Impulse and Hybrid Control Systems: A Viability Approach

A Mini-Course[1]

Jean-Pierre Aubin

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Université de Paris-Dauphine
Centre de Recherche Viabilité, Jeux, Contrôle
F-75775 Paris cx(16), France

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Introduction

A series of papers and books — among which [64, Branicky, Borkar & Mitter], [56, Bensoussan & Menaldi], [161, 162, Matveev & Savkin] and [215, Shaft & Schumacher] to quote a few — aims at embedding several classes of "hybrid systems" or "impulse optimal control problems" into a general framework.

This minicourse contributes to this effort, by presenting impulse control and hybrid systems in the framework of the viability of control systems following [20, Aubin], [28, Aubin-Dordan], [34, 35, Aubin-Haddad] and [37, Aubin, Lygeros, Quincampoix, Sastry & Seube].

Indeed, a key towards success requires simplification of the problem to use only the relevant properties of the problem, the price to pay being more abstraction. Here, the abstraction process amounts to

1. regard control systems as differential inclusions, i.e., differential equations with set-valued right hand sides,

2. regard hybrid control systems as "impulse differential inclusions" that we are about to define.

A first advantage is to summarize the usually protracted description of an hybrid system by only two set-valued maps F — the right-hand side of the differential inclusion governing the continuous evolution of a hybrid system — and R, describing the reset map reinitializing the system when required and a constrained set K inside which the evolution of the "run" or "execution" must remain. Hence, for instance, the existence of a run of an hybrid system for every initial set becomes a viability problem of an adequate auxiliary subset under an impulse differential inclusion, that can be characterized elegantly end efficaciously.

Nameoly, given a control system or a differential game described under the form of a differential inclusion $x' \in F(x)$ and constraints on the states represented by a closed subset K, we say that K *is viable under* F if from any initial state $x_0 \in K$ starts at least one solution of this differential inclusion "viable" in K in the sense that

$$\forall\, t \geq 0,\ \ x(t)\ \in\ K$$

There are no reasons why an arbitrary subset K should be viable under the differential inclusion $x' \in F(x)$.

Hence, the problem of reestablishing viability arises. One can imagine several mechanisms for this purpose:

1. Change either the dynamics or the set of constraints
   (a) either by changing the controls according to feedbacks or dynamic feedbacks that can be constructed (see for instance [8, 10, Aubin]),
   (b) or by changing the dynamics by, for instance, projecting the velocities onto the contingent cones and introducing viability multipliers (see for instance [8, 10, Aubin]),
   (c) or by restricting the constrained set to its viability kernel, which is by definition the largest subset viable under the dynamics,
   (d) or by letting the set of constraints evolve according to mutational equations, as in [13, Aubin].
2. or change the initial conditions by introducing a reset map R mapping any state of K to a (possibly empty) set $R(x) \subset X$ of new "initialized states".

This is the latter strategy we choose to use here: Hence an impulse differential inclusion (and in particular, an impulse control system) is described by a pair (F, R), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the continuous evolution of the system in K and where R, the reset map, governs the discrete switches to new "initial conditions" when the continuous evolution is doomed to leave K.

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called in the "hybrid system" literature a "run" or an "execution".

Namely, let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$. Starting from an initial state $x_0 \in K$, a "run" of the impulse differential inclusion is a "map" $x(\cdot)$ from $[0, T]$ to X which is associated with a non decreasing sequence $\mathcal{T}(x(\cdot)) := \{t_n\}_{n \geq 0}$ of impulse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ such that

1. $x(t_{n+1}) \in R(x(t_n))$ if $t_{n+1} = t_n$,
2. or else, on the interval $[t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ at time t_n until time t_{n+1} at which we take $x(t_{n+1}) \in R(x(^-t_{n+1}))$.

We denote by $\tau_n := t_n - t_{n-1}$ the nth cadence of the run and by $x_n(\cdot) := x(\cdot + t_n)$ the nth motive of the run, a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ on the interval $[0, \tau_n]$. The sequence of states $x(t_n)$ is called the sequence of initialized states.

A run is said to be viable in a closed subset K if for all $t \geq 0$, $x(t) \in K$ and we say that a subset K is viable under (F, R) if from any initial state $x_0 \in K$ starts at least a run to the impulse differential inclusion (F, R) viable in K.

Many examples coming from different fields of knowledge fit this framework:

1. multiple-phase economic dynamics in economics (see for instance [98, Day]),
2. stock management in production theory ([53, 54, 55, Bensoussan & Lions J.-L.] for instance),
3. viability theory, for implementing the extreme version of the "inertia principle" ([8, 10, Aubin])),
4. propagation of the nervous influx along axones of neurons triggering spikes in neurosciences and biological neuron networks[1] (See the "Integrate-and-Fire" models in [67, Bressloff & Coombes], [69, Brette], [73, Burgmann], [102, Destexhe], [197, 198, Shimokawa, Pakdaman & Sato] and [199, Shimokawa, Pakdaman, Takahata, Tanabe & Sato] for instance.),
5. "threshold" impulse control, when "controls jump" when the threshold is about to be trespassed,
6. punctuated evolution as proposed by N. Eldredge and S.J Gould in [107, Eldredge & Gould] in biological evolution,
7. in demographical models, to take into account discontinuous processes such as births and deaths,
8. in issues dealing with "qualitative physics" in Artificial Intelligence and/or "comparative statics" in economics (see for instance [9, Aubin] and [105, Dordan] for a mathematical presentation of these issues and further bibliographical data),
9. in logics, where connections are made between impulse differential inclusions and the μ-calculus (see [92, 93, 94, Davoren], [169, Nerode & Shore] and their references),

[1] instead of the continuous time Hodgkin-Huxley type of systems of differential equations inspired by the propagation of electrical current which are the subject of an abundant literature. See the pioneering [138, Hodgkin & Huxley].

10. in manufacturing and economic production systems, when the jumps are governed by Markov processes instead of set-valued maps (see [112, Filar, Gaitsgory & Haurie], [113, Filar & Haurie], [123, Gershwin], [131, 131, Haurie, Leizarowitz & Van Delft]) and the references herein,

11. and, above all, in automatic control theory where a fast growing literature deals with hybrid "systems".

    Hybrid systems are described by a family of control systems and by a family of viability (or state) constraints indexed by parameters e called "locations". Starting with an initial condition in a set associated with an initial location, the control system associated with the initial location governs the evolution of the state in this set for some time until some impulse time resets the system by imposing a new location, and thus, a new control system, a new constrained set and a new initial condition. One can show that they fit in the above class of impulse systems in a simple and natural way.

The Viability Theorem[2] and its consequences imply that K *is viable under* (F, R) *if and only*[3] *if* $K\backslash R^{-1}(K)$ *is locally viable*[4] *under* F, i.e., in "tangential form"[5], if and only if

$$\forall\, x \in K\backslash R^{-1}(K),\quad F(x) \cap T_K(x) \;\neq\; \emptyset$$

or, equivalently, in dual form, if and only if

$$\forall\, x \in K\backslash R^{-1}(K),\ \forall\, p \in N_K(x),\quad \sigma(F(x), -p) \;\geq\; 0$$

---

[2] See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [8, Aubin].

[3] The subset $K\backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C.

[4] Viability issues ("positively invariance" for hybrid systems) has been studied in pretty much the same spirit in [135, Hespanha & Morse] (Lemma 3) and [63, Branicky].

[5] The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in L$ for every n. The (regular) normal cone $N_L(x) := T_L(x)^- := \{p \in X^* \mid \forall\, v \in T_L(x),\ \langle p, v\rangle \leq 0\}$ is the polar cone to the contingent cone $T_L(x)$ (see for instance [31, Aubin & Frankowska]) or [182, Rockafellar & Wets] for more details). We denote by

$$\forall\, p \in X^\star,\quad \sigma(K, p) \;:=\; \sup_{x \in K} \langle p, x\rangle$$

the support function of K.

The behavior of the run is "summarized" by the "initialization map" $U := U_{(F,R)}$ associating with each initial condition $x_0 \in K$ the set of new initialized conditions $x_1 \in R(x(^-t_1))$ when $x(\cdot)$ ranges over the set of solutions to the differential inclusion $x' \in F(x)$ viable in K until they reach $R^{-1}(K)$ at time $t_1 \geq 0$ at $x(^-t_1) \in R^{-1}(K)$.

Indeed, the sequence of successive initial conditions x_n of a viable run $x(\cdot)$ of the impulse differential inclusion (F,R) — constituting the "discrete component of the run" — is governed by the discrete system $x_n \in U_{(F,R)}(x_{n-1}) \cap K$ starting at x_0. The knowledge of the sequence of initialized states x_n allows us to reconstitute the "continuous component" of the run by solving the differential inclusion $x' \in F(x)$ starting at each reinitialized state x_n and satisfying the end-point condition $x_{n+1} \in R(x(^-t_{n+1}))$, which exists thanks to the definition of the map $U^K_{(F,R)}$.

Assume for a while that the impulse differential inclusion is actually an impulse differential equation (f,r) where the maps f and r are single-valued and that the initialization map is single-valued and differentiable. Then we shall prove that the initialization map is a solution to the system of first-order partial differential inclusions

$$\forall\, i = 1, \dots n,\ \sum_{j=1}^{n} \frac{\partial u_i(x)}{\partial x_j} f_j(x) \ = \ 0$$

or, in a more compact form,

$$0 \ = \ \frac{\partial u}{\partial x} f(x)$$

satisfying the "condition"

$$\forall\, x \in K \cap r^{-1}(K),\ \ r(x) \ = \ u(x)$$

Actually, we shall extend this result to general impulse differential inclusions by characterizing the initialization map $U_{(F,R)}$ as a generalized (set-valued) solution — a Frankowska solution — to the system of first-order partial differential inclusions

$$0 \ \in \ \frac{\partial u}{\partial x} F(x)$$

satisfying the "condition"

$$\forall\, x \in K \cap R^{-1}(K),\ \ R(x) \ \subset \ U(x)$$

These are indeed really Dirichlet boundary condition whenever the reset map R is defined only on the boundary ∂K of a closed subset K and maps ∂K into the interior of K. In this case, resetting initial conditions happens only when the continuous evolution of the state governed by the differential inclusion or the control system is about to leave the domain K. Hence the reset map assigns new initialized states in the interior of K.

Hybrid systems are described by

1. a finite dimensional vector space E of elements e (called "locations"),
2. a set-valued map $K : E \rightsquigarrow X$ associating with any e a subset $K(e) \subset X$
3. a map[6] $f : \text{Graph}(K) \rightsquigarrow X$ with which we associate the differential equation $x'(t) = f(e, x(t))$,
4. a map $r : \text{Graph}(K) \mapsto E \times X$

We shall say that a map $t \mapsto x(t) \in X$ discontinuous at some states $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and absolutely continuous on the intervals $[t_n, t_{n+1}[$ is a run of an hybrid differential equation starting from x_0 if

$$\forall\, n \geq 0, \;\; \forall\, t \in [t_n, t_{n+1}[, \begin{cases} e(t) &= e_n \text{ is a constant location} \\ x'(t) &= f(e_n, x(t)) \\ x(t) &\in K(e_n) \end{cases}$$

and

$$\forall\, n \geq 0, \;\; (e_{n+1}, x_{n+1}) \;=\; r(e_n, x(^{-}t_{n+1})) \;\&\; x(t_{n+1}) \;\in\; K(e_{n+1})$$

at some time t_n.

It is shown that it is enough to look at them as runs to the extended impulse system

$$(e'(t), x'(t)) \;=\; (0, f(e(t)), x(t))$$

which are viable in the graph of the constraint map $e \rightsquigarrow K(e)$.

Indeed, the above system requires that the component $e(t)$ remain equal to some constant parameter e_n as long as the state $x(t)$ is viable in $K(e_n)$ and to switch to another initialized state $x_{n+1} \in K(e_{n+1})$ according to a reset law $(e_{n+1}, x(t_{n+1})) = r(e_n, x(^{-}t_n))$. So, the Impulse Viability Theorem provides a necessary and sufficient condition for the existence of runs of hybrid systems starting from any initial condition $x_0 \in K(e_0)$.

[6] We shall assume later on that the dynamics governing the evolution of the continuous part of the run is set-vlaued. In the case of single-valued map, hybrid systems as we describe them are very close in spririt to differential automata introduced in [209, Taverni].

Actually, we shall study this problem when f is replaced by a set-valued map F describing the dynamics of a differential inclusion and the map r is replaced by a set-valued reset map R encapsuling also uncertainty in the discrete dynamics of the hybrid differential inclusion.

