A Viability Approach to the Inverse Set-Valued Map Theorem

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This contribution, dedicated to Giuseppe da Prato, is far from witnessing the wholehearted recognition and friendship that he deserves.

Abstract

The purpose of this paper is to characterize bu means of viability tools the pseudo-lipschitzianity property of a set-valued map F in a neighborhood of a point of its graph in terms of derivatives of this set-valued map F in a neighborhood of a point of its graph, instead of using the transposes of the derivatives. On the way, we relate these properties to the calmness index of a set-valued map, an extensions of Clarke's calmness of a function, as well as Doyen's Lipschitz kernel of a set-valued map, which is the largest sub-Lipschitz map.

1 Introduction

This paper revisit the Inverse Function Theorem (in finite-dimensional spaces) and its *unexpected* connections with the Viability Theorem and some other concepts of viability theory. These connections between the two theories where discovered by Luc Doyen in [17, Doyen] when he characterized the largest Lipschitz set-valued submap of a given closed map with a given Lipschitz constant, called the Lipschitz kernel of this map.

The concepts of *graphical derivatives* and their use in the extension of the Inverse Function Theorem have been initiated in [1, Aubin] : We recall that the contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n. The contingent derivative of a set-valued map $F : X \rightsquigarrow Y$ is defined through its graph by

$$\text{Graph}(DF(x,y)) \ := \ T_{\text{Graph}(F)}(x,y)$$

Theorem 5.4.3 of [6, Aubin & Frankowska] states that :

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Theorem 1.1 Inverse Set-Valued Map Theorem. *Assume that a set-valued map F satisfies*

$$\begin{cases} (i) & \mathrm{Im}(DF(x_0,y_0)) \;=\; Y \;\; \text{(i.e., } DF(x_0,y_0) \text{ is surjective)} \\ (ii) & (x,y) \rightsquigarrow \mathrm{Graph}(DF(x,y)) \text{ is lower semicontinuous at } (x_0,y_0) \end{cases} \tag{1}$$

Therefore, $\nu_0 := \sup_{\|v\|=1} d(0, DF(x_0,y_0)^{-1}(v))$ *is finite and there exists* $\eta > 0$ *such that, for any* $(x,y) \in \overset{\circ}{B}((x_0,y_0),\eta)$*, and for any* $y_1 \in \overset{\circ}{B}(y, \eta - \|y-y_0\|)$ *there exists a solution* $x_1 \in F^{-1}(y_1)$ *satisfying*

$$\|x_1 - x\| \le \nu_0 \|y_1 - y\|$$

This property of the inverse of the set-valued map is called "pseudo-Lipschitz" property and has a long history. *The purpose of this paper is to characterize this property in terms of derivatives of the F in a neighborhood of a point of its graph* (Theorem 4.3 below). In the recent paper [13, Dontchev, Quincampoix & Zlateva], the authors offered another of this result, still "*inspired by the proof of Theorem 3.2.4 in [3, Aubin] due to Hélène Frankowska*", and provided some applications. They follow the results by A.L. Dontchev, A.S Lewis and R.T. Rockafellar in [14, 15, Dontchev & Rockafellar] and [16, Dontchev, Lewis & Rockafellar].

This is not because this approach helped us to put a final touch to the characterization of "metric regularity" by derivatives of set-valued maps instead of their transposes (called "co-derivatives") that we present it, but rather for displaying even stronger and curious connections between viability theory and set-valued analysis and bring some perspectives.

The Inverse Function Theorem for surjective single-valued maps in Banach spaces is due to Graves in [23, Graves]) and metric regularity (under other names) goes back to [29, Ljusternik].

The first extensions of this theorem to problems nonsmooth maps are due to [11, Clarke], to problems with constraints to [24, Ioffe] and to set-valued maps in finite dimensional spaces using the localization property of Ekeland's Variational Principle due to Gérard Lebourg appeared in [1, 2, Aubin]. It has been adapted to the case of infinite dimensional spaces in [5, Aubin & Frankowska] and in a series of papers by Hélène Frankowska in [19, 20, 21, 22, Frankowska] extending these results in several directions (infinite dimension, higher order derivatives, conical restrictions, etc.).

