Jacobi equations.

In this paper, we shall use only the properties of stochastic differential inclusions proved from firts principles.

Let us mention that in [8, Aubin & Da Prato], the Nagumo Viability Theorem was extended to stochastic differential equations with Lipschitz or monotone dynamics. The stochastic viability theorem has been proved in [10, Aubin & Da Prato] not only for continuous single-valued drift and diffusion coefficients, but also to the case of what can be called stochastic differential inclusions. A characterization of viability in terms of viscosity solutions of second-order Hamilton-Jacobi equations has been given in [10, Buckdahn, Peng, Quincampoix & Rainer].

Let us set $X := \mathbf{R}^n$, $Y := \mathbf{R}^m$, and let us consider a closed subset K of X and a stochastic differential inclusion

$$d\xi \;\in\; F(\xi(t))dt + g(\xi(t))dW(t),$$

where F is a set–valued maps with values in X, and g is a map in $\mathcal{L}(Y,X)$ respectively, and W is an Y–valued Wiener process. It describes the contribution of two kinds of randomness: a contingent one, represented by a set–valued drift, and a stochastic one described by a diffusion.

More generally we shall consider the case of control problems of the form

$$\begin{cases} i) & d\xi \;=\; f(\xi(t),u(t))dt + g(\xi(t),u(t))dW(t) \\ ii) & \text{for almost all } (\omega,t)\in\Omega\times[0,1], \;\; u_\omega(t) \;\in\; U(x_\omega(t)), \end{cases}$$

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[1] which can even be a "fuzzy set-valued map" as in [**?**, Aubin & Dordan].

where the constraints bearing on the controls depend upon the states.

These two problems fit the same following framework. If H is a set–valued map from X to $X \times \mathcal{L}(Y,X)$, we introduce stochastic differential inclusions of the form

$$d\xi \in H(\xi, dt \oplus dW(t)).$$

Let us denote by $\mathbf{I}_0(\xi_0,\gamma,\sigma)$ the It process associated with the drift $\gamma \in L^2(\Omega,\mathcal{F}_t,\mathbf{P};X)$ and the diffusion $\sigma \in L^2(\Omega,\mathcal{F}_t,\mathbf{P});\mathcal{L}(Y,X)$ by

$$\mathbf{I}(\xi_0,\gamma,\sigma)(t) \;:=\; \xi_0 + \int_0^t \gamma(s)ds + \int_0^t \sigma(s)dW(s)$$

We define a solution to such a stochastic differential inclusion to be an It process $\mathbf{I}_0(\xi_0,\gamma,\sigma)$ such that for almost all $(\omega,t) \in \Omega \times [0,\infty[$,

$$(\gamma_\omega(t),\sigma_\omega(t)) \in H(\xi_\omega(t)).$$

The time dependent constraints are defined by a stochastic tube $t \mapsto K(t) \subset X$. We denote by $\mathcal{K}(t)$ the subset

$$\mathcal{K}(t) \;:=\; \{u \in L^2(\Omega,\mathcal{F}_t,P) \mid \text{for almost all } \omega \in \Omega,\;\; u_\omega \in K_\omega(t)\}$$

A subset K is said to be invariant under H if from any initial random variable ξ_0 in K_0, all solution $\xi(\cdot)$ to the stochastic differential inclusion are viable in K, in the sense that

$$\forall\, t \in [0,\infty[,\;\; \text{for almost all } \omega \in \Omega,\;\; \xi_\omega(t) \;\in\; K_\omega(t)$$

For characterizing these properties, we use the concept of stochastic contingent set $\mathcal{T}_K(t,x)$ introduced in [7, Aubin & Da Prato]).

For that purpose, we adapt to the stochastic case the concept of contingent set to a tube. Let us consider a $\mathcal{F}_t$-random variable $\xi \in K(t)$.

We define the *stochastic contingent set* $\mathcal{T}_K(t,x)$ to K at (t,x) (with respect to $\mathcal{F}_t$) as the set of pairs (γ,σ) of $\mathcal{F}_t$-random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of $\mathcal{F}_{t+h_n}$-measurable random variables a^n and b^n such that

$$\begin{cases} i) & \mathbf{E}(\|a^n\|^2) \le \rho^2 \\ ii) & \mathbf{E}(\|b^n\|^2) \le \rho^2 \\ iii) & \mathbf{E}(b^n|\mathcal{F}_\tau) \;=\; 0 \end{cases}$$

and satisfying

$$x + h\gamma + \sigma(W(\tau+h) - W(\tau)) + ha^n + \sqrt{h}b^n \in_{\text{a.s.}} K(t+h_n)$$

almost surely.

We shall prove in essence that when H is Lipschitz, invariance[2] holds true if and only if

$$\forall\, t \geq 0,\ \forall\, \xi \in \mathcal{K}(t),\ \ H(\xi)\ \subset\ \mathcal{T}_K(t,\xi)$$

2 Stochastic Contingent Sets

2.1 Definitions

2.1.1 The Deterministic Case

We say that a set-valued map $K : t \rightsquigarrow K(t)$ from $\mathbf{R}_+$ to X is a tube and that a time-dependent function $t \mapsto x(t)$ is viable in a tube K if

$$\forall\, t \geq 0,\ \ x(t)\ \in\ K(t)$$

We recall the definition of the Bouligand-Severi contingent cone $T_K(t,x)$ to a tube K at a point $x \in K(t)$: A direction $v \in X$ belongs to $T_K(t,x)$ if and only if there exist sequences $h_n > 0$ and $v_n \in X$ converging to 0 and v respectively such that

$$\forall\, n \geq 0,\ \ x + h_n v_n\ \in\ K(t + h_n)$$

When $K(t) = K$ is constant, this boils down to the usual contingent cone $T_K(x)$ to K at $x \in K$.

2.1.2 The Stochastic Case

Let us consider a σ–complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, an increasing family of σ–sub–algebras $\mathcal{F}_t \subset \mathcal{F}$ and two finite dimensional vector-spaces $X := \mathbf{R}^n$, $Y := \mathbf{R}^m$.