We next consider specific viability constraints — called, dynamical inequalities — which are written in the form

$$\forall\, t \in [0,T], \;\; \mathbf{v}(x(t)) \;\leq\; w(t)$$

where $\mathbf{v} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is a given nontrivial nonnegative extended function.

Here, the evolution of $t \mapsto x(t)$ is a run governed by an impulse differential inclusion (F,R) and the evolution of $t \mapsto w(t)$ is governed by some differential equation

$$w'(t) \;=\; -\varphi(x(t), w(t))$$

For instance, taking $\varphi(x,w) = aw$, we would obtain runs along which the loss function decreases exponentially:

$$\forall\, t \geq 0, \;\; \mathbf{v}(x(t)) \;\leq\; \mathbf{v}(x_0)e^{-at}$$

where $a \geq 0$. Such functions can legitimately be called exponential Lyapunov functions.

Let us mention right away the link between dynamical inequalities and viability constraints: Denoting by

$$\mathcal{E}p(\mathbf{v}) \;:=\; \{(x,w) \in X \times \mathbf{R} \mid \mathbf{v}(x) \;\leq\; w\}$$

the epigraph of the function $\mathbf{v}$, the above dynamical inequality constraints can be written

$$\forall\, t \in [0,T], \;\; (x(t), w(t)) \;\in\; \mathcal{E}p(\mathbf{v})$$

Hence we are now in familiar grounds whenever the epigraph of the extended function $\mathbf{v}$ is closed[7]. For applying the Impulse Viability Theorem, it is natural to introduce the contingent cone to the epigraph of the function $\mathbf{v}$ and to observe that it is the epigraph

$$T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) \;=\; \mathcal{E}p(D_{\uparrow}\mathbf{v}(x))$$

[7] Extended functions the epigraph of which are closed are called lower semicontinuous functions.

of the function $D_\uparrow \mathbf{v}(x)$ defined by:

$$\forall\, u \in X, \;\; D_\uparrow \mathbf{v}(x)(u) \;=\; \liminf_{h \to 0+, u' \to u} \frac{\mathbf{v}(x+hu') - \mathbf{v}(x)}{h}$$

It is called the contingent epiderivative of $\mathbf{v}$ at x and is a generalized directional derivative because $D_\uparrow \mathbf{v}(x)(u) = \langle \mathbf{v}'(x), u\rangle$ whenever $\mathbf{v}$ is differentiable.

Introducing the marginal function $\mathbf{u}$ associating with any x the infimum[8] $\mathbf{u}(x) := \inf_{y \in R(x)} \mathbf{v}(y)$ of $\mathbf{v}$ on $R(x)$, we shall prove that the two following statements are equivalent:

1. for any initial state $x_0 \in \mathrm{Dom}(\mathbf{v})$, *there exist a run* $x(\cdot)$ to the impulse differential inclusion (F, R) and a solution to the differential equation $w' = \varphi(x, w)$ satisfying
$$\forall\, t \geq 0, \;\; \mathbf{v}(x(t)) \;\leq\; w(t), \quad w(0) = \mathbf{v}(x(0))$$

2. $\mathbf{v}$ is a contingent solution to the *Hamilton-Jacobi variational inequalities*
$$\forall\, x \;\text{ such that } \mathbf{v}(x) < \mathbf{u}(x), \;\; \inf_{v \in F(x)} D_\uparrow \mathbf{v}(x)(v) + \varphi(x, \mathbf{v}(x)) \;\leq\; 0$$

In other words, the familiar characterization of Lyapunov functions has to be satisfied only for the elements x in the "continuation set"

$$\{x \quad \text{such that } \; \mathbf{v}(x) < \mathbf{u}(x)\}$$

instead of in the whole domain of $\mathbf{v}$.

Assuming that $\inf_{x \in X} \mathbf{v}(x) = 0$, taking $F(x) := B$, the unit ball and $\varphi(x, w) := aw$ with $a > 0$, we obtain an hybrid gradient method for reaching a global minimum of $\mathbf{v}$.

A cadenced run is then defined by constant cadence, initial state and motive, where the value at the end of the cadence is reset at the same reinitialized state. It plays the role of a "discontinuous" periodic solution of a differential inclusion.

We shall prove that if the sequence of reinitialized states of a run converges to some state, then the run converges to a cadenced run starting from this state, and that, under convexity assumptions, that a cadenced run does exist.

[8] setting $\mathbf{u}(x) = +\infty$ whenever $R(x) = \emptyset$.

Existence of periodic solutions to differential automata introduced in [209, Taverni] and an adpatation of the Poincar-Bendixon Theorem to differential automata can be found in [160, Matveev & Savkin].

When K is not viable under an impulse differential inclusion (F, R), we define the reset kernel $\text{Reset}_{(F,R)}(K)$ of K under the impulse differential inclusion (F, R) as the set of initial states from which starts at least one run of the impulse differential inclusion viable in K.

In order to eliminate both solutions to the differential inclusion $x' \in F(x)$ and sequences of the discrete system $x_{n+1} \in R(x_n)$, we assume that the closed subset K is both a repeller under F and a discrete repeller under R.

We shall prove that *the reset kernel* $\text{Reset}_{(F,R)}(K)$ *of* K *under the impulse differential inclusion* (F, R) *is the largest closed subset of* K *viable under the impulse differential inclusion* (F, R).

Actually, we shall provide further characterizations and properties of closed subsets viable under impulse differential inclusions or their reset kernels by using the concept of viable-capture basin of a closed subset C (regarded as a target) $\text{Capt}_F^K(C)$, defined as the set of initial states from which at least one solution to the differential inclusion $x' \in F(x)$ is viable in K until it reaches the target C in finite time.

We shall prove that the reset kernel is the largest "fixed set" of the map $L \mapsto L \cap \text{Capt}_F^K(R^{-1}(L) \cap K)$, i.e., a solution to equation

$$\text{Reset}_{(F,R)}(K) \ := \ \text{Capt}_F^K(R^{-1}(\text{Reset}_{(F,R)}(K)) \cap K)$$

In this case, from any $x_0 \in \text{Reset}_{(F,R)}(K)$ starts a solution viable in $\text{Reset}_{(F,R)}(K)$ until it reaches $R^{-1}(\text{Reset}_{(F,R)}(K))$ at some time t_1 and at some state $x(^-t_1)$. Then there exists a new initial state $x_1 \in R(x(^-t_1)) \cap \text{Reset}_{(F,R)}(K)$ from which one can reiterate the process for obtaining a run locally viable in $\text{Reset}_{(F,R)}(K)$ until it reaches $R^{-1}(\text{Reset}_{(F,R)}(K))$.

Furthermore, we obtain the reset kernel $\text{Reset}_{(F,R)}(K)$ through the "reset kernel algorithm": Defining recursively $K_0 := K$ and, for all $n \geq 0$,

$$K_n \ := \ K_{n-1} \cap \left(\text{Capt}_F^K(R^{-1}(K_{n-1}) \cap K)\right)$$

we shall prove that

$$\text{Reset}_{(F,R)}(K) \ = \ \bigcap_{n \geq 0} K_n$$

This justifies a further study of the viable-capture basins initiated in [78, Cardaliaguet, Quincampoix & Saint-Pierre] and [180, Quincampoix & Veliov] under the name of viability kernel with target and pursued in [17, 19, Aubin] in the framework of invariance envelopes. In particular, we shall characterize *the viable-capture basin* $\mathrm{Capt}_F^K(C)$ *of a closed subset* C *as the largest closed subset* D *satisfying*

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D\backslash C \text{ is locally viable under } F \end{cases}$$

When K is assumed further to be backward invariant (i.e., when all solutions to the backward differential inclusion $x' \in -F(x)$ starting from K are viable in K), then

$$\mathrm{Capt}_F^K(C) \;=\; \mathrm{Capt}_F(C) \;:=\; \mathrm{Capt}_F^X(C)$$

so that $\mathrm{Capt}_F(C)$ is the **unique** closed subset D satisfying

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D\backslash C \text{ is locally viable under } F \\ iii) & D \text{ is backward invariant under } F \end{cases}$$

thanks to a characterization theorem of capture basins given in [17, Aubin], since capture basins are backward invariance envelopes.

When F is assumed to be furthermore Lipschitz, we may reformulate this property by stating that K is the unique "Frankowska extension" of C under F, i.e., the unique closed subset $D \supset C$ which is a repeller under F and satisfies the "tangential conditions"

$$\begin{cases} i) & \forall\, x \in D, \;\; F(x) \subset -T_D(x) \\ ii) & \forall\, x \in D\backslash C, \;\; F(x) \cap T_D(x) \;\neq\; \emptyset \end{cases}$$

or, equivalently, by duality, the "normal conditions"

$$\begin{cases} i) & \forall\, x \in D, \quad \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \;\leq\; 0 \\ ii) & \forall\, x \in D\backslash C \quad \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \;=\; 0 \end{cases}$$

Being viable-capture basins — or, when K is backward invariant, capture basins — the reset kernel $\mathrm{Reset}_{(F,R)}(K)$ of a closed subset K under an impulse differential inclusion (F, R) enjoys the above properties when the target C is $R^{-1}(\mathrm{Reset}_{(F,R)}(K)) \cap K$.

As one of the possible applications, we derive[9] the characterization of the value function of an infinite horizon intertemporal optimal impulse control problem as a

[9] by using the viability approach for usual optimal control used in Chapter 6 of [26, Aubin & Cellina], [8, Aubin] and [33, Aubin & Frankowska].

Frankowska solution — a concept of generalized solution weaker than the concept of viscosity solution — to the quasi-variational inequalities introduced by Alain Bensoussan, Jacques-Louis Lions and many other authors (see [53, 54, 55, Bensoussan & Lions J.-L.] for instance for motivations, examples and a review) and recently extended to hybrid systems in [56, Bensoussan & Menaldi]. See also [43, Barles], [63, Branicky], [64, Branicky, Borkar & Mitter] among other references.

We denote by $\mathcal{R}(x)$ the set of runs $x(\cdot)$ of the control problem

$$\begin{cases} i) & x'(t) \ = \ f(x(t), u(t)) \\ ii) & u(t) \ \in \ P(x(t)) \end{cases}$$

starting from x associated with the switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$, satisfying

1. if $t \in [t_{n-1}, t_n[$, then

$$x(t) \ = \ x_0 + \sum_{k=1}^{n-1} \xi_k + \int_0^t f(x(\tau), u(\tau)) d\tau$$

2. if $t_n = t_{n-1}$,

$$x_n := x(t_n) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^{t_n} f(x(\tau), u(\tau)) d\tau$$

Let us introduce a “cost function” $\mathbf{w} : X \mapsto \mathbf{R} \cup \{+\infty\}$ and a Lagrangian $l : X \times \mathcal{P} \mapsto \mathbf{R}$.

We shall characterize the value function

$$\mathbf{v}(x) := \inf_{(x(\cdot), u(\cdot)) \in \mathcal{R}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k} \mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right)$$

of the above control problem by proving that its epigraph is the reset kernel of $X \times \mathbf{R}_+$ under an auxiliary impulse differential inclusion involving the cost function $\mathbf{w}$. By using the tangential and normal characterizations of the impulse viability kernel, we shall derive that under adequate assumptions, the value function is *the* **unique**

Frankowska[10] *solution* $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *of the quasi-variational inequalities*

$$\begin{cases} i) & 0 \ \le \ \mathbf{v}(x) \ \le \ (\mathbf{v} * \mathbf{w})(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ \le \ 0 \\ iii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p)(\mathbf{v}(x) - (\mathbf{v} * \mathbf{w})(x)) \ = \ 0 \end{cases}$$

where

1. $(\mathbf{v} * \mathbf{w})(x) := \inf_{y \in X}(\mathbf{w}(y) + \mathbf{w}(y + x))$ is the inf-convolution of the functions $\mathbf{v}$ and $\mathbf{w}$,

2. $\partial_{-}\mathbf{v}(x)$ denotes the generalized subgradient of $\mathbf{v}$ at x, as it used both is nonsmooth analysis (see for instance [31, Aubin & Frankowska] or [182, Rockafellar & Wets]) and in the theory of viscosity solutions (see for instance [89, Crandall & Lions P.-L.] and [42, Bardi & Capuzzo-Dolcetta]),

3. $H(x, y, p) := \sup_{u \in P(x)}(\langle p, f(x, u)\rangle + l(x, u)) - ay$ denotes the Hamiltonian associated with the control system and the Lagrangian,

4. $\mathbf{v}$ is unique in the class of lower semicontinuous functions.