Characterizations using the transposes of the derivatives, called co-derivatives, were established by A.D. Ioffe in a series of papers [24, 25, 26, 28, Ioffe] initiated in 1979, and later popularized by B.S. Mordukhovich in [30, Mordukhovich] and many other papers, and co-authors. However, A.D. Ioffe provided also a primal approach bypassing the use of coderivatives and duality, as we do here. These results are surveyed and completed in [28, Ioffe], concluding by the "*contention that primal theorems can potentially produce potentially results could be justifed*". Other results and historical notes can be found in the books [6, Aubin & Frankowska] and [33, Rockafellar & Wets].

The first section deals with calmness concept of a function introduced by F. Clarke which we extend to graphical calmness for set-valued maps, and their viability characterizations. Section 2 deals with extensions of the concept of Doyens's Lipschitz kernel of a set-valued map. The third section concludes the paper by proving the characterization of metric regularity and deriving the Inverse Function Theorem.

2 Graphical Calmness Index

Francis Clarke introduced the concept of calm functions which are in some sense pointwise Lipschitz constants (see [7, 12, 8, 10, Clarke]). This is our starting point for investigating Lipschitz properties of set-valued maps in both a graphical and pointwise perspective by introducing the graphical calmness index of a set-valued map. Finally, we characterize pseudo-Lipschitz maps.

Definition 2.1 Graphical Calmness Index. *Let us consider a set-valued map F, $T > 0$, a closed convex cone P. We denote by $P(1)$ the subset of $u \in P$ such that $\|u\| = 1$. We introduce the following functions defined on the graph of F by*

$$\left\{ \begin{array}{lll} (i) & \delta_F(T,u;x,y) \; := \; \sup_{0<t\leq T} \dfrac{d(y, F(x+tu))}{t} \\ \text{and} & \delta_F^P(T;x,y) \; := \; \sup_{u\in P(1)} \delta_F(T,u;(x,y)) \\ (ii) & \delta_F(u;x,y) \; := \; \sup_{t>0} \dfrac{d(y, F(x+tu))}{t} \\ \text{and} & \delta_F^P(x,y) \; := \; \sup_{u\in P(1)} \delta_F(u;(x,y)) \end{array} \right.$$

When the closed convex cone $P = X$ is equal to, we set

$$\delta_F(T;x,y) \; := \; \delta_F^X(T;x,y) \;\&\; \delta_F(x,y) \; := \; \delta_F^X(x,y)$$

The function $\delta_F(T;x,y)$ is called the local pointwise graphical calmness index of F *and the function $\delta_F(T;x,y)$ is the* pointwise graphical calmness index of F.

We observe that

$$\left\{ \begin{array}{ll} (i) & \delta_F^P(u;x,y) \; = \; \sup_{t\geq 0} \delta_F^P(T,u;x,y) =: \delta_F(\infty,u;x,y) \\ (ii) & \delta_F^P(x,y) \; = \; \sup_{t\geq 0} \delta_F^P(T;x,y) =: \delta_F(\infty;x,y) \end{array} \right.$$

This definition is motivated by the following observation :

Lemma 2.2 Characterization of Calmness Indexes. *Let us consider a set-valued map F. The graphical calmness indexes are characterized by*

$$\left\{ \begin{array}{ll} (i) & \delta_F^P(T;x,y) \; = \; \sup_{x_1\in(x+P)\cap B(x,T) \;\&\; x_1\neq x} \dfrac{d(y,F(x_1))}{\|x_1 - x\|} \\ (ii) & \delta_F^P(x,y) \; = \; \sup_{x_1\in(x+P) \;\&\; x_1\neq x} \dfrac{d(y,F(x_1))}{\|x_1 - x\|} \end{array} \right. \tag{2}$$

and, when $P = X$, by

$$\delta_F(T;x,y) \; = \; \sup_{\|x_1-x\|\leq T \;\&\; x_1\neq x} \frac{d(y,F(x_1))}{\|x_1 - x\|} \;\&\; \delta_F(x,y) \; = \; \sup_{x_1\neq x} \frac{d(y,F(x_1))}{\|x_1 - x\|}$$