Let us denote by $\mathbf{I}_\tau(\xi_\tau, \gamma, \sigma)$ the It process associated with the drift $\gamma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; X)$ and the diffusion $\sigma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}); \mathcal{L}(Y, X)$ by

$$\mathbf{I}_\tau(\xi_\tau, \gamma, \sigma)(t)\ :=\ \xi_\tau + \int_\tau^t \gamma(s)ds + \int_\tau^t \sigma(s)dW(s)$$

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[2] whereas when H is Marchaud, viability holds true if and only if

$$\forall\, t \geq 0,\ \forall\, \xi \in \mathcal{K}(t),\ \ H(\xi)\ \cap\ \mathcal{T}_K(t,\xi)\ \neq\ \emptyset$$

The time-dependent constraints are defined by a closed adapted stochastic tube $K : t \in \mathbf{R}_+ \rightsquigarrow K(t) \subset X$. We denote by $\mathcal{K}(t)$ the subset

$$\mathcal{K}(t) \ := \ \{u \in L^2(\Omega, \mathcal{F}_t, P) \mid \text{for almost all } \omega \in \Omega, \ \ u_\omega \in K_\omega(t)\}$$

Definition 2.1 *We shall say that a adapted stochastic process $\xi(\cdot)$ is* viable in the stochastic tube K *if*

$$\forall\, t \geq 0, \ \ x(t) \ \in \ \mathcal{K}(t) \tag{1}$$

i.e., if and only if

$$\forall\, t \geq 0, \ \ \text{for almost all } \omega \in \Omega, \ \ \xi_\omega(t) \ \in \ K_\omega(t)$$

Definition 2.2 (Stochastic Contingent Set) *Let us consider a $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathcal{K}(t)$. We define the* stochastic contingent set $\mathcal{T}_K(t, \xi)$ *to K at $\xi \in K(t)$ as the set of pairs $(\gamma, \sigma) \in \mathbf{R}^n \times \mathcal{L}(Y, X)$ satisfying the following property: For any $\alpha, \rho > 0$, there exist $h \in]0, \alpha[$ and $\mathcal{F}_{t+h}$-measurable random variable a^h and martingale b^h with values in $\mathbf{R}^n$, and $\mathcal{L}(Y, X)$ such that*

$$\left\{ \begin{array}{ll} i) & \mathbf{E}(\|a^h\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^h\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^h|\mathcal{F}_t) \ = \ 0 \end{array} \right. \tag{2}$$

and satisfying

$$\xi + h\gamma + \sigma(W(t+h) - W(t)) + ha^h + \sqrt{h} b^h \in \mathcal{K}(t+h)$$

The stochastic tangent set $\mathcal{S}_K(t, \xi)$ *to K at $\xi \in K(t)$ (with respect to $\mathcal{F}_t$) as the set of pairs (γ, σ) of $\mathcal{F}_t$-random variables satisfying the following property: There exist adapted continuous processes $a(s)$ and $b(s)$ converging to 0 when $s \to t$ such that, for h small enough,*

$$\xi + \int_t^{t+h} (\gamma + a(s))\, ds + \int_t^{t+h} (v + b(s))\, dW(s) \ \in \ \mathcal{K}(t) \tag{3}$$

We see at once that

$$\mathcal{S}_K(t, \xi) \ \subset \ \mathcal{T}_K(t, \xi)$$

Indeed, we set

$$a^h \ := \ \frac{1}{h} \int_t^{t+h} a(s) ds$$

and

$$b^h \ := \ \frac{1}{\sqrt{h}} \int_t^{t+h} b(s) dW(s)$$

and we observe that

$$\mathbf{E}(b^h|\mathcal{F}_t) \;=\; 0$$

that

$$\begin{cases} \mathbf{E}\left(\left\|a^h\right\|^2\right) \;=\; \frac{1}{h^2}\mathbf{E}\left(\left\|\int_t^{t+h} a(s)ds\right\|^2\right) \\ \leq\; \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|(a(s)\|^2)\right)ds \end{cases}$$

converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

In the same way,

$$\begin{cases} \mathbf{E}\left(\left\|b^h\right\|^2\right) \;=\; \frac{1}{h}\mathbf{E}\left(\left\|\int_t^{t+h} b(s)dW(s)\right\|^2\right) \\ =\; \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|b(s)\|^2\right)ds \end{cases}$$

converges also to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

We also observe that, denoting by $\mathbf{E}(K)$ the tube $t \rightsquigarrow \mathbf{E}(K(t))$,

$$\text{if } \;(\gamma,\sigma) \in \mathcal{T}_K(t,x), \;\text{ then }\; \mathbf{E}(\gamma) \in T_{\mathbf{E}(K)}(t,\mathbf{E}(x))$$

where $T_{\mathbf{E}(K)}(\mathbf{E}(x))$ denotes the *contingent cone* to the expectation of K at $\mathbf{E}(x) \in \mathbf{E}(K(t))$, because, by taking the expectation in both sides of formula (3), we infer that

$$\mathbf{E}(x) + h_n\mathbf{E}(\gamma) + h_n\mathbf{E}(a^n) \;\in\; \mathbf{E}(K)(t+h_n)$$

2.2 The Fundamental Estimate

Lemma 2.3 *Let K be a stochastic tube, $\xi(0) \in \mathcal{K}(0)$ a $\mathcal{F}_0$-adapted stochastic process.*

We define an It process $\xi := \mathbf{I}_0(\xi_0,\gamma,\sigma)$

$$\xi(t) \;=\; \xi_0 + \int_0^t \gamma(s)ds + \int_0^t \sigma(s)dW(s)$$

and we choose a $\mathcal{F}_0$- measurable projection $\eta_0 \in \Pi_{K(0)}(\xi(0))$. Assume that there exist $\gamma_0 \in X$ and $\sigma_0 \in X$ such that

$$\begin{cases} i) & \frac{1}{h}\int_0^h \mathbf{E}\left(\|(\gamma(s)-\gamma_0\|^2)\right)ds \;\to\; 0 \\ ii) & \frac{1}{h}\int_0^h \mathbf{E}\left(\|(\sigma(s)-\sigma_0\|^2)\right)ds \;\to\; 0 \end{cases}$$

Then, for any pair of $\mathcal{F}_0$-random variable (γ,σ) in the stochastic contingent set $\mathcal{T}_K(0,\eta 0)$, the following estimate

$$\left\{\begin{array}{l}\displaystyle\liminf_{t_n\to 0}\frac{\mathbf{E}(d^2_{K(t_n)}(\xi(t_n)) - \mathbf{E}(d^2_K(\xi(0))}{t_n}\\ \\ \leq\; 2\mathbf{E}\left(\langle\xi(0)-\eta_0,\gamma_0-\gamma\rangle\right)+\mathbf{E}(\|\sigma_0-\sigma\|^2)\end{array}\right.$$

holds true.