Knowing the value function $\mathbf{v}$, we introduce the two regulation maps $\mathbf{R}_{(f,P)}$ and $\mathbf{R}_{\mathbf{w}}$ defined by

$$\mathbf{R}_{(f,P)}(x) \ := \ \{u \in P(x) \mid D_{\uparrow}\mathbf{v}(x)(f(x, u)) + l(x, u) - a\mathbf{v}(x) \le 0\}$$

and

$$\mathbf{R}_{\mathbf{w}}(x) \ := \ \{y \in X \mid \mathbf{w}(y) + \mathbf{v}(y + x) = (\mathbf{v} * \mathbf{w})(x)\}$$

Therefore, an optimal run is obtained in the following way: Starting from x_0 such that $\mathbf{v}(x_0) < (\mathbf{v} * \mathbf{w})(x_0)$, we choose a solution to the control system

$$\begin{cases} i) & x'(t) \ = \ f(x(t), u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}_{(f,P,l)}(x(t)) \end{cases}$$

[10] Hélène Frankowska proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is invariant and backward viable under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant ([119, 120, 122, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of [89, Crandall & Lions P.-L.]. See also [50, Barron & Jensen] and [42, Bardi & Capuzzo-Dolcetta] for more details.

until the time $t_1 \geq 0$ when $\mathbf{v}(x(^{-}t_1)) = (\mathbf{v} * \mathbf{w})(x(^{-}t_1))$.

At this stage, we take for reinitialized sate $x_1 \in \mathbf{R_w}(x(^{-}t_1)$, and we reiterate the process.

In other words, the optimal impulse control problem with infinite horizon conceals an underlying impulse control system $((f, \mathbf{R}_{(f,P,l)}), \mathbf{R_w})$ the runs of which are the optimal runs of the initial control problem.

This application to optimal impulse control is presented not so much for providing the existence of a generalized solution to the quasi variational inequalities to which the value function is a solution, but to convince the reader that the simple framework we propose allows us to cover many problems arising in the realm of hybrid systems in control theory, of stock management in production theory, of multiple-phase dynamical economies in economics, of the propagation of the nervous influx along axons of neurons triggering spikes in neurosciences, etc. It evidences that the tangential and normal characterizations of the viability of a set or of its reset kernel are implying the theorems characterizing value functions as solutions to quasi variational inequalities, and thus, share the same features. This suggests that this approach based on viability theory and set-valued analysis provides some potential for studying those problems which can be formulated as viability problems for impulse differential inclusions.

Chapter 1

Definitions and Elementary Properties

1.1 Runs of Impulse Differential Inclusions

We define “runs” in the following way:

We define “runs” in the following way:

Definition 1.1.1 *Let us set* $x(^{-}t) := \lim_{\tau \mapsto t-} x(\tau)$. *We say that a sequence*

$$\vec{x}(\cdot) \ := \ (\tau_n, x_n, x_n(\cdot))_{n \geq 0} \ \in \ (\mathbf{R}_+ \times X)^{\mathbf{N}} \times \prod_{n=0}^{\infty} \mathcal{C}(0, \tau_n; X)$$

is a run $\vec{x}(\cdot)$ *made of*

1. *a finite or infinite sequence* $\tau(\vec{x}(\cdot)) := \{\tau_n\}_n$ *of nonnegative* cadences $\tau_n \in [0, +\infty[$,
2. *a sequence of* reinitialized states $x_n \in X$,
3. *a sequence of* motives $x_n(\cdot) \in \mathcal{C}(0, \tau_n; X)$.

and by the formulas

$$\begin{cases} i) & \text{the sequence } \mathcal{T}(\vec{x}(\cdot)) := \{t_n\}_{n \geq 0} \text{ of impulse times } t_{n+1} := t_n + \tau_n, \ t_0 = 0 \\ ii) & \forall \, t \in [t_n, t_{n+1}], \ \ \vec{x}(t) \ := \ x_n(t - t_n) \ \& \ u(t) \ := \ u_n(t - t_n) \end{cases} \tag{1.1}$$

Naturally, if $\tau_n = 0$*, i.e., when* $t_{n+1} = t_n$*, we identify the motive* $x_n(\cdot)$ *with the reinitialized state* $x_n(\cdot) \equiv x_n \in \mathcal{C}(0,0;X) \equiv X$*, so that runs can be time-dependent functions, sequences, or hybrids of them.*

If the sequence of cadences is finite and stops at τ_N*, we agree that the* N*th motive* $(x_N(\cdot), u_N(\cdot))$ *is taken on* $[0, +\infty[$ *and we agree to set* $\tau_{N+1} = +\infty$*.*

We associate with a run $\vec{x}(\cdot)$ *its sequence of* switches $\mathbf{s}(\vec{x}(\cdot)) := (\mathbf{s}_n(\vec{x}(\cdot)))_{n\geq 0}$ *defined by*

$$\mathbf{s}_n(\vec{x}(\cdot)) \; := \; x_n - \vec{x}(^-t_n)$$

We denote by $\mathcal{T}(\vec{x}(\cdot);t)$ *the set of impulse times* $t_n \leq t$ *less than or equal to* t*.*

The life expectation $T(\vec{x}(\cdot))$ *of a run* $\vec{x}(\cdot)$ *is defined by*

$$T(\vec{x}(\cdot)) \; := \; \sum_{n=0}^{+\infty} \tau_n \; = \; \lim_{n \to +\infty} t_n \; \leq \; +\infty$$

so that its domain of definition is the interval $[0, T(\vec{x}(\cdot))]$*. The run is said to be*

1. finite *if for some* p *and for all* $n \geq p$*,* $t_n = t_p$*, ,*
2. a Zeno run *if its life expectation* $T(\vec{x}(\cdot)) < +\infty$ *is finite,*
3. strict *is all the cadences* $\tau_n > 0$ *are strictly positive, i.e., if the sequence of impulse times* $t_n \in T$ *is strictly increasing,*
4. exhaustive *if it is not finite and if its life expectation* $T(\vec{x}(\cdot)) = +\infty$ *is infinite (non Zeno).*

We say that a run $\vec{x}(\cdot)$ *is viable in* K *if for any* $t \geq 0$*,* $\vec{x}(t) \in K$*.*

Let us just point out that

$$\forall\, n \geq 0, \;\; \vec{x}(t_n) \; = \; x_n \; := \; \vec{x}(^-t_n) + \mathbf{s}_n(\vec{x}(\cdot))$$

Let us consider a finite dimensional vector space X, a closed subset $K \subset X$ and a set-valued map $F : X \rightsquigarrow X$, with which we associate the differential inclusion $x'(t) \in F(x(t))$. We say that K is viable under F if, from any initial state $x_0 \in K$ starts a solution $x(\cdot)$ to the differential inclusion viable in K in the sense that

$$\forall\, t \geq 0, \;\; x(t) \in K$$

When K is not viable under F, we can maintain the viability by "resetting" the differential inclusion whenever the solution is doomed to leave K by fixing a new initialized state through a reset map $R : X \rightsquigarrow X$. We then may still obtain viable evolutions, called "runs", which are discontinuous time-dependent function at a increasing sequence of "switching times".

Let us set $x(^{-}t) := \lim_{t \mapsto t-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t-\eta, t[$ where $\eta > 0$,and, for consistency purposes, $x(s) = x(^{-}t)$ if $s = t$.

Definition 1.1.2 *Let us consider a finite dimensional vector space X, a closed subset $K \subset X$, a set-valued map $F : X \rightsquigarrow X$ and a set-valued map $R : X \rightsquigarrow X$, regarded as a* reset map[1]*. Then the pair (F, R) governs a run $\vec{x}(\cdot)$ of an impulse differential inclusion if*

$$\begin{cases} i) & \forall\, n \geq 0, \;\; x_n(\cdot) \;\in\; \mathcal{S}_F(x_n) \\ ii) & \forall\, n \geq 0 \;\; \textit{such that}\; \tau_n < +\infty, \;\; x_{n+1} \;\in\; R(x_n(\tau_n)) \end{cases} \tag{1.2}$$

We shall denote by $\mathcal{R}(x) := \mathcal{R}^K_{(F,R)}(x)$ the set of runs of the impulse differential inclusion (F, R) starting from $x \in K$ viable in K.

We shall say that the impulse differential inclusion is strict (resp. exhaustive) on K if for all $x \in X$, all runs $\vec{x}(\cdot) \in \mathcal{R}_{(F,R)}(x)$ viable in K are strict (resp. exhaustive).

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in R(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

A necessary condition for a run $x(\cdot)$ to be a solution to the impulse differential inclusion is that, for any switching or impulse time t_n, $x(^{-}t_n)$ belongs to the domain $\mathrm{Dom}(R) = R^{-1}(X)$ in order to be able to obtain a nonempty set $R(x(^{-}t_n))$ in which we chose the new initialized state $x(t_{n+1})$.

Proposition 1.1.3 *A run $x(\cdot)$ of an impulse differential inclusion (F, R) is associated with a sequence of impulse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and defined by: for all $j = 1, \ldots n-1$,*

1. *if $t_j = t_{j-1}$,*

$$x_j \;:=\; x(t_j) \;\in\; x_0 + \sum_{k=1}^{j} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^{t_j} F(x(\tau))d\tau$$

[1]When $\Phi : X \rightsquigarrow X$ is defined on X, we associate with it its "graphical restriction" to $K \times K$ (again denoted by) Φ defined on $L := K \cap \Phi^{-1}(K)$ and associating with x the subset $\Phi(x) \cap K$.

2. if $t_j > t_{j-1}$,

$$\begin{cases} i) \quad \forall\, t \in [t_{j-1}, t_j[, \;\; x(t) \;\in\; x_0 + \sum_{k=1}^{j-1} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^t F(x(\tau))d\tau \\ ii) \quad x_j \;:=\; x(t_j) \;\in\; x_0 + \sum_{k=1}^{j} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^{t_j} F(x(\tau))d\tau \end{cases}$$

where $\mathbf{s}(\vec{x}(\cdot))$ *is the associated switching sequence.*

Proof — We shall prove this statement recursively by assuming that we have constructed the viable run $x(\cdot)$ on the interval $[0, t_n[$ through the switching times $0 \leq t_1 \leq \cdots t_{n-1} \leq t_n \leq \cdots$ satisfying for all $j = 1, \ldots n-1$

$$\forall\, t \in [t_{j-1}, t_j[, \;\; x(t) \;\in\; x_0 + \sum_{k=1}^{j-1} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^t F(x(\tau))d\tau$$

At time t_n, we choose next $x_n \in R(x(^{-}t_n))$ through the reset map and we observe that

$$\begin{cases} R(x(^{-}t_n)) \;:=\; x(^{-}t_n) + \mathbf{s}_n(\vec{x}(\cdot)) \\ = \; x_0 + \sum_{k=1}^{n-1} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^{t_n} F(x(\tau))d\tau + \mathbf{s}_n(\vec{x}(\cdot)) \end{cases}$$

Either $t_{n+1} = t_n$, and thus $x(^{-}t_n) = x(t_n)$ and

$$x_{n+1} \;:=\; x(t_{n+1}) \;\in\; R(x(^{-}t_n)) \;=\; x_0 + \sum_{k=1}^{n+1} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^{t_{n+1}} F(x(\tau))d\tau$$

or else, we take a solution to the differential inclusion $x' \in F(x)$ starting at time t_n from x_n

$$\forall\, t \in [t_n, t_{n+1}[, \begin{cases} x(t) & \in & x(^{-}t_n) + \mathbf{s}_n(\vec{x}(\cdot)) + \int_{t_n}^t F(x(\tau))d\tau \\ & = & x_0 + \sum_{k=1}^{n} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^t F(x(\tau))d\tau \end{cases}$$

This completes the proof. $\square$

Remark — Denoting by $\delta(t)$ the Dirac measure at time t, we may denote symbolically the impulse differential inclusion described by (F,R) by

$$x'(t) \in F(x(t)) + \sum_{k\geq 0} \mathbf{s}_k(\vec{x}(\cdot))\delta(t_k)$$

where $t_0 \leq t_1 \cdots \leq \cdots \leq t_n \leq \cdots$ denotes a sequence of "switching" times and $x(^-t_k) \in K$ a sequence of elements of K. This notation may be misleading since it can convey the feeling that switching times are prescribed *a priori*. However, this notation is used in [53, 54, 55, Bensoussan & Lions J.-L.].