Proof — Indeed,

$$\frac{d(y, F(x+tu))}{t} = \frac{d(y, F(x_1))}{\|x_1 - x\|}$$

either by associating with any u such that $u \in P(1)$ and any $0 < t \leq T$ elements $x_1 := x + tu$ and $y_1 := y_t \in F(x_1)$ or by associating with x_1 and $y_1 \in F(x_1)$ elements $u := \frac{x_1 - x}{\|x_1 - x\|}$, $t := \|x_1 - x\|$, $y_1 := y\|x_1 - x\| \in F(x(\|x_1 - x\|)) = F(x_1)$. ■

Remark : Lower semicontinuous upper bound of the calmness index — Actually, the graphical calmness indexes enjoy a viability characterization. For that purpose, we introduce the auxiliary system

$$\begin{cases} (i) & x'(t) = u \\ (ii) & y'(t) \in z(t)B \\ (iii) & z'(t) = 0 \end{cases} \tag{3}$$

and the viability kernels (see Definition 4.1.1, p.121 of [3, Aubin])

$$\mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+) = \bigcap_{T \geq 0} \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+)(T)$$

The graphical calmness indexes are related to viability kernels by

$$\begin{cases} (i) & \delta_F(T, u; x, y) \leq \gamma_F(T, u; x, y) := \inf_{(x,y,z) \in \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+)(T)} z \\ (ii) & \delta_F(u; x, y) \leq \gamma_F(u; x, y) := \inf_{(x,y,z) \in \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+)} z \end{cases}$$

Indeed, to say that $(x, y, z) \in \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+)(T)$ means that there exists $v(t) \in B$ such that $t \mapsto \left(x + tu, y(t) := y + z\int_0^t v(\tau)d\tau, z\right)$ is viable in $\mathrm{Graph}(F) \times \mathbb{R}_+$ on the interval $[0, T]$, i.e., such that for all $t \leq T$, $y \in F(x + tu) + ztB$, or, equivalently, such that $d(y, F(x + tu)) \leq zt$. Therefore,

$$\delta_F(T, u; x, y) \leq \inf_{(x,y,z) \in \mathrm{Viab}_{(3)(T)}(\mathrm{Graph}(F) \times \mathbb{R}_+)} z =: \gamma_F(u; x, y)$$

They are lower semicontinuous whenever the graph of F is closed : in this case

$$\begin{cases} (i) & \mathcal{E}p(\gamma_F(u; \cdot)) = \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+) \\ (ii) & \mathcal{E}p(\gamma_F^P(\cdot)) = \bigcap_{u \in P(1)} \mathrm{Viab}_{(3)}(\mathrm{Graph}(F) \times \mathbb{R}_+) \end{cases}$$

Their epigraphs being viability kernels, they can be computed by the Saint-Pierre Viability Kernel Algorithm introduced in [34, Saint-Pierre]. ■

3 Lipschitz Kernel of a Set-Valued Map

We shall derive the following links between the derivative and the calmness index of F :

Lemma 3.1 Calmness Indexes and Contingent Derivatives. *The local pointwise graphical index is linked to the contingent derivative $DF(x, y)$ is the following way*

$$d(0, DF(x, y)(u)) \;\leq\; \inf_{T>0} \delta_F(T, u; x, y)$$

Proof — Indeed, assume that $\delta := \inf_{T>0} \delta_F(T, u; x, y)$ is finite. Then, for any $\varepsilon > 0$, there exists $T(\varepsilon; x, y)$ such that for all $h \leq T(\varepsilon; x, y)$, $d(y, F(x+hu)) \leq h(\delta+\varepsilon)$. This means that there exists $v_h \in B$ such that $y + (\delta + \varepsilon)v_h \in F(x + hu)$, a subsequence $v_{h_n} \in B$ of which converges to some $v \in B$. This implies that $(\delta+\varepsilon)v \in DF(x, y)(u) \subset D^{\star\star}F(x, y)(u)$. Therefore,

$$d(0, DF(x, y)(u)) \;\leq\; \|(\delta + \varepsilon)v\| \;\leq\; \inf_{T>0} \delta_F(T, u; x, y) + \varepsilon$$

from which the desired inequality ensues by letting ε converge to 0. ■

Let us introduce the convexified contingent derivative $D^{\star\star}F(x, y)$ of the set-valued map $F : X \rightsquigarrow Y$ at a point $(x, y) \in \text{Graph}(F)$ of its graph :