Proof — Let us set $x=\xi(0)$, choose a projection $y:=\eta_0\in\Pi_K(0,x)$ and take (γ,σ) in the stochastic contingent set $\mathcal{T}_K(0,y)$. This means that there exist sequences $t_n>0$ converging to 0 and $\mathcal{F}_{t_n}$-measurable a^n and b^n satisfying the assumptions (2) and

$$\forall\, n\geq 0,\ \text{for almost all}\ \omega\in\Omega,\ y_\omega+\sigma_\omega W_\omega(t_n)+\gamma_\omega t_n+t_n a^n_\omega+\sqrt{t_n}b^n_\omega\in K_\omega(t_n)$$

Therefore

$$\left\{\begin{array}{l} d^2_{K(t_n)}(\xi(t_n))-d^2_K(\xi(0))\;\leq\\ \\ \left\|x+\int_0^{t_n}\gamma(s)ds+\int_0^{t_n}\sigma(s)dW(s)-y-\sigma W(t_n)-\gamma t_n-t_n a^n-\sqrt{t_n}b^n\right\|^2\\ -\;\|x-y\|^2\\ \\ =\left\|(x-y)+\int_0^{t_n}(\gamma(s)-\gamma)ds+\int_0^{t_n}(\sigma(s)-\sigma)dW(s)-t_n a^n-\sqrt{t_n}b^n\right\|^2\\ -\;\|x-y\|^2\\ \\ =:\; I\end{array}\right.$$

The latter term can be split in the following way:

$$\begin{cases} I \;=\; 2 < x-y, \int_0^{t_n}(\sigma(s)-\sigma)dW(s) > & I_1 \\ + \, 2 < x-y, \int_0^{t_n}(\gamma(s)-\gamma)ds > & I_2 \\ + \, \| \int_0^{t_n}(\sigma(s)-\sigma)dW(s))\|^2 & I_3 \\ + \, \| \int_0^{t_n}(\gamma(s)-\gamma)ds\|^2 & I_4 \\ +2\left\langle \int_0^{t_n}(\sigma(s)-\sigma)dW(s), \int_0^{t_n}(\gamma(s)-\gamma)ds \right\rangle & I_5 \\ -2\left\langle x-y+\int_0^{t_n}(\gamma(s)-\gamma)ds + \int_0^{t_n}(\sigma(s)-\sigma)dW(s), t_n a^n \right\rangle & I_6 \\ -2\left\langle x-y+\int_0^{t_n}(\gamma(s)-\gamma)ds + \int_0^{t_n}(\sigma(s)-\sigma)dW(s), \sqrt{t_n} b^n \right\rangle & I_7 \\ + \, \|t_n a^n + \sqrt{t_n} b^n\|^2 & I_8 \end{cases}$$

We take the expectation in both sides of this inequality and estimate each term of the right hand-side. First, we observe that

$$\mathbf{E}\left(\left\langle x-y, \int_0^{t_n}(\sigma(s)-\sigma)dW(s)|\mathcal{F}_0\right\rangle\right) \;=\; 0$$

so that the expectation of the first term I_1 of the right-hand side of the above inequality vanishes.

The second term I_2 is estimated by $2t_n\alpha_n$ where

$$\alpha_n \;:=\; \mathbf{E}\left(\left\langle x-y, \frac{1}{t_n}\int_0^{t_n}(\gamma(s)-\gamma)ds\right\rangle\right)$$

converges to

$$\alpha \;:=\; \mathbf{E}\left(\langle x-y, \gamma_0-\gamma\rangle\right)$$

The third term I_3 is estimated by $t_n\beta_n$ where

$$\begin{cases} \beta_n := \frac{1}{t_n}\mathbf{E}\left(\| \int_0^{t_n}(\sigma(s)-\sigma)dW(s))\|^2\right) \\ = \; \frac{1}{t_n}\int_0^{t_n}\mathbf{E}\left(\|\sigma(s)-\sigma\|^2\right)ds \end{cases}$$

converges to

$$\beta \;:=\; \mathbf{E}(\|\sigma_0-\sigma\|^2)$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$. The fourth term I_4 is easily estimated by $t_n\delta_n$ where

$$\left\{\begin{array}{l} \delta_n \;:=\; \frac{1}{t_n}\mathbf{E}\left(\| \int_0^{t_n}(\gamma(s)-\gamma)ds\|^2\right) \\ \\ \le\; \int_0^{t_n}\mathbf{E}\left(\|\gamma(s)-\gamma\|^2\right) ds \;\le\; ct_n \end{array}\right.$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \le t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

By the Cauchy-Schwarz inequality, the term I_5 is estimated by $2t_n\eta_n$ where

$$\left\{\begin{array}{l} \eta_n \;:=\; \frac{1}{t_n}\mathbf{E}\left(\left\langle \int_0^{t_n}(\sigma(s)-\sigma)dW(s), \int_0^{t_n}(\gamma(s)-\gamma)ds\right\rangle\right) \\ \\ \le\; \frac{1}{t_n}\mathbf{E}(\| \int_0^{t_n}(\sigma(s)-\sigma)dW(s)\|^2)^{1/2}\mathbf{E}(\| \int_0^{t_n}(\gamma(s)-\gamma)ds\|^2)^{1/2} \\ \\ =\; \frac{1}{t_n}\left(\int_0^{t_n}\mathbf{E}(\|\sigma(s)-\sigma\|^2 ds)\right)^{1/2}(\mathbf{E}(\| \int_0^{t_n}(\gamma(s)-\gamma)ds\|^2)^{1/2} \\ \\ \le \sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n}\mathbf{E}(\|\sigma(s)-\sigma\|^2)\right)^{1/2}\left(t_n\int_0^{t_n}(\mathbf{E}(\|\gamma(s)-\gamma\|^2)\right)^{1/2} \\ \\ \le\; ct_n^{\frac{1}{2}} \end{array}\right.$$

We now estimate the three latter terms involving the errors a^n and b^n.

We begin with I_6. First,

$$\mathbf{E}\left(\langle x-y, a^n\rangle\right) \;\le\; \mathbf{E}\left(\|x-y\|^2\right)^{\frac{1}{2}}\left(\mathbf{E}(\|a^n\|^2)\right)^{\frac{1}{2}}$$

which converges to 0 by assumption (2)i).

Now, the Cauchy-Schwarz inequality implies that

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(\gamma(s)-\gamma)ds, a^n\right\rangle\right) \\ \\ \le\; \mathbf{E}\left(\left\|\int_0^{t_n}(\gamma(s)-\gamma)ds\right\|^2\right)^{\frac{1}{2}}\mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

Finally, the stochastic term is estimated in the following way:

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(\sigma(s)-\sigma)dW(s), a^n\right\rangle\right) \\ \\ \le\; \mathbf{E}\left(\left\|\int_0^{t_n}(\sigma(s)-\sigma)dW(s)\right\|^2\right)^{\frac{1}{2}}\mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \\ \\ =\; \left(\int_0^{t_n}\mathbf{E}(\|\sigma(s)-\sigma)\|^2 ds\right)^{\frac{1}{2}}\mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

which obviously converges to 0.