Replacing the series of Dirac measures by Radon measures, other concepts of impulsive differential equations, inclusions and control problems — in which controls are measures or distributions — were introduced in [5, Ananina-Filppova], [144, 145, Kurzhanski], [194, 195, Sesekin] and presented in the book [221, Zavalishchin & Sesekin], by [65, 66, Bressan & Rampazzo], [90, Dal Maso G. & Rampazzo], based on reparametrization and graph completion techniques for defining discontinuous solutions to such impulsive control systems. A different approach was next proposed by [174, Pereira, Silva & Vinter], [175, Pereira & Silva], [205, 206, Silva & Vinter], [218, Vinter & Pereira] for defining a concept of "robust solution" allowing them to provide necessary conditions for optimality. □

Remark: Constant Switching Maps — When $S \subset X$ is a closed subset, regarded as a switching set, we associate with it the reset map R defined by

$$\forall\, x \in K, \;\; R(x) \;:=\; x + S$$

We observe that $R^{-1}(y) = y - S$. A run $x(\cdot)$ of (F,R) where R is associated with a constant switching map $S \subset X$ is defined by: for all $j = 1, \ldots n-1$,

$$\begin{cases} i) \quad \forall\, t \in [t_{j-1}, t_j[, \;\; x(t) \;\in\; x_0 + (j-1)S + \int_0^t F(x(\tau))d\tau \\ ii) \quad x_j \;:=\; x(t_j) \;\in\; x_0 + jS + \int_0^{t_j} F(x(\tau))d\tau \end{cases}$$

Remark — If for some n, $t_m = t_n$ for all $m \geq n$, then the finite run stops at time t_n as a sequence of the discrete system $x(t_{m+1}) \in R(t_m)$. If the sequence of impulse times is finite and stop at time t_n, then the run ends as a solution to the differential inclusion on the interval $[t_n, T]$ if T is finite — this happens when $\lim_{t\mapsto T-} \|x(t)\| = +\infty$ — or on the interval $[t_n, +\infty[$ otherwise (the run does not blow up in finite time).

We can exclude the latter case by assuming that K is a repeller under F — in the sense that all solutions to the differential inclusion $x' \in F(x)$ starting from K leave K in finite time — and the former one by assuming that K is a repeller under R — in the sense that all sequences to the discrete inclusion $x_{n+1} \in R(x_n)$ starting from K leave K in finite time.

Linear growth conditions on F is one of the sufficient conditions forbidding the explosion of solutions in finite time, an assumption that we shall make throughout.

1.2 The Initialization Map

The behavior of a run is "summarized" by the "initialization map" $U := U_{(F,R)}$ associating with each initial condition $x_0 \in K$ the set of new initialized conditions $x_1 \in R(x(^-t_1))$ when $x(\cdot)$ ranges over the set of solutions to the differential inclusion $x' \in F(x)$ viable in K until they reach $R^{-1}(K)$ at time $t_1 \geq 0$ at $x(^-t_1) \in R^{-1}(K)$.

Indeed, the sequence of successive initial conditions x_n of a viable run $x(\cdot)$ of the impulse differential inclusion (F, R) — constituting the "discrete component of the run" — is governed by the discrete system $x_n \in U_{(F,R)}(x_{n-1}) \cap K$ starting at x_0. The knowledge of the sequence of initialized states x_n allows us to reconstitute the "continuous component" of the run by solving the differential inclusion $x' \in F(x)$ starting at each reinitialized state x_n.

We need to introduce some definitions and notations.

We shall denote by $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0,\infty,X)$ the solution map (or set-valued flow) mapping an initial state x to the set $\mathcal{S}_F(x)$ of solutions to the differential inclusion $x' \in F(x)$ starting at x and by $\mathcal{S}_F^K : K \rightsquigarrow \mathcal{C}(0,\infty,K)$ the viable solution map mapping an initial state $x \in K$ to the set $\mathcal{S}_F^K(x)$ of solutions to the differential inclusion $x' \in F(x)$ starting at $x \in K$ and viable in K.

We next denote by $\vartheta_F^K(t,x) := \bigcup_{x(\cdot)\in\mathcal{S}_F^K(x)}\{x(t)\}$ & $\vartheta_F^K(t,C) := \bigcup_{x\in C}\vartheta_F^K(t,x)$ the K-viable reachable maps of $x \in K$ and $C \subset K$ respectively. We set $\vartheta_F := \vartheta_F^X$ when there are no viability constraints.

Definition 1.2.1 *Let $C \subset K \subset X$. We shall set*

$$\Gamma_C^K(t,x) \ := \ C \cap \vartheta_F^K(t,x)$$

and

$$\mathbf{T}_C^K(x) \ := \ \{t \geq 0 \mid \Gamma_C^K(t,x) \neq \emptyset\}$$

We see at once that

$$\mathbf{T}_C^K(x) \subset [\omega_C^{F^\flat}(x), \tau_C^{F^\sharp}(x)] \subset [\omega_C^{F^\flat}(x), \tau_K^{F^\sharp}(x)]$$

Definition 1.2.2 *Let (F,R) describe an impulsive differential inclusion. The set-valued map $\Gamma^K := \Gamma_{(F,R)}^K : K \rightsquigarrow K \cap R^{-1}(K)$ is defined by*

$$\Gamma_{(F,R)}^K(t,x) := \Gamma_{R^{-1}(K)}^K(t,x) = \vartheta_F^K(t,x) \cap R^{-1}(K)$$

of the elements c of the stopping set $K \cap R^{-1}(K)$ through which the solutions to the differential inclusion $x' \in F(x)$ starting at x and viable in K until they reach $R^{-1}(K)$. We set

$$\mathbf{T}_{(F,R)}^K(x) := \{t \geq 0 \quad \text{such that } \Gamma_{(F,R)}^K(t,x) \neq \emptyset\}$$

We associate with the dynamics (F,R) of the impulse differential inclusion the initialization map $U_{(F,R)} : K \rightsquigarrow X$

$$U_{(F,R)}(x) = \bigcup_{t \in \mathbf{T}_{(F,R)}^K(x)} R(\Gamma_{(F,R)}^K(t,x))$$

and we shall set $U := U_{(F,R)}$ when no ambiguity arises.

First, we single out the following property:

Proposition 1.2.3 *A subset K is viable under the impulse differential inclusion (F,R) if and only if it is discretely viable under the initialization map $U_{F,R}$.*

Proof — Assume that K is viable under (F,R) and prove that K is viable under $U_{(F,R)}$. Take any $x_0 \in K$. By definition, there exists a run $x(\cdot)$ associated with a sequence $\mathcal{T}(x(\cdot)) := \{t_n\}$ of impulse times viable in K. Then the sequence $\vec{x} : n \to x(t_n)$ is a solution of the discrete dynamical system $U_{(F,R)}$, obviously viable in K.

Conversely, assume that K is viable under $U_{(F,R)}$ and prove that K is viable under impulse differential inclusion (F,R). Let x_0 given in K and a solution $\vec{x} : n \to x_n \in U_{(F,R)}(x_{n-1}) \cap K$ to the discrete dynamical system $U_{(F,R)}$.

By definition of the initialization map $U_{(F,R)}$, one can associate with $c_{n-1} \in R^{-1}(x_n) \cap \vartheta_F^K(\tau_{n-1}, x_{n-1})$ where $\tau_n \geq 0$ and $c_{n-1} = x_n(\tau_{n-1})$ is the value at time

$\tau_{n-1} \in \mathbf{T}^K_{R^{-1}(x_n)\cap K}(x_{n-1}) \subset \mathbf{T}^K_{(F,R)}(x_n)$ of a solution $x_n(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at time t_{n-1} from x_{n-1}. Setting $t_n := t_{n-1}+\tau_{n-1}$ and $x(t) := x_n(t)$ if $t \in [t_{n-1}, t_n]$, we have checked that $x(\cdot)$ is a run to the impulse differential inclusion (F, R) associated with the sequence $\{t_n\}_{n\geq 0}$ of impulse times t_n starting from x_0 and viable in K. □

In other words, knowing the initialization map $U_{(F,R)}$, we can reconstruct a viable run of the impulse differential inclusion (F, R) by

1. taking for sequence of initialized states $x(t_n) := x_n$ a sequence $\vec{x} = \{x_n\}$ of the discrete system $U_{(F,R)}$,
2. a sequence of impulse times $t_n \in t_{n-1} + \mathbf{T}^K_{R^{-1}(x_n)\cap K}(x_{n-1})$,
3. and, for every n and every $t \in [t_{n-1}, t_n]$, a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting from x_{n-1} at time t_{n-1} reaching $c_n \in R^{-1}(x_n)$ at time $t_n \geq t_{n-1}$.

Remark: Poincaré Return Map — The initialization map $U_{(F,R)}$ plays in some way the role of a return map à la Poincaré: Actually, the Poincaré return map of a differential inclusion $x' \in F(x)$ is the initialization map of the special impulse differential inclusion $(F, \mathbf{1}_C)$ where $C \subset X$ is a hyperplane and $\mathbf{1}_C$ the map defined by

$$\mathbf{1}_C(x) \;:=\; \begin{cases} x & \text{if} \quad x \in C \\ \emptyset & \text{if} \quad x \notin C \end{cases}$$

because in this case $U_{(F,\mathbf{1}_C)}(x) = \vartheta^K_F(t, x) \cap C$. The value $U_{(F,\mathbf{1}_C)}(x)$ provides all the points of the trajectories of the solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ crossing the hyperplane C.

For defining the Poincaré first-return map, we need to use the concept of chronector defined below. □

1.3 Hybrid Differential Inclusions

"Hybrid differential inclusion", as they are called by engineers, or "multiple-phase dynamical economies", as they are called by economists — may be regarded as auxiliary impulse differential inclusions.

Definition 1.3.1 *An hybrid differential inclusion (K, F, R) is defined by*

1. *a finite dimensional vector space E of states e called* locations*,*
2. *a set-valued map $K : E \rightsquigarrow X$ associating with any location e a (possibly empty) subset $K(e) \subset X$*
3. *a set-valued map $F : \mathrm{Graph}(K) \rightsquigarrow X$ with which we associate the differential inclusion $x'(t) \in F(e, x(t))$,*
4. *a set-valued map $R : \mathrm{Graph}(K) \rightsquigarrow E \times X$*

Definition 1.3.2 *We shall say that a (set-valued) map $t \mapsto x(t) \in X$ discontinuous at points $t_0 := 0 \le t_1 \le \cdots \le t_n \le \cdots$ is a* run *of an hybrid differential inclusion if for each impulse time t_n*

1. *either $t_{n+1} = t_n$, $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ and $x(t_{n+1}) \in K(e(t_{n+1}))$,*
2. *or $t_{n+1} > t_n$, and, for all $t \in [t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x'(t) \in F(e(t_n), x(t))$ viable in $K(e(t_n))$ and we take $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ and $x(t_{n+1}) \in K(e(t_{n+1}))$.*

A map $x(\cdot)$ is a run of the hybrid differential inclusions if and only if $(e(\cdot), x(\cdot))$ is a run of the following auxiliary system of impulse differential inclusions

$$\begin{cases} i) & e'(t) \; = \; 0 \\ ii) & x'(t) \; \in \; F(e(t), x(t)) \end{cases}$$

viable in $\mathrm{Graph}(K)$.

Indeed the locations remaining constant in the intervals $[t_n, t_{n+1}[$ since their velocities are equal to 0.

Example: Case of a Discrete Set of Locations We shall say a the domain of the set-valued map K is discrete if $\mathrm{Dom}(K) := \{e_i\}_{i=1,\ldots,n,\ldots}$ is denumerable. Then

$$\forall \, i = 1, \ldots, n, \ldots, \; \exists \, \varepsilon_i > 0 \mid B(e_i, \varepsilon_i) \cap (\mathrm{Dom}(K) \backslash \{e_i\}) \; = \; \emptyset$$

A set-valued map $e \in E \rightsquigarrow K(e) \subset X$ is defined on a discrete set $\mathrm{Dom}(R)$ if

$$\forall \, e \in E \backslash \mathcal{E}, \;\; K(e) \; = \; \emptyset$$

□

1.4 Examples

1.4.1 Discrete Dynamical Systems

One of the first questions that arises is whether we can "compare with a continuous system" a discrete dynamical system

$$\forall\, n \geq 0, \;\; x_{n+1} \;\in\; \Phi_n(x_n)$$

where for all $n \geq 0$, $\Phi_n : X \rightsquigarrow X$. This is naturally possible by associated with such a solution $\vec{x} := \{x_n\}$ of this discrete dynamical system, and thus, by "regarding" such a discrete system as an impulse differential inclusion. For that purpose, we introduce a strictly increasing sequence $\mathcal{T} := \{t_n\}_{n\geq 0}$ such that $\lim_{n\to+\infty} t_n = +\infty$ "realizing a time implementation" of the above discrete dynamical system.