Definition 3.2 Convexified Contingent Derivative. *The* convexified contingent derivative $D^{\star\star}F(x, y)$ *of the set-valued map* $F : X \rightsquigarrow Y$ *at a point* $(x, y) \in \text{Graph}(F)$ *of its graph is defined by*

$$\text{Graph}(D^{\star\star}F(x, y)) \;:=\; \overline{co}\left(T_{\text{Graph}(F)}(x, y)\right)$$

The constant

$$\lambda^P(D^{\star\star}F(x, y)) \;:=\; \sup_{u\in P(1)} d(0, D^{\star\star}F(x, y)(u))$$

is called the $\|P\|$-pointwise Lipschitz modulus *of the closed convex process* $D^{\star\star}F(x, y)$*. When* $P = X$*, we set* $\lambda(D^{\star\star}F(x, y)) := \lambda^X(D^{\star\star}F(x, y))$*.*

The graph of $D^{\star\star}F(x, y)$ being a closed convex cone, the convexified contingent derivative is a *closed convex process*, according to the terminology introduced by Rockafellar in [32, Rockafellar] to designate the set-valued analogues of continuous linear operators.

The function $u \mapsto d(0, D^{\star\star}F(x, y)(u))$ is convex, lower semicontinuous (when the dimension of X is finite) and positively homogenous :

$$\begin{cases} \forall\, u \in P, \;\; d(0, D^{\star\star}F(x,y)(u)) \; = \; d\left(0, D^{\star\star}F(x,y)\left(\frac{u}{\|u\|}\right)\right)\|u\| \\ \leq \; \lambda^P(D^{\star\star}F(x,y))\|u\| \end{cases}$$

When the domain $\mathrm{Dom}(DF(x,y))$ of the contingent derivative is the whole space X, this implies that $\mathrm{Dom}(DF^{\star\star}(x,y)) = X$, and thus, that the function $u \mapsto d(0, D^{\star\star}F(x,y)(u))$ is also finite. In this case, we deduce the set-valued analogue of the Banach Closed Graph Theorem due (in the case of arbitrary Banach spaces) to Robinson in [31, Robinson] and Ursescu in [35, Ursescu] :

Theorem 3.3 Robinson-Ursescu Theorem. *If the domain of the closed convex process $D^{\star\star}F(x,y)$ is the whole space X, then it is Lipschitz : $\lambda(D^{\star\star}F(x,y)) < +\infty$.*

Proof — Since the function $u \mapsto d(0, D^{\star\star}F(x,y)(u))$ is finite, the whole space X is the union of the sections

$$S_n \; := \; \{u \mid \; d(0, D^{\star\star}F(x,y)(u)) \leq n\}$$

which are which are closed for $u \mapsto d(0, D^{\star\star}F(x,y)(u))$ is lower semicontinuous. Baire's Theorem implies that the interior of one of these sections is not empty, and actually, that there exists a ball ηB of radius $\eta > 0$ contained in S_n because the function $u \mapsto d(0, D^{\star\star}F(x,y)(u))$ is convex and positively homogeneous. Therefore,

$$\forall\, u \in B, \;\; d(0, D^{\star\star}F(x,y)(u)) \; \leq \; \frac{n}{\eta}$$

and thus

$$\lambda(D^{\star\star}F(x,y)) \; := \; \sup_{\|u\|=1} d(0, D^{\star\star}F(x,y)(u)) \; \leq \; \frac{n}{\eta}\|u\|$$