We continue with I_7. We have

$$\mathbf{E}\left\langle x-y, \frac{1}{\sqrt{t_n}}b^n\right\rangle \;=\; 0$$

since $\mathbf{E}(b^n|\mathcal{F}_0)=0$.

The Cauchy-Schwarz inequality implies that

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(\gamma(s)-\gamma)ds, \frac{1}{\sqrt{t_n}}b^n\right\rangle\right) \\ \\ \leq\; \mathbf{E}\left(\left\|\int_0^{t_n}(\gamma(s)-\gamma)ds\right\|^2\right)^{\frac{1}{2}}\mathbf{E}\left(\|\frac{1}{\sqrt{t_n}}b^n\|^2\right)^{\frac{1}{2}} \\ \\ =\; \frac{1}{\sqrt{t_n}}\left(t_n\int_0^{t_n}\mathbf{E}(\|(\gamma(s)-\gamma)\|^2)ds\right)^{\frac{1}{2}}\mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

Finally, the worst term of I_7 is estimated in the following way:

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(\sigma(s)-\sigma)dW(s), \frac{1}{\sqrt{t_n}}b^n\right\rangle\right) \\ \\ \leq\; \frac{1}{\sqrt{t_n}}\sqrt{\mathbf{E}\left(\left\|\int_0^{t_n}(\sigma(s)-\sigma)dW(s)\right\|^2\right)}\sqrt{\mathbf{E}\left(\|b^n\|^2\right)} \\ \\ =\; \left(\frac{1}{t_n}\int_0^{t_n}\mathbf{E}(\|\sigma(s)-\sigma)\|^2ds\right)^{\frac{1}{2}}\mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

which converges to 0 by assumption (2)ii).

It remains to estimate the last term of I_8. There is no difficulty because

$$\frac{1}{t_n}\mathbf{E}\left(\left\|t_n a^n+\sqrt{t_n}b^n\right\|^2\right) \;=\; \mathbf{E}\left(\left\|\sqrt{t_n}a^n+b^n\right\|^2\right)$$

converges to 0.

Putting everything together, we deduce the inequality of the lemma.

3 Stochastic Differential Inclusions

3.1 Definitions

Let us consider a σ–complete probability space $(\Omega,\mathcal{F},\mathbf{P})$, a filtration of σ-sub-algebras $\mathcal{F}_t\subset\mathcal{F}$ and two finite dimensional vector-spaces $X:=\mathbf{R}^d$, $Y:=\mathbf{R}^c$.

Let us denote by $\mathbf{I}_0(\xi_0, \gamma, \sigma)$ the It process associated with the drift $\gamma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; X)$ and the diffusion $\sigma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; \mathcal{L}(Y, X))$ by

$$\mathbf{I}_0(\xi_0, \gamma, \sigma)(t) \ := \ \xi_0 + \int_0^t \gamma(s) ds + \int_0^t \sigma(s) dW(s)$$

Definition 3.1 *We associate with a set-valued map $H : X \mapsto X \times \mathcal{L}(Y, X)$ the stochastic differential inclusion denoted by*

$$d\xi(t) \ \in \ H(\xi(t), dt \oplus dW(t)) \tag{4}$$

We shall say that an It process $\mathbf{I}_0(\xi_0, \gamma, \sigma)$ is a solution to such a stochastic differential inclusion if for almost all $(\omega, t) \in \Omega \times [0, \infty[$,

$$\text{for almost all } (t, \omega) \in [0, T] \times \Omega, \ \ (\gamma_\omega(t), \sigma_\omega(t)) \in H(\xi_\omega(t)). \tag{5}$$

We sball denote by $\mathcal{S}_H(\xi_0)$ the set of It processes $\mathbf{I}_0(\xi_0, \gamma, \sigma)$ solutions to the stochastic differential inclusion $d\xi(t) \in H(\xi(t), dt \oplus dW(t))$.

In other words, an It process $\mathbf{I}(\xi_0, \gamma, \sigma)$ is a solution to the stochastic differential inclusion (5) starting at ξ_0 if and only if there exist a drift $\gamma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; X)$ and a diffusion $\sigma \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}); \mathcal{L}(Y, X)$ such that

$$\text{for almost all } t \geq 0, \ \ \left(\mathbf{I}\left(\xi_0, \gamma, \sigma\right)(t), \gamma(t), \sigma(t)\right) \ \in_{\text{a.s.}} \ \text{Graph}(H)$$

Let us consider the subset

$$\left\{ \begin{array}{l} \mathcal{H}(\xi_0) \ := \ \{(\gamma, \sigma) \in L^2(\Omega, \mathcal{F}_t, \mathbf{P}; X) \times L^2(\Omega, \mathcal{F}_t, \mathbf{P}); \mathcal{L}(Y, X) \\ \text{such that for almost all } t \geq 0, \\ \left(\mathbf{I}\left(\xi_0, \gamma, \sigma\right)(t), \gamma(t), \sigma(t)\right) \ \in_{\text{a.s.}} \ \text{Graph}(H)\} \end{array} \right.$$

Hence, the set of solutions to the stochastic differential inclusion is equal, to

$$\mathcal{S}_H(\xi_0) \ = \ \mathbf{I}\left(\xi_0, \gamma, \sigma\right)_{(\gamma, \sigma) \in \mathcal{H}(\xi_0)}$$

In particular, in the case when $H := f \times g$ describes the dynamics of a stochastic differential equation with Lipschitz or monotone coefficients, we obtain

$$\mathcal{H}(\xi_0) \ = \ (f(\xi(\cdot)), g(\xi(\cdot)))$$

where $\xi(\cdot)$ is the (unique) solution to the stochastic differential equation starting at ξ_0.

Definition 3.2 *We associate with a set-valued map H and a subset $\mathcal{K} \subset L^2(\Omega, \mathcal{F}_s, \mathbf{P})$ its* reachable sets *$\vartheta_H(t, K)$ defined by*

$$\vartheta_H(t, s, \mathcal{K}) \ := \ \{\xi(t) \ \in \ \mathcal{S}_H(\xi_s)\}_{\xi_s \in \mathcal{K}}$$

The set-valued map $t \in \mathbf{R}_+ \mapsto \vartheta_H(t, s, \mathcal{K}) \subset L^2(\Omega, \mathcal{F}_t, \mathbf{P})$ is called the stochastic reachable tube.