Next, we introduce

1. the new state space $\mathbf{R} \times X$ and $K := \mathbf{R}_+ \times X$,
2. the dynamical system $F(t,x) \;:=\; (1,0)$,
3. the reset map

$$R(t,x) \;:= \begin{cases} \emptyset & \text{if} \;\; t \notin \mathcal{T} \\ \{t_n\} \times \Phi_n(x) & \text{if} \;\; t := t_n \in \mathcal{T} \end{cases}$$

The runs of the above impulse differential inclusion are the maps $t \mapsto (t, x(t))$ associated with the (prescribed) impulse times $t_n \in \mathcal{T}$ such that

$$\forall\, n \geq 0,\; \forall\, t \in [t_n, t_{n+1}[,\;\; x(t) = x_n \;\in\; \Phi_{n-1}(x_{n-1})$$

In other words, $x(t)$ is the piecewise constant function on each interval $[t_n, t_{n+1}[$ the values of which are governed by the discrete dynamical system.

1.4.2 Hybrid "Structural" Systems

The problem now is whether one can combine a discrete dynamical system

$$\forall\, n \geq 0, \;\; x_{n+1} \;\in\; \Phi_n(x_n)$$

and a differential inclusion

$$y'(t) \in F(t, x_n, y(t))$$

where $F : \mathbf{R}_+ \times X \times Y \rightsquigarrow Y$ governs the evolution of a state variable $y(\cdot)$ depending upon the discrete evolution of the "causal" or "structuring" variables x_n, whereas the evolution of the state variable $y(t)$ does not influence the evolution of the causal variables x_n.

As above, we introduce a strictly increasing sequence $\mathcal{T} := \{t_n\}_{n\geq 0}$ such that $\lim_{n\to+\infty} t_n = +\infty$ "realizing a time implementation" of the above discrete dynamical system.

Next, we introduce

1. the new state space $\mathbf{R} \times X \times Y$ and $K := \mathbf{R}_+ \times X$,
2. the dynamical system $\widehat{F}(t,x,y) \; := \; \{1\} \times \{0\} \times F(t,x,y)$,
3. the reset map

$$\widehat{R}(t,x,y) \; := \left\{ \begin{array}{ll} \emptyset & \text{if} \;\; t \notin \mathcal{T} \\ \{t_n\} \times \Phi_n(x) \times \{y\} & \text{if} \;\; t := t_n \in \mathcal{T} \end{array} \right.$$

A run $t \mapsto \widehat{x}(t) := (t, x(t), y(t))$ of the above auxiliary hybrid differential inclusion is thus defined by

$$\forall\, n \geq 0, \forall\, t \in [t_n, t_{n+1}[, \left\{ \begin{array}{lcl} x(t) & = & x(t_n) \\ y(t) & \in & y(t_n) + \int_{t_n}^{t} F(\tau, x(t_n), y(\tau)) d\tau \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} x(t_{n+1}) & \in & \Phi_n(x(t_n)) \\ y(t_{n+1}) & = & y(^{-}t_{n+1}) \end{array} \right.$$

The viability issues of structural hybrid systems involve closed subsets $\mathcal{H} \subset \mathbf{R} \times X \times Y$, which can be regarded as graphs of either set-valued maps $H : (t,x) \in \mathbf{R} \times X \rightsquigarrow H(t,x) \subset Y$ or of set-valued maps $U : (t,y) \in \mathbf{R} \times Y \rightsquigarrow U(t,y) \subset X$ defined by

$$y \;\in\; H(t,x) \;\text{ if and only if }\; x \in U(t,y) \;\text{ if and only if }\; (t,x,y) \;\in\; \mathcal{H}$$

The viability property — also called the tracking property in the control literature — requires that starting at time $t_0 = 0$, at x_0 and $y_0 \in H(0, x_0)$, the evolutions $n \mapsto x_n$ of the discrete system and $t \mapsto x(t)$ of the continuous system are related by

$$\forall \;\; \geq 0, \left\{ \begin{array}{l} \forall\, t \in [t_n, t_{n+1}[, \;\; y(t) \in H(t, x_n) \\ y(t_{n+1}) \;\in\; H(t_{n+1}, x_{n+1}) \text{ where } x_{n+1} \;\in\; \Phi_n(x_n) \end{array} \right.$$

(the evolution of the discrete system is tracked by the one of the continuous system) or, equivalently, by

$$\forall \ \geq 0, \begin{cases} \forall\, t \in [t_n, t_{n+1}[, \quad x_n \in U(t, y(t)) \\ x_{n+1} \ \in \ U(t_{n+1}, y(t_{n+1})) \cap \Phi_n(x_n) \end{cases}$$

(the evolution of the continuous system is tracked by the one of the discrete system). See for instance Chapter 8 of [8, Aubin] for pairs of differential inclusions.

When H (or U) is surjective, the continuous system (or the discrete system) is called an exosystem in the control literature or an abstraction (as in [171, Pappas & Sastry] and [173, Pappas] for instance) of the other system.

1.4.3 Finite-Difference Approximations

Given a increasing sequence $\mathcal{T} := \{t_n\}_{n\geq 0}$ of discrete times, we can regard the Euler polygonal approximations of solutions

$$\forall\, n \geq 0,\ \forall\, t \in [t_n, t_{n+1}[, \quad x(t) \ \in \ x(t_n) + (t - t_n)F(t_n, x(t_n))$$

of differential inclusions $x' \in F(t,x)$. Such a solution is the component $x(t)$ of the run $(t \mapsto t, x(t), y(t))$ of the hybrid differential inclusion $(\widehat{F}, \widehat{R})$ defined by

$$\widehat{F}(t,x,y) \ := \ (1, y, 0)$$

and

$$\begin{cases} \text{if} & t \notin \mathcal{T}, & \widehat{R}(t,x,y) \ := \ \emptyset \\ \text{if} & t := t_n \in \mathcal{T} & \\ \text{and if} & (t_n, x + (t_{n+1} - t_n)y, y) \notin \mathrm{Graph}(F), & \widehat{R}(t,x,y) \ := \ \emptyset \\ \text{if} & t := t_n \in \mathcal{T} & \\ \text{and if} & (t_n, x + (t_{n+1} - t_n)y, y) \in \mathrm{Graph}(F), & \\ & \widehat{R}(t,x,y) \ := \ (t_n, x + (t_{n+1} - t_n)y, y) & \end{cases}$$

1.5 Propagation of Nervous Influx in a Neuron

This section deals with the behavior of the propagation of the nervous influx along the ionic channels of a neuron and along a network of such biological neurons.

We postpone the problem: Why do we need to study, model or simulate the propagation of the nervous influx ? We shall answer this question in the framework of adaptation.

For the time, we shall present a brief summary of what is known about this topic. Research has been mainly conducted in two directions:

1. the theories of Louis Lapicque in 1907 (see [146, Lapicque]) and Hodgkin-Huxley equations (see [138, Hodgkin-Huxley]) in 1958 using metaphors from electricity,
2. formal neuron networks, that has been taken over by computer scientists for pattern recognition and data analysis purposes (see for instance [9, Aubin] for more references).

Actually, we shall bypass these "classical" studies and attempt to present mathematical metaphors of the propagation of the nervous influx in the framework of the very recent field of impulse differential equations and differential inclusions and hybrid systems. Indeed, we shall regard a biological neuron as a network of ionic channels, and thus, a network of biological neurons as a network of networks of ionic channels.

Roughly speaking, the propagation of the nervous impulse along a neuron is actually the propagation of ions along each side of the membrane of the neuron. When the number of ions at the entrance of a channel is high enough, then the channel opens and, instantaneously, the ions cross the channel (impulse), resetting new initial conditions at the channel, from which ions propagates until they reach the next ionic channel.

The same phenomenon happens for the propagation of the nervous influx from one presynaptic neuron to a postsynaptic neuron through a synapse. When the number of neurotransmitters at the presynaptic neuron is high enough, they are instantaneously released in the synapse, that they cross until they reach the receptors of the postsynaptic neuron. Then the postsynaptic neuron is depolarized, triggering the propagation of ions along it.

Actually, this falls in an even basic problem, that we shall call molecular semiology.

1.5.1 Biological Semiology

We start from the fact that an organism must **adapt itself to environment constraints**, by perceiving it and recognizing it through "metaphors" with "conceptual controls".

Even if we assume that the basic principle is simple (biological communication), its organization became more and more complex in the course of evolution. Indeed, the improvement consists in adding new structures to old ones, without always destroying them.

The understanding and mastering of the basic structures may someday allow a possible computer translation by economizing the useless part of complexity which results from (non teleological) biological evolution, retaining only what was relevant to allow adaptation.

This problem of adaptation is not taken into account explicitly in most studies of neural networks.

This chemical communication process, by sending macromolecule from an emitter cell to a receptor cell, the emission and the reception of such a molecule modifying the chemical properties of these cells, could constitute a relevant framework to many biological phenomena at several levels.

At the basic level, we face a biological molecular semiology where messages are sent

1. from proteins to proteins
2. from nucleic acids (DNA and RNA) to proteins
3. from proteins to nucleic acids (DNA and RNA) in order to activate some sites of the DNA

We shall deal here only with the communications between proteins, leaving aside the other two modes of communication involved in the genetic and epigenetic components of the molecualr semiologic system.

One can consider "adaptive systems" as systems formed of receptors (of the environment), effectors (on the environment) and intermediate (or processing) cells. Their behavior consists in acting at each instant to turn an unfavorable state of the environment into a more favorable one.

The perception of the environment by the emitters provokes the transmission of molecular messages towards the cells of the system.

These messages go from one cell to the others, the reception by an intermediate cell inducing the emission of another message.

The action starts after the reception of the message by the effectors. It modifies the state of the environment which is again perceived by the system and leads to a new action by the system.

At this level, the endocrine system and the nervous system may be distinguished by the mode of transport of the chemical messenger. This transport is slow and non-specific in the case of the endocrine system : hormones are carried by the blood;

it is fast and specific in the case of the nervous system : neurotransmitters have to cross a synapse (the place where the axon of a "presynaptic" neuron meets the dendrite of a "postsynaptic" one), which is only 0.02μ wide. When the pulse-coded information sent by the presynaptic neuron reaches a certain threshold value, it releases neurotransmitters in the synapse, inducing an electrical response on the postsynaptic membrane after about 10^{-3} seconds.

1.5.2 The Hodgkin-Huxley Equations

Since it is assumed that the membrane capacity per unit area is constant in most cases, we shall take it equal to 1 in the sequel. The state variable denoted by x denotes the displacement of the membrane potential from its resting value (negative displacement describes depolarization). The Hodgkin-Huxley equation states that the total membrane current is the sum of an ionic current $f(t)$ and of a capacity current $C\frac{dx}{dt}$. The summation means that these two currents are in parallel.

The ionic current is further split as the sum of currents $g_k(x - \xi_k)$ where the coefficients g_k are ionic conductances measuring the permeability of the membrane at ionic channels k and where the ξ_k are the equilibrium potentials for the ions (sodium and potassium).

The propagation of the nervous influx along the axon of a neuron obeys Hodgkin-Huxley equation of the form

$$\frac{dx}{dt} = \sum_{k=1}^{K} g_k(x(t), u_k(t))(x(t) - \xi_k) + f(t)$$

the indices k denote the ionic channels, the u_k the proportion of molecules circulating in the kth channel, and $x - \xi_k$ the difference of potential at the kth channel.

Usually, the evolutions of the u_k obey equations of the form

$$\frac{du_k}{dt} = \alpha_k(x)(1 - u_k) - \beta_k(x)u_k$$

where the functions α_k and β_k are transfer rates in the two opposite directions of the ionic channel and the u_k the proportion of molecules inside the axon.

We quote from [138, Hodgkin & Huxley]: Our experiments suggest that the ionic conductances are functions of time and membrane potential ... The influence of membrane potential can be summarized by stating

- first, that depolarization causes a transient increase in sodium conductance and a slower but maintained increase in potassium conductance;

- secondly, that these changes are graded and that they can be reversed by repolarizing the membrane.

1.5.3 Neurons as Networks of Channels

It has been observed that when $x(t)$ reaches a threshold $x(^{-}t_1) = y$ at impulse time t_1, then both the membrane potential and the ionic conductances are reset instantaneously at new values and the neuron triggers the release of some neurotransmitters.

Instead of designing a continuous model showing continuous impulses, we suggest to design the discrete resetting laws to be coupled with the continuous law governing the evolution of the membrane potential.

This leads us to consider a neuron as a network of ionic channels k on the membrane of a neuron. We denote by a_k the number of molecules inside the neuron at channel k, by b_k the number of molecules outside the neuron and $x_k := \varphi_k(a_k, b_k)$ the "difference of potential" at the kth channel of the neuron (for instance $\varphi_k(a_k, b_k) = a_k - b_k$).