This means that the Lipschitz modulus is finite. ■

We shall prove that under adequate assumptions, this inequality is actually an equality and, as a very important byproduct, we provide Doyen's viability characterization of Lipschitz maps in [17, 18, Doyen] :

Theorem 3.4 Doyen's Viability Characterization of Lipschitz Maps. *Let us set*

$$\Phi_{(u,\lambda)} := \{u\} \times \lambda B \subset X \times Y$$

Assume that the graph of $F : X \rightsquigarrow Y$ is closed.
The following conditions are equivalent :

1. *F is λ-Lipschitz on P,*
2. $\sup_{(x,y)\in\text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y)) < +\infty$,
3. *For all $u \in P(1)$, the graph of F is viable under the maps $\Phi_{(u,\lambda)}$:*

$$\text{Graph}(F) = \bigcap_{u\in P(1)} \text{Viab}_{\Phi_{(u,\lambda)}}(\text{Graph}(F))$$

In this case,

$$\sup_{(x,y)\in\text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y)) = \sup_{(x,y)\in\text{Graph}(F)} \delta_F^P(x,y)$$

Proof — We shall prove successively that if F is λ-Lipschitz on P, then that inequality

$$\sup_{(x,y)\in\text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y)) < +\infty$$

holds true, that this inequality implies that the graph of F is viable under the maps $\Phi_{(u,\lambda)}$ when u ranges over $P(1)$, and that this implies that F is λ-Lipschitz on P.

1. If F is λ-Lipschitz on P, then for every $(x,y) \in \text{Graph}(F)$ and for every $u \in P(1)$, $\delta_F(u;x,y) \leq \lambda$. Lemma 3.1 implies that for every $(x,y) \in \text{Graph}(F)$,

   $$\forall\, u \in P,\ \ d(0, D^{\star\star}F(x,y)(u)) \leq d(0, DF(x,y)(u)) \leq \inf_{T>0} \delta_F(T,u;x,y) \leq \lambda$$

   and thus,

   $$\sup_{(x,y)\in\text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y)) \leq \sup_{(x,y)\in\text{Graph}(F)} \delta_F^P(x,y)$$

   This amounts to saying that the viability tangential condition

   $$\forall\, u \in P, \quad \Phi_{(u,\lambda)} \cap \overline{\text{co}}\left(T_{\text{Graph}(F)}(x,y)\right) \neq \emptyset$$

   holds true, or, equivalently, that inequality $\sup_{(x,y)\in\text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y)) \leq \lambda < +\infty$ is satisfied.
2. By Viability Theorem 3.3.4, p.91 and 3.2.4, p. 85 of [3, Aubin], this means that the graph of F is viable under the constant set-valued map $\Phi_{(u,\lambda)}$.

3. If the graph of F is viable under the maps $\Phi_{(u,\lambda)}$ when u ranges over $P(1)$, then for any $upb = P(1)$, there exists $\tau \mapsto v(\tau) \in B$ such that the evolution
$$(x(\cdot), y(\cdot)) : \; t \mapsto \left(x + tu, y + \lambda \int_0^t v(\tau) d\tau \right)$$
governed by differential inclusion $(x'(t), y'(t)) \in \Phi_{(u,\lambda)}$ is viable in Graph(F). This implies
$$\forall \, (x,y) \in \text{Graph}(F), \; \forall \, t \geq 0, \; \forall \, u \in P(1), \;\; d(y, F(x+tu)) \; \leq \; \lambda \left\| \int_0^t v(\tau) d\tau \right\| \; tl = \; \lambda t$$
and thus, that $\delta_F^P(x,y) \leq \lambda$. Hence
$$\sup_{(x,y) \in \text{Graph}(F)} \delta_F^P(x,y) \; \leq \; \sup_{(x,y) \in \text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y))$$
This means that F is Lipschitz on P and implies that $\sup_{(x,y) \in \text{Graph}(F)} \delta_F^P(x,y) = \sup_{(x,y) \in \text{Graph}(F)} \lambda^P(D^{\star\star}F(x,y))$ holds true, since we have proved the opposite inequality in the first step of the proof. ■