Definition 3.3 *We shall say that it is* Lipschitz *if there exists a constant λ such that, for all $x, y \in X$,*

$$H(y) \ \subset \ H(x) + \lambda\|x - y\|(B_X \times B_{\mathcal{L}(Y,X)})$$

4 The Localization Theorem

Proposition 4.1 *Let us consider a sequence $\{A_k\}_k$ of mutually disjoint subset $A_k \in \mathcal{F}_0$ such that*

$$\Omega \ = \ \bigcup_{k=1}^{\infty} A_k$$

Then

$$\sum_{k=1}^{\infty} \mathcal{S}_H\left(\xi_0^k\right)\chi_{A_k} \ \subset \ \mathcal{S}_H\left(\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}\right)$$

Therefore, in the case when $H := f \times g$ is Lipschitz and single-valued, we deduce that

$$\sum_{k=1}^{\infty} \mathcal{S}_{f\times g}\left(\xi_0^k\right)\chi_{A_k} \ = \ \mathcal{S}_{f\times g}\left(\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}\right)$$

Proof — Indeed, consider solutions $\mathbf{I}(\xi_0^k, \gamma^k, \sigma^k) \in \mathcal{S}_H(\xi_0^k)$ to the stochastic differential inclusion. Then we observe that for all $t \geq 0$,

$$\mathbf{I}\left(\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}, \sum_{k=1}^{\infty} \gamma^k \chi_{A_k}, \sum_{k=1}^{\infty} \sigma^k \chi_{A_k}\right)(t) \ := \ \sum_{k=1}^{\infty} \mathbf{I}\left(\xi_0^k, \gamma^k, \sigma^k\right)(t)\chi_{A_k}$$

On the other hand, since for almost all $t \geq 0$,

$$\left(\mathbf{I}\left(\xi_0^k, \gamma^k, \sigma^k\right)(t), \gamma^k(t), \sigma^k(t)\right) \ \in_{\text{a.s.}} \ \text{Graph}(H)$$

we infer that

$$\left(\sum_{k=1}^{\infty} \mathbf{I}\left(\xi_0^k, \gamma^k, \sigma^k\right)(t)\chi_{A_k}, \sum_{k=1}^{\infty} \gamma^k(t)\chi_{A_k}, \sum_{k=1}^{\infty} \sigma^k(t)\chi_{A_k}\right) \ \in_{\text{a.s.}} \ \text{Graph}(H)$$

These two facts imply that

$$\mathbf{I}\left(\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}, \sum_{k=1}^{\infty} \gamma^k \chi_{A_k}, \sum_{k=1}^{\infty} \sigma^k \chi_{A_k}\right)(t) \ \in \text{a.s.} \ \ \mathrm{Graph}(H)$$

and thus, that $\mathbf{I}\left(\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}, \sum_{k=1}^{\infty} \gamma^k \chi_{A_k}, \sum_{k=1}^{\infty} \sigma^k \chi_{A_k}\right)(t)$ is a solution to the stochastic differential inclusion starting at $\sum_{k=1}^{\infty} \xi_0^k \chi_{A_k}$.

4.1 It Norms

We identify the space $\mathbf{I}(\Omega)$ of It processes with the space

$$\mathbf{I}(\Omega) \ := \ L^2(\Omega, \mathcal{F}_0, X) \times L^2(\Omega, \mathcal{F}_\cdot, X) \times L^2(\Omega, \mathcal{F}_\cdot, \mathcal{L}(Y, X))$$

through the map $\mathbf{I}_0$.

Definition 4.2

$$\|I_0(\xi_0, \gamma, \sigma)(t)\|_{\mathbf{I}} \ := \sqrt{\mathbf{E}(\|\xi_0\|^2) + \mathbf{E}\left(\int_0^t \|\gamma\|^2\right) + \mathbf{E}\left(\int_0^t \|\sigma\|_2^2\right)}$$

the It norm *of the It process (which plays the role of the Sobolev norm in the deterministic world).*

We observe the following

Lemma 4.3 *There exists a constant c such that, for every It process, inequality*

$$\mathbf{E}\left(\|\mathbf{I}_0(\xi_0, \gamma, \sigma)(t)\|^2\right) \ \leq \ 3(1+t) \ \|\mathbf{I}_0(\xi_0, \gamma, \sigma)(t)\|_{\mathbf{I}}^2$$

holds true

Proof — We obtain

$$\left\{ \ \mathbf{E}\left(\|\mathbf{I}_0(\xi_0, \gamma, \sigma)(t)\|^2\right) \ \leq \ 3\left(\mathbf{E}(\|\xi_0\|^2) + t\mathbf{E}\left(\int_0^t \|\gamma(s)\|^2\right) + \mathbf{E}\left(\int_0^t \|\sigma(s)\|_2^2\right)\right)\right.$$

because

$$\begin{cases} \mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \ \leq \ t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds \\ \text{and} \\ \mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) \ = \ \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds \end{cases}$$

5 The Da Prato-Frankowska Theorem

5.1 The Theorem

Theorem 5.1 *Let us assume that H is Lipschitz. Let us consider a given It process $\eta := \mathbf{I}_0(\eta_0, \delta, \tau)$. Then, there exist $T > 0$ and a constant $c(T) > 0$ such that, from any initial $\mathcal{F}_0$-random variable ξ_0 starts a solution $\xi := \mathbf{I}_0(\xi_0, \gamma, \sigma)$ to the stochastic differential inclusion $d\xi \in H(\xi, dt \oplus dW(t))$ satisfying the following estimate*

$$\begin{cases} \forall\, t \in [0,T], \;\; \|\mathbf{I}_0(\xi_0, \gamma, \sigma)(t) - \mathbf{I}_0(\eta_0, \delta, \tau)(t)\|_{\mathbf{I}}^2 \\ \\ \leq \; c(T) \left(t\mathbf{E}(\|\xi_0 - \eta_0\|^2) + \int_0^t \mathbf{E}(d((\delta(s), \tau(s)), H(\mathbf{I}_0(\eta_0, \delta, \tau)(s)))^2) ds \right) \end{cases}$$

The proof of this theorem needs the following Lemma:

Lemma 5.2 *Let us consider an integrable function $e(\cdot)$. Then*

$$\int_0^t dt_1 \int_0^{t_1} e(s) \frac{(t_1 - s)^{n-1}}{(n-1)!} ds \; = \; \int_0^t e(s) \frac{(t-s)^n}{n!} ds \tag{6}$$