The channel is open if $x_k \geq y_k$ and closed otherwise, where y_k is a threshold. In this case, the molecules travel instantaneously across the channel according to the discrete rule

$$(a_k(t), b_k(t)) \;:=\; \rho_k(a_k(^{-}t), b_k(^{-}t))$$

Otherwise, the propagation of molecules at each side of the membrane is described by

$$\left\{ \begin{array}{lll} i) & a_k'(t) & = \sum_l u_k^l a_l(t) \\ ii) & b_k'(t) & = \sum_l v_k^l b_l(t) \end{array} \right. \tag{1.3}$$

Therefore, starting with an initial condition, a "run" $x(t) := (x_k(t))_k$ evolves according to the system of differential equations (1.3). It is associated with a sequence of impulse times $t_0 = 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ at which the potential is reset according to the rule

$$\forall\, k, \;\; (a_k(t_n), b_k(t_n)) \;:=\; \rho_k(a_k(^{-}t_n), b_k(^{-}t_n))$$

whenever for an index k, $x_k(^{-}t_n) = y_k$.

Observe that, at impulse time t_n, only the channels $k \in \Theta((a_k(^{-}t_n), b_k(^{-}t_n)))$ such that $\varphi_k(a_k(^{-}t_n), b_k(^{-}t_n)) = y_k$ are open at time t_n, whereas the molecules $(a_k(t), b_k(t))$ continue to evolve continuously across time t_n.

Hence, given

1. the depolarizing constraints $x_k = y_k$ indicating the threshold when the molecules cross the ionic channel k

2. the continuous evolution law of propagation of molecules inside and outside the neuron

3. the discrete reset law giving the new state of molecules crossing each channel at the impulse time when the threshold is reached

we obtain an impulse control system.

Now, we identify presynaptic and postsynaptic synapses as special channels receiving and emitting molecules respectively: Denoting by E and S the sets of presynaptic and postsynaptic synapses, given an initial state of neurotransmitters at the presynaptic synapses at initial time 0, one can determine the values of the "runs"

$$(a_k(t), b_k(t))_{k\in S} \ = \ \vartheta(t, (a_{j_0}, b_{j_0})_{j\in E})$$

at the postsynaptic synapses determined by the impulse control system

$$\forall\, j \in E, \ \ (a_j(0), b_j(0)) \ = \ (a_{j_0}, b_{j_0})$$

at the synapses of the presynaptic neurons.

We associate with any t the set $\Theta_S(t, (a_{j_0}, b_{j_0})_{j\in E})$ of postsynaptic synapses $k \in S$ such that

$$\varphi_k(a_k(t), b_k(t)) \ = \ x_k(t) \ = \ \vartheta_k$$

which release neurotransmitters at time t.

In summary, the map $\Theta_S(t, \cdot)$ maps the initial presynaptic activity $(a_{j_0}, b_{j_0})_{j\in E}$ to the postsynaptic activity

$$(a_k(t), b_k(t))_{k\in S} \ := \ \Theta_S(t, (a_{j_0}, b_{j_0})_{j\in E})$$

at each instant t, describing which are the neurons firing at this instant, if any.

These evolutionary systems are particular instances of "threshold systems" that we next define.

Chapter 2

The Basic Theory

2.1 Characterization of Impulse Viability

Definition 2.1.1 *We shall say that a subset K is viable under an impulse differential inclusion (F, R) if from any initial state x of K starts at least one run viable in K.*

We shall say that a subset K is invariant under an impulse differential inclusion (F, R) if from any initial state x of K, all runs starting from x are viable in K.

2.1.1 Impulse Viability

Let us consider the dynamics (F, R) of an impulse differential inclusion and a closed subset $K \subset X$.

1. From any initial state $x_0 \in \mathrm{Viab}_F(K)$ starts at least a solution to the differential inclusion $x' \in F(x)$ viable in K, and thus, a "continuous time-dependent" run,

2. From any initial state $x_0 \in Viab_R^N(K)$ starts at least a solution to the discrete system $x_{n+1} \in R(x_n)$ viable in K for N iterations,

so that what remains to study for really hybrid runs is what happens when the initial state ranges over $K \backslash (\mathrm{Viab}_F(K) \cup Viab_R^1(K)) = K \backslash (\mathrm{Viab}_F(K) \cup R^{-1}(K))$:

We may regard the 1-discrete viability kernel $Viab_R^1(K) = K \cap R^{-1}(K)$ as the target of the impulse differential inclusion.

Theorem 2.1.2 *Let (F, R) be an impulse differential inclusion and $K \subset X$ be a closed subset. Assume that F is Marchaud and that R is upper semicontinuous with compact images*[1]*. Then the following statements are equivalent*

1. *K is viable under (F, R),*
2. *$K \backslash R^{-1}(K)$ is locally viable under F,*
3. *K, F and R are linked through the tangential condition*

$$\forall\, x \in K \backslash R^{-1}(K), \quad F(x) \cap T_K(x) \neq \emptyset$$

or, equivalently, in dual form, through the normal condition

$$\forall\, x \in K \backslash R^{-1}(K),\ \forall\, p \in N_K(x), \quad \sigma(F(x), -p) \ \geq\ 0$$

Proof — By the Viability Theorem, the two latter conditions are equivalent. It remains to prove that K is viable under (F, R) if and only if $K \backslash R^{-1}(K)$ is locally viable under F. This is what we shall prove next,

Indeed, if K is viable under (F, R), then from any $x_0 \in K \backslash R^{-1}(K)$ starts at least a solution to the differential inclusion $x' \in F(x)$ viable in K

1. either forever if x_0 belongs to the viability kernel $\text{Viab}_F(K)$ of K
2. or until it reaches at some time $t_1 > 0$ a state $x({}^-t_1)$ in $C := K \cap R^{-1}(K)$.

This shows that $K \backslash R^{-1}(K)$ is locally viable.

Conversely, let us assume that $K \backslash R^{-1}(K)$ is locally viable and take an initial state $x_0 \in K \backslash R^{-1}(K)$.

If x_0 belongs to $\text{Viab}_F(K)$, then at least one solution starting from x_0 is viable in K, and thus, defines a run viable in K.

If x_0 does not belong to $\text{Viab}_F(K)$, all solutions leave K in finite time before (possibly) reaching the viability kernel. It is then enough to prove that at least one of them reaches $R^{-1}(K)$ before leaving K. This is the case of a solution $x^\sharp(\cdot) \in \mathcal{S}_F(x)$ which maximizes $\tau_K(x(\cdot))$, i.e., which satisfies

$$\tau_K^{F^\sharp}(x) \ :=\ \sup_{x(\cdot) \in \mathcal{S}_F(x)} \tau_K^F(x(\cdot)) \ =\ \tau_K^F(x^\sharp(x))$$

---

[1] This assumption implies that $R^{-1}(K)$ is closed, which is the property we really need. It remains true when we assume only that the subsets $K \cap (R(x) + B)$ are compact, where B denotes the unit ball.

leaves $K\backslash(\text{Viab}_F(K) \cup R^{-1}(K))$ through $R^{-1}(K)$. This solution exists by Proposition 6.1.24 since K is closed and F is Marchaud. Next, we claim that $x^\sharp := x^\sharp(\tau_K^{F^\sharp}(x)) \in K\backslash\text{Viab}_F(K)$. Otherwise, if $x^\sharp$ would belong to the viability kernel, it could be concatenated with a solution viable in K for ever, so that the initial state x_0 would belong the viability kernel, which is not the case.

Furthermore, $x^\sharp$ belongs to $R^{-1}(K)$. If not, $x^\sharp$ would belong to $K\backslash R^{-1}(K)$ which is assumed to be locally viable. Then one could associate with $x^\sharp \in K\backslash(\text{Viab}_F(K) \cup R^{-1}(K))$ a solution $y(\cdot) \in \mathcal{S}_F(x^\sharp)$ and $T > 0$ such that $y(\tau) \in K\backslash(\text{Viab}_F(K) \cup R^{-1}(K))$ for all $\tau \in [0, T]$. Concatenating this solution to $x^\sharp(\cdot)$, we obtain a solution viable in K on an interval $[0, \tau_K^{F^\sharp}(x) + T]$, which contradicts the definition of $x^\sharp(\cdot)$.

Therefore $x^\sharp$ belongs to $K \cap R^{-1}(K) = Viab_R^1(K)$ so that there exists $x_1^\sharp \in K \cap R(x^\sharp)$. If $x_1^\sharp$ belongs to $Viab_R(K)$, then at least a solution to the discrete system $x_{n+1} \in R(x_n)$ remains viable in K forever. If it belongs to some $Egress_R(K) := Viab_R^N(K)\backslash Viab_R^{N+1}(K)$, at least a solution to the discrete system $x_{n+1} \in R(x_n)$ remains viable in K for N iterations and $x_{N+1} \notin R^{-1}(K)$. Either $x_{N+1}^\sharp \in \text{Viab}_F(K)$, then a solution to the differential inclusion $x' \in F(x)$ starts from $x_1^\sharp$ at time $\tau_K^{F^\sharp}(x)$ and remains in K forever. Or it belongs to some $K\backslash(\text{Viab}_F(K) \cup R^{-1}(K))$, and from x_{N+1} starts at least a solution to the differential inclusion $x' \in F(x)$ viable in K until it reaches $R^{-1}(K)$. □

Remark: Other Equivalent Formulations of Viability — The characterizations of the above Theorem 2.1.2 are also equivalent to state that the *the (continuous) egress set of* K *under* F *and the (discrete) egress set* $K\backslash R^{-1}(K)$ *of* K *under* F *are disjoint*:

$$K\backslash R^{-1}(K) \cap \text{Egress}_F(K) = \emptyset$$

or, equivalently,

$$\text{Egress}_F(K) \subset R^{-1}(K)$$

Since the egress set of K under F is not easily characterized and does not enjoy nice topological properties, these characterizations are less useful than the ones provided by Theorem 2.1.2 which involve tangential or normal conditions. □

2.1.2 The Stability Theorem

Proposition 2.1.3 *If*

$$\text{Viab}(K) \cap R(K) = \emptyset \tag{2.1}$$

then the cadences τ_n *of a run* $x(\cdot)$ *starting from* $K \backslash \mathrm{Viab}(K)$ *and viable in* K *are finite.*

If we assume furthermore that $R(K) \cap K$ *is compact and that* F *is Marchaud, then there exist a finite scalar* $\overline{T} < +\infty$ *such that the cadences* τ_n *of any run* $x(\cdot)$ *starting from* $K \backslash R^{-1}(K)$ *and viable in* K *belong to the interval* $[0, \overline{T}]$.

Proof — Let us consider a run $x(\cdot)$ associated with sequences $\{\tau_n\}_n$ of cadences, $\{x_n\}_n \in K$ of reinitialized conditions and $\{x_n(\cdot)\}_n \in \mathcal{S}(x_n)$ of motives viable in K on the intervals $[0, \tau_{n+1}[$. Since any reinitialized state x_n belongs to $K \cap R(K)$, and thus, disjoint from $\mathrm{Viab}(K)$, all solutions $y(\cdot) \in \mathcal{S}(x_n)$ starting from x_n leave K in finite time. Therefore, since the run $x(\cdot)$ is viable in K, each motive $x_n(\cdot) \in \mathcal{S}(x_n)$ reaches $K \cap R^{-1}(K)$ in finite time τ_n before leaving K.

Now, if we assume that F is Marchaud, we introduce

$$\overline{T} \ := \ \sup_{x \in K \cap R(K)} \tau_K^\sharp(x)$$

By Theorem 6.1.24, the subset $K \cap R(K)$ being compact, the exit function being finite and upper semicontinuous, we infer that $\overline{T}$ is finite. □

Remark — Naturally, one cannot exclude that all cadences τ_n are equal to 0 for $n \geq N$ or that they congerge to 0. One can exclude the first case by assuming that K is a discrete repeller under R and the second one by using Theorem 2.2.1 or 2.2.2, for instance. □

I $(0, \infty; X) \ := \ [\mathbf{\Phi}_+$

Therefore, a run $x(\cdot) \in \mathcal{R}_{(\mathcal{S},R)}(x_0)$ can be regarded as a sequence

$$x(\cdot) \ := \ \{(\tau_n, x_n, x_n(\cdot))\}_{n \geq 0} \ \in \ \mathcal{I}(0, \infty; X) \ := \ [\mathbf{R}_+ \times X \times \mathcal{C}(0, \infty; X)]^{\mathbf{N}}$$

of initial states $x_n \in X$, of cadences τ_n and of motives $x_n(\cdot)$ satisfying, for every $n \geq 0$,

$$\begin{cases} i) & (x_n, x_n(\cdot)) \ \in \ \mathrm{Graph}(\mathcal{S}) \\ ii) & (x_n(\tau_n), x_{n+1}) \ \in \ \mathrm{Graph}(R) \end{cases}$$

and $x(0) \ = \ x_0$.