This viability characterization of Lipschitz maps leads to the concept of Doyen's Lipschitz Kernel of a map whenever F is not Lipschitz :

Proposition 3.5 Doyen's Lipschitz Kernel of a Set-Valued Map. *Let $F : X \rightsquigarrow Y$ be a closed set-valued map and $\lambda > 0$ be a finite constant. There exists a largest λ-Lipschitz set-valued map $\mathbb{L}_\lambda^\sharp(F)$ contained in F, called the* Lipschitz kernel *of F. It is given by the formula*
$$\mathbb{L}_\lambda^\sharp(F) \; = \; \bigcap_{n \geq 1} \mathbb{L}_\lambda^n(F)$$
where
– $\mathcal{L}_\lambda$ is defined on subsets $\mathcal{F} \subset X \times Y$ by
$$\mathcal{L}_\lambda(\mathcal{F}) := \bigcap_{u \in P(1)} \text{Viab}_{\Phi_{(u,\lambda)}}(\mathcal{F})$$
– $\mathbb{L}_\lambda^n(F)$ is defined recursively by
$$\text{Graph}(\mathbb{L}_\lambda^1(F)) \; := \; \mathcal{L}_\lambda(\text{Graph}(F)) \; \& \; \mathbb{L}_\lambda^n(F) \; = \; \mathbb{L}_\lambda^1(\mathbb{L}_\lambda^{n-1}(F))$$

Proof — By Theorem 3.4, the graph $\mathcal{F} := \text{Graph}(F)$ of a λ-Lipschitz F is a fixed point of the map $\mathcal{L}_\lambda$ defined on graphs $\mathcal{F} \subset X \times Y$ by
$$\mathcal{L}_\lambda(\mathcal{F}) := \bigcap_{u \in P(1)} \text{Viab}_{\Phi_{(u,\lambda)}}(\mathcal{F})$$

Section 7 of [4, Aubin & Catté] implies that the map $\mathcal{F} \mapsto \mathcal{L}_\lambda(\mathcal{F})$ is a pre-opening. Then the largest fixed point of $\mathcal{L}_\lambda$ contained in $\mathcal{F}$ is equal to $\mathbb{L}_\lambda^\sharp(\mathcal{F})$ where $\mathbb{L}_\lambda\sharp$ is the opening associated with the pre-opening $\mathcal{L}_\lambda^\sharp$. The opening algorithm provides formula $\mathbb{L}_\lambda^\sharp(F) = \bigcap_{n\geq 1} \mathbb{L}_\lambda^n(F)$. ■

4 Pseudo-Lipschitz Maps

The calmness indexes are pointwise indexes and the Lipschitz property is global. It is important to investigate an intermediate situation by "localizing" calmness indexes and Lipschitz constants in a neighborhood of a point $(x_0, y_0) \in \text{Graph}(F)$ of an element of the graph of F. For that purpose, we introduce the following definition :

Definition 4.1 Pseudo-Lipschitz Modulus. *The function* α_F *defined on the graph of* F *by*

$$\alpha_F(x_0, y_0) \ := \limsup_{(x,y)\mapsto_{\text{Graph}(F)}(x_0,y_0),\ T\to 0+} \delta_F(T; x, y)$$

is called the pseudo-Lipschitz modulus *of* F.

This definition is motivated by the following property :

Proposition 4.2 Pseudo-Lipschitz Property. *If* $\alpha_F(x_0, y_0) < +\infty$, *then, for any* $\varepsilon > 0$, *there exist* $\eta > 0$ *such that*

$$\begin{cases} \forall\, x, x_1 \in B\left(x_0, \dfrac{\eta}{2}\right),\ \forall\, y \in\ F(x) \cap B(y_0, \eta), \\ d(y, F(x_1)) \ \leq\ (\alpha_F(x_0, y_0) + \varepsilon)\|x_1 - x\| \end{cases}$$

This is called the pseudo-Lipschitz property *of* F *at* $(x_0, y_0) \in \text{Graph}(F)$ *(also called the Aubin property by Rockafellar and Wets in [33, Rockafellar & Wets]).*