Proof — We observe that

$$\begin{cases} \int_0^t dt_1 \int_0^{t_1} e(s) \frac{(t_1 - s)^{n-1}}{(n-1)!} ds = \int_0^t e(s) ds \int_s^t \frac{(t_1 - s)^{n-1}}{(n-1)!} dt_1 \\ \\ = \int_0^t e(s) ds \left[\frac{(t_1 - s)^n}{n!} \right]_s^t = \int_0^t e(s) \frac{(t-s)^n}{n!} ds \;\; \square \end{cases}$$

Proof of Theorem 5.1

1. — *Construction of approximate solutions*

By Corollary 8.2.13 of SET-VALUED ANALYSIS, [11, Aubin & Frankowska], we can associate with any stochastic process $\zeta(t)$ and any pair $(\gamma(t), \sigma(t))$ in $X \times \mathcal{L}(Y, X)$ the nonempty set $G(\zeta, \gamma, \sigma)$ of measurable selection $(\gamma_\propto(t), \sigma_\propto(t)) \in \mathcal{H}(\zeta(t))$ such that

$$\|\gamma_\propto(t) - \gamma(t)\|^2 + \|\sigma_\propto(t) - \sigma(t)\|_2^2 \; = \; d((\gamma(t), \sigma(t)), \mathcal{H}(\zeta(t)))^2$$

Starting with the It process $\eta := \mathbf{I}_0(\eta_0, \delta, \tau)$, we set

$$d(s)^2 \; := \; \mathbf{E}(d((\delta(s), \tau(s)), H(\mathbf{I}_0(\eta_0, \delta, \tau)(s)))^2)$$

Next, we take

$$(\gamma_1, \sigma_1) \;\in\; G(\eta_0, \delta, \tau))$$

and define

$$\xi_1 \;=\; \mathbf{I}_0(\xi_0, \gamma_1, \sigma_1)$$

We observe that

$$\|\xi_1(t) - \eta(t)\|_{\mathbf{I}}^2 \;\leq\; \mathbf{E}(\|\xi_0 - \eta_0\|^2) + d(t)^2$$

We then choose inductively for $n \geq 1$

$$(\gamma_{n+1}, \sigma_{n+1}) \;\in\; G(\xi_n, \gamma_n, \sigma_n))$$

and define

$$\xi_{n+1} \;=\; \mathbf{I}_0(\xi_0, \gamma_{n+1}, \sigma_{n+1})$$

We denote by λ the Lipschitz constant of H and infer that

$$\begin{cases} \|\xi_n(t) - \xi_{n-1}(t)\|_{\mathbf{I}}^2 \\ = \mathbf{E}\left(\int_0^t \|(\gamma_n(s) - \gamma_{n-1}(s))\|^2\, ds + \int_0^t \|(\sigma_n(s) - \sigma_n(s))\|^2\, ds\right) \\ = \mathbf{E}\left(\int_0^t d((\gamma_n(s), \sigma_n(s)), H(\xi_{n-1}(s)))^2 ds\right) \\ \leq \lambda^2 \int_0^t \mathbf{E}(\|\xi_{n-1}(s) - \xi_{n-2}(s)\|^2) ds \\ \leq 3\lambda^2(1+t) \int_0^t \|\xi_{n-1}(s) - \xi_{n-2}(s)\|_{\mathbf{I}}^2 ds \end{cases}$$

thanks to Lemma 4.3.

Therefore, setting $\nu := 3\lambda^2(1+t)$ and iterating these inequalities, we obtain

$$\begin{cases} \|\xi_n(t) - \xi_{n-1}(t)\|_{\mathbf{I}}^2 \;\leq \\ \nu^n \int_0^t dt \int_0^{t_1} dt_1 \ldots \int_0^{t_{n-1}} \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds \\ \leq \int_0^t \frac{\nu^n (t-s)^n}{n!} \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds \\ \leq \frac{\nu^n t^n}{n!} \int_0^t \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds \end{cases}$$

thanks to Lemma 5.2. Let $T > 0$ be such that the series$\sqrt{\frac{\nu^n t^n}{n!}}$ converges (This happens at least whenever $2\lambda^2 T(1+T) < 1$) and let us set

$$c(t) \;:=\; \sum_{n=0}^{+\infty} \sqrt{\frac{\nu^n t^n}{n!}}$$

Consequently,

$$\begin{cases} \|\xi_p(t) - \xi_q(t)\|_{\mathbf{I}} \ \leq \ \sum\limits_{n=q}^{p-1} \|\xi_{n+1}(t) - \xi_n(t)\|_{\mathbf{I}} \\ \\ \leq \sum\limits_{n=q}^{p-1} \left(\sqrt{\frac{\nu^n t^n}{n!}}\right) \sqrt{\int_0^t \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds} \end{cases}$$

This implies that the sequence of It processes $\xi_n := \mathbf{I}_0(\xi_0, \gamma_n, \sigma_n)$ is a Cauchy sequence in the complete space $\mathbf{I}(\Omega)$. Therefore, the sequences γ_n and σ_n converge to L^2 function γ and matrix σ, and consequently, for every $t \in [0, T]$, the It processes $\xi_n(t)$ converge to $\xi(t) := \mathbf{I}_0(\xi_0, \gamma, \sigma)(t)$ almost surely and also, that ξ_n converges to ξ in $L^2([0, T] \times \Omega, X)$.

By taking $q = 0$ in the preceding inequalities, we obtain

$$\begin{cases} \|\xi_p(t) - \eta(t))\|_{\mathbf{I}} \ \leq \left(\sum\limits_{n=0}^{p-1} \sqrt{\frac{\nu^n t^n}{n!}}\right) \sqrt{\int_0^t \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds} \\ \leq \ c(t) \sqrt{\int_0^t \|\xi_1(s) - \eta(s)\|_{\mathbf{I}}^2 ds} \end{cases}$$

which implies the estimate we were looking for.

3. — *The limit is a solution*

Since the functions $(\xi_{n-1}, \gamma_n, \sigma_n)$ satisfy

$$\text{for almost all } t \geq 0, \ \ (\xi_{n-1}(t), \gamma_n(t), \sigma_n(t)) \ \in_{\text{a.s.}} \ \text{Graph}(H)$$

and converge to (ξ, γ, σ) in $L^2([0, T] \times \Omega, X)^2 \times L^2([0, T] \times \Omega, \mathcal{L}(X, Y))$, we infer that the limit satisfies

$$\text{for almost all } \ t \geq 0, \ \ (\xi(t), \gamma(t), \sigma(t)) \ \in_{\text{a.s.}} \ \text{Graph}(H)$$

that means that $\xi(t)$ is a solution to the stochastic differential inclusion.