Theorem 2.1.4 *Let us assume that the subset* K *is closed, that* F *is Marchaud, that* R *is upper semicontinuous, that* $R(K)$ *is compact and that (2.1):*

$$\mathrm{Viab}(K) \cap R(K) \ = \ \emptyset$$

holds true. Then the solution map $\mathcal{R}^K_{(F,R)}$ *is upper semicompact on* $K \backslash \mathrm{Viab}(K)$.

Proof — Let us consider a sequence of initial states $x_0^\varepsilon \in K\backslash\text{Viab}(K)$ converging to $x_0 \in K\backslash\text{Viab}(K)$ and a sequence of runs

$$x^\varepsilon(\cdot) \;=\; \{(x_n^\varepsilon, \tau_n^\varepsilon, x_n^\varepsilon)\}_{n\geq 0} \;\in\; \mathcal{R}_{(\mathcal{S},R)}^K(x_0^\varepsilon)$$

viable in K.

We can identify the graph $\mathcal{G}$ of the solution map $\mathcal{R}_{(F,R)}^K$ to a subset

$$\mathcal{G} \;\subset\; K \times (K \times \mathbf{R}_+ \times \text{Graph}(\mathcal{S}))^{\mathbf{N}}$$

We supply it with the product topology. Under our assumptions, for any compact subset L of K, Theorem 6.1.6 implies that the graph of $\mathcal{S}|_L$ is compact. Hence the graph $\mathcal{G}_L$ of the restriction $\mathcal{R}_{(F,R)}^K|_L$ to L of the solution map $\mathcal{R}_{(F,R)}^K$ satisfies

$$\mathcal{G}_L \;\subset\; L \times ([0,\beta] \times \text{Graph}(\mathcal{S}|_L))^{\mathbf{N}}$$

which is a product of compacts. By the Tychonof Theorem, it itself a compact subset.

Therefore, a subsequence (again denoted by) $\{(\tau_n^\varepsilon, x_n^\varepsilon, x_n^\varepsilon(\cdot))\}_{n\geq 0}$ converges to some sequence $\{(\tau_n, x_n, x_n(\cdot))\}_{n\geq 0}$. This means that for every $n \geq 0$, τ_n^ε converges to $\tau_n \in [0,\beta]$ and $(x_n^\varepsilon, x_n^\varepsilon(\cdot)) \in \text{Graph}((\mathcal{S}))$ to some $(x_n, x_n(\cdot)) \in \text{Graph}((\mathcal{S}))$ Consequently, the sequences $x_n^\varepsilon(\tau_n^\varepsilon)$ converge $x_n(\tau_n)$ and thus, since the graph of the reset map R is closed, inclusions

$$\forall\, n \geq 0,\;\; (x_{n+1}^\varepsilon, x_n^\varepsilon(\tau_n^\varepsilon)) \;\in\; \text{Graph}(R)$$

imply that

$$\forall\, n \geq 0,\;\; (x_{n+1}, x_n(\tau_n)) \;\in\; \text{Graph}(R)$$

Therefore, the sequence $\{(x_n, \tau_n, x_n(\cdot))\}_{n\geq 0}$ defines a run $x(\cdot)$ of the impulse differential inclusion (F,R) viable in K. □

2.1.3 Hybrid Viability

Let us consider a hybrid differential inclusion (K,F,R) (see Definition 1.3.1).

We first observe that

$$\text{Viab}_{\{0\}\times F}(\text{Graph}(K)) \;=\; \text{Graph}(e \rightsquigarrow \text{Viab}_{F(e,\cdot)}(K(e)))$$

and that the graph of K is a repeller if and only if for every e, the set $K(e)$ is a repeller under $F(e,\cdot)$.

On the other hand, we introduce the decreasing sequence of set-valued maps $K_N : E \rightsquigarrow X$ defined by $K_0 = K$ and

$$\forall\, N \geq 1,\ \ \text{Graph}(K_N) \ := \ Viab_R^N(\text{Graph}(K))$$

We observe that whenever $R(e,x) := R_E(e) \times R_X(x)$, the maps K_N can be defined recursively through the formulas

$$\forall\, e \in E,\ \ K_{N+1}(e) \ = \ K_N(e) \cap R_X^{-1}(K_N(R_E(e))$$

We begin by providing necessary and condition for the existence of solutions to discrete hybrid differential inclusions:

Theorem 2.1.5 *Assume that the domain* $\text{Dom}(K) := \{e_i\}_{i=1,\ldots,n}$ *is a discrete set. Hence the discrete hybrid differential inclusion* (F,R) *has a solution for every initial state if and only if*

$$\forall\, i = 1,\ldots,n,\ \forall\, x \in K(e_i)\backslash K_1(e_i),\ \ F(e_i,x) \cap T_{K(e_i)}(x) \neq \emptyset$$

For nondiscrete hybrid differential inclusions, we need the definition of the contingent derivative $DK(e,x) : E \rightsquigarrow X$ of a set-valued map $K : E \rightsquigarrow X$ at a point (e,x) of its graph: It can be defined by

$$\text{Graph}(DK(e,x)) \ := \ T_{\text{Graph}(K)}(e,x)$$

In the case of hybrid differential inclusions, we can apply Theorem 2.1.2 for characterizing the existence of solutions of hybrid differential inclusions:

Theorem 2.1.6 *Let* (K,F,R) *be a hybrid differential inclusion. Assume that* F *is Marchaud and that* R *is upper semicontinuous with compact images. Then the hybrid differential inclusion has a solution for every initial state if and only if*

$$\forall\, e \in E, \forall\, x \in K(e)\backslash K_1(e),\ \ F(e,x) \cap DK(e,x)(0) \neq \emptyset$$

It remains to prove Theorem 2.1.5:
Proof of Theorem 2.1.5 — Since the domain $\text{Dom}(K) := \{e_i\}_{i=1,\ldots,n,\ldots}$ is a discrete set, then

$$\forall\, i = 1,\ldots,n,\ldots,\ \exists\, \varepsilon_i > 0 \mid B(e_i,\varepsilon_i) \cap (\text{Dom}(K)\backslash\{e_i\}) \ = \ \emptyset$$

Therefore, one observes readily that

$$\forall\, i = 1, \ldots, n, \ldots,\ \forall\, x \in K(e_i),\ \ DK(e_i, x)(0) \ = \ T_{K(e_i)}(x)$$

so that the hybrid differential inclusion has a solution for every initial state if and only if

$$\forall\, i = 1, \ldots, n, \ldots,\ \forall\, x \in K(e_i) \backslash K_1(e_i),\ \ F(e_i, x) \cap T_{K(e_i)}(x) \neq \emptyset$$

2.1.4 Characterization of Viable Qualitative Systems

Hence, we can apply Theorem 2.1.2 for characterizing the existence of solutions of qualitative systems:

Theorem 2.1.7 *Let (K, f, Q, R) be a qualitative system. Assume that f is continuous with linear growth and that the graphs of the qualitative map $Q : Z \rightsquigarrow X$ and the reset map $R : Z \rightsquigarrow Z$ are closed. Then the qualitative system has a solution for every initial state if and only if*

$$\forall\, u \in \mathrm{Dom}(Q), \forall\, x \in Q(u) \backslash Q(R(u)),\ \ f(x, u) \in DQ(x, u)(0)$$

When the domain $\mathrm{Dom}(Q) := \{u_i\}_{i=1,\ldots n,\ldots}$ *of Q is discrete, the above necessary and sufficient condition can be written*

$$\forall\, i = 1, \ldots, n, \ldots,\ \forall\, x \in Q(u_i) \backslash Q(R(u_i)),\ \ f(x, u_i) \in T_{Q(u_i)}(x)$$

2.1.5 Characterization of Viability in terms of Capture Basins

We now characterize the viability of a closed subset under an impulse differential inclusion in terms of capture basins:

Proposition 2.1.8 *Assume that $F : X \rightsquigarrow X$ is Marchaud and that K is a closed repeller. Then K is viable under the impulse differential inclusion (F, R) if and only if*

$$K \ = \ \mathrm{Capt}_F^K(R^{-1}(K) \cap K) \ = \ \mathrm{Capt}_F^K(\mathit{Viab}_R^1(K))$$

Proof — Assume that K is viable under (F, R). Then, for any $x_0 \in K$, either x_0 belongs to $R^{-1}(K)$ and we take $x_1 \in R(x_0) \cap K$ as the next initial state, or from x_0 starts one solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on some interval $[0,t[$ such that $R(x(^-t)) \cap K \neq \emptyset$. This means that $x(^-t)$ belongs to $R^{-1}(K) \cap K$ so that x_0 belongs to $\mathrm{Capt}_F^K(R^{-1}(K) \cap K)$. Hence

$$K \subset \mathrm{Capt}_F^K(R^{-1}(K) \cap K)$$

Conversely, if the above property holds true, from any $x_0 \in K$ starts a solution to the differential inclusion $x' \in F(x)$ reaching $R^{-1}(K) \cap K$ at some finite time $T \geq 0$ at some $x(^-T) \in R^{-1}(K) \cap K$ before leaving K. This implies that $R(x(^-T)) \cap K$ is not empty. On the other hand, the values $x(\tau)$ belong to $\mathrm{Capt}_F^K(R^{-1}(K) \cap K)$ when $\tau \in [0,T]$, and thus, to K. □

We observe that whenever K is backward invariant, then the capture basin $\mathrm{Capt}_F(R^{-1}(K) \cap K)$ of $R^{-1}(K) \cap K$ is contained in K and the capture basin of $R^{-1}(K) \cap K$ is the smallest backward invariant subset containing K. Therefore, in this case,

$$\mathrm{Capt}_F^K(C) = \mathrm{Capt}_F(C) := \mathrm{Capt}_F^X(C)$$

so that K is viable under (F,R) if and only if

$$K = \mathrm{Capt}_F(R^{-1}(K) \cap K)$$

We deduce the following consequence: Assume that $F : X \rightsquigarrow X$ is Marchaud and Lipschitz and that K is a closed repeller which is backward invariant under F. Then K is viable under the impulse differential inclusion (F,R) if and only if

$$\begin{cases} i) & \forall\, x \in K \backslash R^{-1}(K), \;\; F(x) \cap T_K(x) \neq \emptyset \\ ii) & \forall\, x \in K, \;\; F(x) \subset -T_K(x) \end{cases}$$

or, in a dual formulation,

$$\begin{cases} i) & \forall\, x \in K \backslash R^{-1}(K),\ \forall\, p \in N_K(x), \;\; \sigma(F(x), -p) = 0 \\ ii) & \forall\, x \in K,\ \forall\, p \in N_K(x), \;\; \sigma(F(x), -p) \leq 0 \end{cases}$$

2.2 Non Zeno Impulse Differential Inclusions

In order to avoid runs which are always solutions to the differential inclusion $x' \in F(x)$, we have to assume that $\mathrm{Viab}_F(K) = \emptyset$, i.e., that K is a repeller under F.

In order to avoid more than one iteration of the resetting process, we have to assume that $Viab_R^2(K) = \emptyset$.

If both these conditions are satisfied, then the sequence of impulse times is strictly increasing.

We shall prove that when the closed subset is compact and the 2-discrete viability kernel $Viab_R^2(K)$ is empty, every run of the impulse differential inclusion is a non Zeno run: the sequence of impulse times is (strictly) increasing and goes to infinity:

Theorem 2.2.1 *Let (F, R) be an impulse differential inclusion where F is Marchaud and that R is upper semicontinuous with compact images and K be a closed subset viable under (F, R). Assume further that K_1 is compact and that the 2-discrete viability kernel $Viab_R^2(K)$ is empty, every run of the impulse differential inclusion — which exists thanks to Theorem 2.1.2 — is a strict non Zeno run.*

Proof — Since $Viab_R^2(K)$ is empty, then $Viab_R^1(K) = Egress_R^2(K)$, so that for every $x \in Viab_R^1(K)$, $R(x) \cap Viab_R^1(K)$ is empty. Since R is assumed to be upper semicontinuous with compact images and since K is compact, we infer that

$$R(Viab_R^1(K)) \cap Viab_R^1(K) = \emptyset$$

Let us set $K_1 := Viab_R^1(K)$ and denote by $\omega_{K_1}^{F^\flat}$ the hitting function of K_1. Since F is Marchaud, Proposition 6.1.24 implies that it is lower semicontinuous. It is strictly positive on $E_0 := Egress_R(K) = K \backslash R^{-1}(K)(x)$. Since R is upper semicontinuous with compact images, the "Maximum Theorem" implies that $\varphi(x) := \inf_{y \in R(x)} \omega_{K_1}^{F^\flat}(y)$ is lower semicontinuous, and strictly positive on K_1. Since K_1 is compact, then $\overline{\tau} := \min_{x \in K_1} \varphi(x) > 0$.