Proof — Indeed, to say that $\alpha_F(x_0, y_0) < +\infty$ amounts to saying that for any ε, there exist η such that for any $(x, y) \in \text{Graph}(F) \cap B((x_0, y_0), \eta)$ and $T \leq \eta$,

$$\delta_F(T; x, y) \ := \sup_{\|x_1 - x\|\leq T\ \&\ x_1\neq x} \frac{d(y, F(x_1))}{\|x_1 - x\|} \ \leq\ \alpha_F((x_0, y_0)) + \varepsilon$$

Therefore, if both x and x_1 belong to $B\left(x_0, \frac{\eta}{2}\right)$ and if $y \in B(y_0, \eta)$, then $x_1 \in B(x, \eta)$ and we infer that

$$d(y, F(x_1)) \;\leq\; (\alpha_F((x_0, y_0)) + \varepsilon)\|x_1 - x\| \;\blacksquare$$

As for Lipschitz maps, we provide now a differential characterization of pseudo-Lipschitz modulus in terms of Lipschitz constants of the contingent derivatives :

Theorem 4.3 Differential Characterization of Pseudo-Lipschitzianity. *Assume that the graph of F is closed. Then F is pseudo-Lipschitz around $(x_0, y_0) \in \mathrm{Graph}(F)$ if and only if*

$$\limsup_{(x,y) \to_{\mathrm{Graph}(F)} (x_0, y_0)} \lambda(D^{\star\star}F(x,y)) \;<\; +\infty$$

is finite and in this case,

$$\alpha_F(x_0, y_0) \;:=\; \limsup_{(x,y) \to_{\mathrm{Graph}(F)} (x_0, y_0)} \lambda(D^{\star\star}F(x,y))$$

Consequently, if a set-valued map F satisfies

$$\left\{ \begin{array}{ll} (i) & \mathrm{Dom}(DF(x_0, y_0)) \;=\; X \\ (ii) & (x,y) \rightsquigarrow \mathrm{Graph}(DF(x,y)) \text{ is lower semicontinuous at } (x_0, y_0) \end{array} \right. \quad (4)$$

then $\alpha_F(x_0, y_0) \;=\; \lambda_{D^{\star\star}F}(x_0, y_0)$ is finite and thus, F is pseudo-Lipschitz around (x_0, y_0).

Proof — Let us set

$$\left\{ \begin{array}{l} \mathcal{F}_\eta := \mathrm{Graph}(F) \cap B((x_0, y_0), \eta) \\ \lambda_\eta \;:=\; \sup_{(x,y) \in \mathrm{Graph}(F) \cap B((x_0,y_0),\eta)} \lambda(D^{\star\star}F(x,y)) \\ \Phi_{(u,\lambda_\eta)} := \{u\} \times \lambda_\eta B \\ T(\eta) \;:=\; \frac{\eta}{2\max(1, \lambda_\eta)} \end{array} \right.$$

By construction, we observe that

$$\forall\, (x,y) \in \mathcal{F}_\eta, \quad \Phi_{(u,\lambda_\eta)} \cap \overline{\mathrm{co}}\left(T_{\mathcal{F}_\eta}(x,y)\right) \;\neq\; \emptyset$$

Local Viability Theorem 3.3.4., p.90 of [3, Aubin] implies that there exists $\tau \mapsto v(\tau) \in B$ such that the evolution

$$(x(\cdot), y(\cdot)) : \; t \mapsto \left(x + tu, y + \lambda_\eta \int_0^t v(\tau) d\tau \right)$$

governed by differential inclusion $(x'(t), y'(t)) \in \Phi_{(u,\lambda_\eta)}$ is viable in $\mathcal{F}_\eta$ on the interval $[0, T(\eta)]$ because $B\left((x,y), \frac{\eta}{2}\right) \subset B((x_0,y_0),\eta)$ whenever $(x,y) \in B\left((x_0,y_0), \frac{\eta}{2}\right)$ and $\|\Phi_{(u,\lambda_\eta)}\| \leq \max(1, \lambda_\eta)$. This implies that

$$\begin{cases} \forall\,(x,y) \in B\left((x_0,y_0), \frac{\eta}{2}\right),\ \forall\, t \leq T(\eta),\ \forall\, u \in P(1), \\ \dfrac{d(y, F(x+tu))}{t} \ \leq\ \lambda_\eta \ :=\ \sup_{(x,y)\in\mathcal{F}_\eta} \lambda(D^{\star\star}F(x,y)) \end{cases}$$