5.2 Existence from Given Initial Drifts and Diffusion

As a consequence, we obtain the useful

Theorem 5.3 *Let us assume that H is Lipschitz. Then, there exist $T > 0$ and a constant $c(T) > 0$ such that such that, for any $\mathcal{F}_0$-random variable ξ_0 and* *any $\mathcal{F}_0$-random variables $(\gamma_0, \sigma_0) \in H(\xi_0)$, there exists a solution a solution $\xi := \mathbf{I}_0(\xi_0, \gamma, \sigma)$ to the stochastic differential inclusion $d\xi \in H(\xi, dt \oplus dW(t))$ starting from ξ_0 and satisfying*

$$\lim_{s\to 0} \frac{1}{s}\int_0^s \mathbf{E}(\|\gamma(s)-\gamma_0\|^2) \;=\; 0 \;\;\&\;\; \lim_{s\to 0} \frac{1}{s}\int_0^s \mathbf{E}(\|\sigma(s)-\sigma_0\|^2) \;=\; 0$$

Proof — Indeed, by associating with $(\gamma_0, \sigma_0) \in H(\xi_0)$ the It process $\eta := \mathbf{I}_0(\xi_0, \sigma_0, \tau_0)$, the Da Prato-Frankowska Theorem 5.1 implies the existence of a solution $\xi := \mathbf{I}_0(\xi_0, \gamma, \sigma)$ to the stochastic differential inclusion such that

$$\|\xi(t) - \eta(t)\|_{\mathbf{I}}^2 \;\le\; c(T)\int_0^t \mathbf{E}(d(\gamma_0, \sigma_0, H(\mathbf{I}_0(\eta_0, \gamma_0, \sigma_0))(s))^2)ds$$

But, H being Lipschitz, inclusion

$$(\gamma_0, \sigma_0) \;\in\; H(\xi_0) \;\subset\; H(\mathbf{I}_0(\xi_0, \gamma_0, \sigma_0)(s)) + \lambda\|\mathbf{I}(\xi_0, \gamma_0, \sigma_0)(s) - \xi_0\|B$$

holds true almost everywhere. Therefore,

$$\left\{\begin{array}{l} \mathbf{E}(d(\gamma_0, \sigma_0, H(\mathbf{I}_0(\eta_0, \gamma_0, \sigma_0))(s))^2)) \;\le\; \lambda^2\mathbf{E}(\|\mathbf{I}(0, \gamma_0, \sigma_0)(s)\|^2) \\ \le\; 2s(1+s)\lambda^2(\mathbf{E}(\|\gamma_0\|^2) + \mathbf{E}(\|\sigma_0\|^2)) \end{array}\right.$$

Consequently, by Theorem 5.1,

$$\left\{\begin{array}{l} \mathbf{E}\left(\int_0^s \|\gamma - \gamma_0\|^2\right) + \mathbf{E}\left(\int_0^s \|\sigma - \sigma_0\|^2\right) \\ \le\; 2c(T)\lambda^2 \int_0^s \tau(1+\tau)(\mathbf{E}(\|\gamma_0\|^2) + \mathbf{E}(\|\sigma_0\|^2))d\tau \\ =\; 2c(T)\lambda^2 t^2(1 + \frac{2}{3}t)(\mathbf{E}(\|\gamma_0\|^2) + \mathbf{E}(\|\sigma_0\|^2)) \end{array}\right.$$

Dividing by $s > 0$ and letting s converge to 0, we infer that both $\frac{1}{s}\int_0^s \mathbf{E}(\|\gamma(s) - \gamma_0\|^2)$ and $\frac{1}{s}\int_0^s \mathbf{E}(\|\sigma(s) - \sigma_0\|^2)$ converge to 0.

6 The Invariance Theorem for Stochastic Differential Inclusions

6.1 Sufficient Condition

Theorem 6.1 *Assume that $H : X \rightsquigarrow X \times \mathcal{L}(Y, X)$ is Lipschitz. Let $\mathcal{K}$ be a set-valued random variable such that, for all $t \ge 0$*

$$\forall\, \xi \in \mathcal{K}, \;\; H(\xi) \;\in\; \mathcal{T}_K(t, \xi) \tag{7}$$

Then the set-valued random variable $\mathcal{K}$ is invariant under H.

Proof — Let us assume that $\mathcal{K}$ satisfies condition (7), ξ_0 belong to $\mathcal{K}$ and let $\xi(\cdot) := \mathbf{I}_0(\xi_0, \gamma, \sigma)$ be any solution to differential inclusion (4) starting at ξ_0 and defined on some interval $[0, T]$. Let us set $g(t) := \mathbf{E}(d^2_{K(t)}(\xi(t)))$. Since the solution to the stochastic differential inclusion can be written for any $h \geq 0$

$$\xi(t+h) \;=\; \xi(t) + \int_t^{t+h} \gamma(s)ds + \int_t^{t+h} \sigma(s)dW(s)$$

we deduce from Lemma 2.3 that for any $\mathcal{F}_t$-measurable selection $\eta(t)$ of $\Pi_K(\xi(t))$ and any $(\delta, \tau) \in \mathcal{T}(t, \eta(t))$, the following inequality

$$\begin{cases} D_\uparrow g(t)(1) \;:=\; \liminf\limits_{h \to 0+} \left(\dfrac{\mathbf{E}(d^2_{K(t+h)}(\xi(t+h)) - \mathbf{E}(d^2_{K(t)}(\xi(t)))}{h} \right) \\ \leq\; 2\mathbf{E}\left(\langle \xi(t) - \eta(t), \gamma(t) - \delta \rangle\right) + \mathbf{E}(\|\sigma(t) - \tau\|^2) \\ 2\sqrt{\mathbf{E}(d^2_K(\xi(t)))} \;\cdot \sqrt{\mathbf{E}(\|\gamma(t) - \delta\|^2)} + \mathbf{E}(\|\sigma(t) - \tau\|^2) \end{cases}$$

holds true.

Since H is assumed to be Lipschitz, we deduce that

$$H(\xi(t)) \;\subset\; H(\eta(t)) + \lambda d_K(\xi(t))(B_X \times B_{\mathcal{L}(Y,X)})$$

We infer that

$$D_\uparrow g(t)(1) \;\leq\; (2\lambda + \lambda^2)\mathbf{E}(d^2_K(\xi(t))) = (2\lambda + \lambda^2)g(t)$$

Then g is a solution to

$$\forall\, t \in [0, T], \;\; D_\uparrow g(t)(1) \;\leq\; (2\lambda + \lambda^2)\lambda g(t) \quad \& \quad g(0) = 0$$

We deduce from Lemma 6.2 below that $g(t) = 0$ for all $t \in [0, T]$, and therefore, that $\xi(t)$ is viable in K on $[0, T]$.

Lemma 6.2 *Let $g : \mathbf{R}_+ \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial lower semicontinuous function satisfying*

$$\forall\, x \in \mathbf{R}_+, \;\; D_\uparrow g(t)(1) \;\leq\; a(t)g(t) + b(t)$$

Then

$$\forall\, t \geq 0, \;\; g(t) \;\leq\; e^{\int_0^t a(\tau)d\tau} g(0) + \int_0^t e^{\int_s^t a(\tau)d\tau} b(s)ds$$

Proof — This is a trivial consequence of the Nagumo Theorem for the system of differential equations

$$\begin{cases} i) & \tau'(t) \; = \; 1 \\ ii) & x'(t) \; = \; a(t)x(t) + b(t) \end{cases}$$

subjected to the viability constraint

$$\forall\, t \geq 0, \;\; (\tau(t), x(t)) \; \in \; \mathcal{E}p(g)$$

The tangential condition is

$$\forall\, \tau \geq 0, \;\; (1, a(\tau)g(\tau) + b(\tau)) \; \in \; T_{\mathcal{E}p(g)}(\tau, g(\tau)) \; = \; \mathcal{E}p(D_{\uparrow}g(\tau))$$

i.e., inequality

$$\forall\, \tau \geq 0, \;\; D_{\uparrow}g(\tau)(1) \; \leq \; a(\tau)g(\tau) + b(\tau)$$

Therefore, starting from $(0, g(0))$, the solution

$$(t, e^{\int_0^t a(\tau)d\tau} g(0) + \int_0^t e^{\int_s^t a(\tau)d\tau} b(s)ds) \; \in \; \mathcal{E}p(g)$$

to this system of differential equations belongs to the epigraph of g. This implies the inequality we were looking for. □

Remark on Sufficient Conditions — By imposing tangent conditions not only on $\mathcal{K}(t)$, but on all random variables, viable or not, we obtain the following

Lemma 6.3 *Assume that for every $\mathcal{F}_t$-random variable ξ, there exists a $\mathcal{F}_t$-measurable projection $\eta \in \Pi_K(\xi)$ such that*

$$H(\xi) \; \in \; \mathcal{T}_K(t, \eta) \tag{8}$$

Then the set-valued random variable K is invariant under H.

Proof — The proof of Lemma 6.3 is also based on Lemma 2.3.

Let us consider a solution $\xi := \mathbf{I}_0(\xi_0, \gamma, \sigma) \in \mathcal{S}_H(\xi_0)$ to the stochastic differential inclusion $d\xi \in H(\xi, dt \oplus dW(t))$. Since the solution to the stochastic differential inclusion can be written for any $h \geq 0$

$$\xi(t+h) \; = \; \xi(t) + \int_t^{t+h} \gamma(s)ds + \int_t^{t+h} \sigma(s)dW(s)$$

we deduce from Lemma 2.3 that for any $\mathcal{F}_t$-measurable selection $\eta(t)$ of $\Pi_K(\xi(t))$ and any $(\delta, \tau) \in \mathcal{T}(t, \eta(t))$, the following inequality

$$\left\{\begin{array}{l}\liminf_{h\to 0+}\left(\mathbf{E}(d^2_{K(t+h)}(\xi(t+h))-\mathbf{E}(d^2_K(\xi(t)))\right)/h\\[2ex] \leq\ 2\mathbf{E}\left(\langle\xi(t)-\eta(t),\gamma(t)-\delta\rangle\right)+\mathbf{E}(\|\sigma(t)-\tau\|^2)\end{array}\right.$$

holds true. Since there exists a selection $\eta(t)$ of $\Pi_K(\xi(t))$ such that we can take $\delta := \gamma(t)$ and $\tau := \sigma(t)$ by assumption, we infer that setting

$$g(t)\ :=\ \mathbf{E}\left(d^2_{K(t)}(\xi(t))\right)$$

the contingent epiderivative $D_\uparrow g(t)(1)$ is non positive.

This implies that $g(t) \leq 0$ for all $t \in [0,T]$, and thus, that $g(t)\ \leq\ g(0)\ =\ 0$. $\square$

6.2 Necessary Conditions

Let K be a set-valued random variable.

Theorem 6.4 *Let us assume that $H : X \rightsquigarrow X \times L(Y,X)$ is Lipschitz. If the stochastic tube K is invariant under H, then*

$$\forall\,\xi\in\mathcal{K},\ \ H(\xi)\ \in\ \mathcal{S}_K(t,\xi)\ \subset\ \mathcal{T}_K(t,\xi) \tag{9}$$

Proof — By Theorem 5.3 (Corollary 2.2 of [20, Da Prato & Frankowska]), for every $(\gamma_0,\sigma_0)\in H(\xi_0)$, there exists an It process $\mathbf{I}_0(\xi_0,\gamma,\sigma)$ to the stochastic differential inclusion (7) starting from ξ_0 and satisfying

$$\lim_{s\to 0}\frac{1}{s}\int_0^s\mathbf{E}(\|\gamma(s)-\gamma_0\|^2)\ =\ 0\ \ \&\ \ \lim_{s\to 0}\frac{1}{s}\int_0^s\mathbf{E}(\|\sigma(s)-\sigma_0\|^2)\ =\ 0$$

Since the stochastic tube $\mathcal{K}$ is invariant under H, this It process is viable in K.

Therefore, setting $\xi(t) := \mathbf{I}_0(\xi_0,\gamma,\sigma)(t)$, we obtain

$$\xi(h)\ =\ \xi_0+\int_0^h\gamma(s)ds+\int_0^h\sigma(s)dW(s) \tag{10}$$

which can be written it in the form

$$\xi(h)\ =\ \xi(0)+h\gamma_0+\sigma_0W(h)+\int_0^h a(s)ds+\int_0^h b(s)dW(s)\ \in_{\text{a.s.}}\ K(h)$$

where

$$\left\{\begin{array}{lcl}a(s)&=&\gamma(s)-\gamma_0\\ b(s)&=&\sigma(s)-\sigma_0\end{array}\right.$$

converge to 0 in the sense that

$$\lim_{s\to 0}\frac{1}{s}\int_0^s \mathbf{E}(a(s)^2) \;=\; 0 \;\;\&\;\; \lim_{s\to 0}\frac{1}{s}\int_0^s \mathbf{E}(b(s)^2) \;=\; 0$$

by Theorem 5.3.

This means that (γ_0,σ_0) belongs to the stochastic tangent set $\mathcal{S}_K(0,\xi_0)$ to the stochastic tube K at $\xi_0\in K_0$. □

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