In other words,

$$\begin{cases} \sup_{(x,y)\in B\left((x_0,y_0),\frac{\eta}{2}\right)} \delta_F(T(\eta);(x,y)) \ :=\ \sup_{(x,y)\in B\left((x_0,y_0),\frac{\eta}{2}\right)} \sup_{t\leq T(\eta)} \sup_{\forall\, u\in P(1)} \dfrac{d(0, DF(x+tu))}{t} \\ \leq\ \sup_{(x,y)\in \mathrm{Graph}(F)\cap B((x_0,y_0),\eta)} \lambda(D^{\star\star}F(x,y)) \end{cases}$$

By letting η converges to 0, we infer that

$$\alpha_F(x_0,y_0) \ :=\ \limsup_{(x,y)\to_{\mathrm{Graph}(F)}(x_0,y_0),\ T\mapsto 0} \delta_F(T;(x,y)) \ \leq\ \limsup_{(x,y)\to_{\mathrm{Graph}(F)}(x_0,y_0)} \lambda(D^{\star\star}F(x,y))$$

The converse inequality follows from Lemma 3.1. This completes the proof of the first statement.

Since $(x,y) \rightsquigarrow \mathrm{Graph}(DF(x,y))$ is lower semicontinuous at (x_0,y_0), we know that $DF(x_0,y_0) = DF^{\star\star}(x_0,y_0)$ is a closed convex process (see Theorem 4.1.10 of **Set-Valued Analysis**, [6, Aubin & Frankowska]). The Robinson-Ursescu Closed Graph Theorem 3.3 (see Theorem 2.2.6 of **Set-Valued Analysis**, [6, Aubin & Frankowska]) states that since $\mathrm{Dom}(DF(x_0,y_0)) = X$, the closed convex process $DF(x_0,y_0)$ is Lipschitz with a finite Lipschitz constant $\lambda_{D^{\star\star}F}(x_0,y_0) = \sup_{u\in P(1)} d(0, DF(x_0,y_0)(u))$.

On the other hand, since $(x,y) \rightsquigarrow \mathrm{Graph}(DF(x,y))$ is lower semicontinuous at (x_0,y_0), we infer that

$$\limsup_{(x,y)\to_{\mathrm{Graph}(F)}(x_0,y_0)} \lambda(D^{\star\star}F(x,y)) \ \leq\ \lambda(D^{\star\star}F(x_0,y_0))$$

Indeed, we associate with any $u \in P(1)$ an element $v_0 \in DF(x_0,y_0)(u) = D^{\star\star}F(x_0,y_0(u))$ achieving $\|v_0\| = d(0, D^{\star\star}F(x_0,y_0)(u))$. Since the graph of DF is lower semicontinuous at (x_0,y_0), we can associate with any sequence (x_n,y_n) converging to (x_0,y_0) elements (u_n,v_n) converging to (u,v_0). Hence $d(0, D^{\star\star}F(x_n,y_n)(u_n)) \leq \|v_n\| \leq \|v_0\| + \varepsilon$ for n large enough. Consequently, $d(0, D^{\star\star}F(x,y)(u))$ is upper semicontinuous at (x_0,y_0), and so is its supremum over the compact sphere $u \in P(1)$.

Therefore, this inequality together with the first statement of the theorem imply that $\alpha_F(x_0,y_0) \leq \lambda(D^{\star\star}F(x_0,y_0))$ is finite, and thus, that F is pseudo-Lipschitz around (x_0,y_0). ■

Replacing F by its inverse F^{-1}, Theorem 4.3 is nothing else than the 1982 Inverse Theorem 1.1 for set-valued maps stated in the introduction.

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