Now, if $x(\cdot)$ is a run of the impulse differential inclusion (F, R), the sequence of initialized values $x(t_n) \in R(x(^-t_n))$ belongs to $R(K_1)$, so that $x(^-t_{n+1})$ reaches $K_1 := K \cap R^{-1}(K)$ at time t_{n+1} larger than or equal to $t_n + \overline{\tau}$. Therefore, the strictly increasing sequence of impulse times t_n cannot converge in finite time. □

More generally, is we assume only that K is a discrete repeller under R, the sequence of impulse times of every run is non increasing and cannot converge in finite time (such a run can be called a exhaustive or a non Zeno run):

Theorem 2.2.2 *Let (F, R) be an impulse differential inclusion where F is Marchaud and R be upper semicontinuous with compact images and K be a closed subset viable under (F, R). Assume further that K is a discrete repeller under R and that K_1 is compact. Then every run of the impulse differential inclusion — which exists thanks to Theorem 2.1.2 — is a non Zeno run.*

2.3 Characterization of the Substratum and the Initialization Map

2.3.1 Characterization of the Substratum

We begin by characterizing the graph of the substratum $\Gamma^K_{(F,R)}$:

Theorem 2.3.1 *Let us assume that F is Marchaud, that $C \subset R$ is closed and that the graph of $R : C \rightsquigarrow X$ is closed.*

Then the substratum $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ is the **unique** *set-valued map with closed graph satisfying*

$$\forall\, x \in K, \;\; \Gamma^K_{(F,R)}(0,x) \; := \; R(x) \cap K$$

and, for any $T > 0$

1. *for any $y \in \Gamma^K_{(F,R)}(T,x)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on $[0,T]$ such that*

$$\forall\, t \in [0,T], \;\; y \in \Gamma^K_{(F,R)}(T-t, x(t))$$

2. *for any $y \in K \backslash \Gamma^K_{(F,R)}(T,x)$, for every solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on $[0,T]$, then*

$$\forall\, t \in [0,T], \;\; y \in K \backslash \Gamma^K_{(F,R)}(T-t, x(t))$$

As a consequence, for any $T > 0$ and for any $y \in \partial_K \Gamma^K_{(F,R)}(T,x)$, for every solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on $[0,T]$, then

$$\forall\, t \in [0,T], \;\; y \in \partial_K \Gamma^K_{(F,R)}(T-t, x(t))$$

For proving Theorem 2.3.1, we shall first observe that the graph of the substratum of (K, F, R) is a viable-capture basin and next, deduce the above results from the characterization of viable-capture basins. Let us recall that we denoted by $R^{|K}_{|K}$ the graphical restriction of R to $K \times K$ defined by

$$R^{|K}_{|K}(x) \; := \begin{cases} R(x) \cap K & \text{if} \;\; x \in K \\ \emptyset & \text{if} \;\; x \notin K \end{cases}$$

the graph of which is equal to

$$\text{Graph}(R^{|K}_{|K}) \; = \; \text{Graph}(R) \cap (K \times K)$$

Lemma 2.3.2 *The graph of the substratum* $\Gamma^K_{(F,R)}$ *of* (K,F,R) *is the viable-capture basin of* $\{0\}\times \mathrm{Graph}(R^{|K}_{|K})$ *under the set-valued map* $\{-1\}\times F\times\{0\}$*:*

$$\mathrm{Graph}(\Gamma^K_{(F,R)}) \;=\; \mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}\left(\{0\}\times \mathrm{Graph}(R^{|K}_{|K})\right)$$

and $\forall\, x\in C,\;\; \Gamma^K_{(F,R)}(0,x) = R(x)\cap K$*.*

Proof — Indeed, to say (T,x,y) belongs to the viable-capture basin

$$\mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}\left(\{0\}\times \mathrm{Graph}(R^{|K}_{|K})\right)$$

means that there exists a solution $x(\cdot)\in\mathcal{S}_F(x)$ and $\bar t\in[0,T]$ such that

$$\begin{cases} i) & \forall\, t\in[0,\bar t],\;\; (T-t,x(t),y)\in \mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}(\{0\}\times \mathrm{Graph}(R^{|K}_{|K}) \\ ii) & (T-\bar t, y, x(\bar t)) \;\in\; \{0\}\times \mathrm{Graph}(R^{|K}_{|K}) \end{cases}$$

i.e., if and only if $\bar t = T$ and

$$\begin{cases} i) & \forall\, t\in[0,T[,\;\; x(t) \;\in\; K \\ ii) & y \;\in\; R(x(T))\cap K \end{cases}$$

This is equivalent to say that $y\in\Gamma^K_{(F,R)}(T,x)\cap K$.

Consequently, to say that y belongs to $\Gamma^K_{(F,R)}(0,x)$ means that $y\in R(x)\cap K$. □

Proof of Theorem 2.3.1 — We observe first that the map $\{-1\}\times F\times\{0\}$: $\mathbf{R}\times X\times X \rightsquigarrow \mathbf{R}\times X\times X$ is Marchaud and that $\mathbf{R}_+\times K\times K$ is a repeller under this map since any solution $(T-t,x(t),y)$ starting at (T,x,y) leaves $\mathbf{R}_+\times K\times K$ at time T. Theorem 6.3.10 states that the viable-capture basin

$$\mathrm{Graph}(\Gamma^K_{(F,R)}) \;=\; \mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}\left(\{0\}\times \mathrm{Graph}(R^{|K}_{|K})\right)$$

is the unique closed subset $\mathcal{V}\subset\mathbf{R}\times K\times K$ containing $\{0\}\times\mathrm{Graph}(R^{|K}_{|K})$ satisfying

1. $\mathcal{V}\backslash(\{0\}\times\mathrm{Graph}(R^{|K}_{|K}))$ is locally viable under $\{-1\}\times F\times\{0\}$
2. and
$$\mathrm{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}(\mathcal{V}) \;=\; \mathcal{V}$$
This states that whenever $(T,x,y)\in(\mathbf{R}_+\times K\times K)\backslash\mathcal{V}$, all solutions to the differential inclusion $(t',x',y')\in\{-1\}\times F(x)\times\{0\}$ leave $(\mathbf{R}_+\times K\times K)$ before possibly reaching the target $\{0\}\times\mathrm{Graph}(R^{|K}_{|K})$.

The first statement means that whenever (T, x, y) belongs to $\mathcal{V}$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ such that $(T-t, x(t), y)$ belongs to $\mathcal{V}$ until it reaches $\{0\} \times \text{Graph}(R_{|K}^{|K})$. This is equivalent to saying that

$$\forall\, t \in [0, T], \;\; y \in \Gamma_{(F,R)}^{K}(T-t, x(t))$$

The second statement means that whenever (T, x, y) does not belong to $\mathcal{V}$, all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ are such that $(T-t, x(t), y)$ do not belong to $\mathcal{V}$ whenever $(T-t, x(t), y) \in \mathbf{R}_+ \times K \times K$, i.e., whenever $x(\cdot)$ is viable in K on the interval $[0, T]$. This is equivalent to saying that for all solutions to $x' \in F(x)$ viable in K on the interval $[0, T]$,

$$\forall\, t \in [0, T], \;\; y \in K \backslash \Gamma_{(F,R)}^{K}(T-t, x(t))$$

Let us consider now $y \in \partial \Gamma_{(F,R)}^{K}(T, x)$ where $T > 0$. This means that there exists a sequence $y_n \in K$ such that $y_n \in K \backslash \Gamma_{(F,R)}^{K}(T, x)$ converges to y. Hence (T, x, y_n) does not belong to the capture basin of $\{0\} \times \text{Graph}(R_{|K}^{|K})$ viable in $\mathbf{R}_+ \times K \times K$. Therefore we know that for any solution $x(\cdot) \in \mathcal{S}(x)$ viable in K on $[0, T]$, for any $t \in [0, T]$, $y_n \in K \backslash \Gamma_{(F,R)}^{K}(T-t, x(t))$ and, in particular, that $y_n \in K \backslash \Gamma_{(F,R)}^{K}(0, x(T)) = R(x(T))$. Taking any solution $x(\cdot) \in \mathcal{S}(x)$ satisfying (**??**) and the limit when $n \to +\infty$, we infer that

$$\forall\, t \in [0, T], \;\; y \in \partial_K \Gamma_{(F,R)}^{K}(T-t, x(t))$$

and that

$$y \in \partial_K R(x(T))$$

2.3.2 Characterization of the Initialization Map

We proceed in the same way for characterizing the graph of the initialization map $U_{(F,R)}^{K}$:

Theorem 2.3.3 *Let us assume that F is Marchaud, that $C \subset R$ is closed and that the graph of R is closed and that $\text{Viab}_F(K) \subset R^{-1}(K)$.*

Then the initialization map $U_{(F,R)}^{K} : K \rightsquigarrow K$ is the **unique** *set-valued map with closed graph satisfying*

$$\forall\, x \in C, \;\; U_{(F,R)}^{K}(x) \;:=\; R(x) \cap K$$

and, for any $x \in K \backslash C$,

1. *for any* $y \in U^K_{(F,R)}(x)$*, there exist* $T \geq 0$ *and a solution* $x(\cdot)$ *to the differential inclusion* $x' \in F(x)$ *viable in* K *on* $[0,T]$ *such that*

$$\forall\, t \in [0,T], \;\; y \in U^K_{(F,R)}(x(t))$$

2. *for any* $y \in K\backslash U^K_{(F,R)}(x)$*, for every solution* $x(\cdot)$ *to the differential inclusion* $x' \in F(x)$ *viable in* K *on* $[0,T]$*, then*

$$\forall\, t \in [0,T], \;\; y \in K\backslash U^K_{(F,R)}(x(t))$$

As a consequence, for any $T > 0$ *and for any* $y \in \partial_K U^K_{(F,R)}(x)$*, for every solution* $x(\cdot)$ *to the differential inclusion* $x' \in F(x)$ *viable in* K *on* $[0,T]$*, then*

$$\forall\, t \in [0,T], \;\; y \in \partial_K U^K_{(F,R)}(x(t))$$

We shall first observe that the graph of the initialization map of (K,F,R) is a viable-capture basin and next, deduce the above results from the characterization of viable-capture basins.

Lemma 2.3.4 *The graph of the initialization map* $U^K_{(F,R)}$ *of* (K,F,R) *is the viable-capture basin of* $\mathrm{Graph}(R^{|K}_{|K})$ *under the set-valued map* $F \times \{0\}$*:*

$$\mathrm{Graph}(U^K_{(F,R)}) \;=\; \mathrm{Capt}^{K\times K}_{F\times\{0\}}\left(\mathrm{Graph}(R^{|K}_{|K})\right)$$

and $\forall\, x \in C, \;\; U^K_{(F,R)}(x) = R(x) \cap K$*.*

Proof — Indeed, to say (x,y) belongs to the viable-capture basin

$$\mathrm{Capt}^{K\times K}_{F\times\{0\}}\left(\mathrm{Graph}(R^{|K}_{|K})\right)$$

means that there exist $T \geq 0$ and a solution $x(\cdot) \in \mathcal{S}_F(x)$ and $\bar{t} \in [0,T]$ such that

$$\begin{cases} i) & \forall\, t \in [0,T[, \;\; (x(t),y) \in \mathrm{Capt}^{K\times K}_{F\times\{0\}}(\mathrm{Graph}(R^{|K}_{|K}) \\ ii) & (y, x(T)) \;\in\; \mathrm{Graph}(R^{|K}_{|K}) \end{cases}$$

i.e., if and only if

$$\begin{cases} i) & \forall\, t \in [0,T[, \;\; x(t) \;\in\; K \\ ii) & y \;\in\; R(x(T)) \cap K \end{cases}$$

This is equivalent to say that $y \in U^K_{(F,R)}(x) \cap K$. $\square$

Proof of Theorem 2.3.3 — We observe first that the map $F \times \{0\} : X \times X \rightsquigarrow \mathbf{R} \times X \times X$ is Marchaud.

Theorem 6.3.13 states that the viable-capture basin

$$\mathrm{Graph}(U^K_{(F,R)}) \;=\; \mathrm{Capt}^{K\times K}_{F\times\{0\}}\left(\mathrm{Graph}(R^{|K}_{|K})\right)$$

is the unique closed subset $\mathcal{V} \subset K \times K$ containing $\{0\} \times \mathrm{Graph}(R^{|K}_{|K})$ satisfying

1. $\mathcal{V}\backslash(\{0\} \times \mathrm{Graph}(R^{|K}_{|K}))$ is locally viable under $F \times \{0\}$
2. and
$$\mathrm{Capt}^{K\times K}_{F\times\{0\}}(\mathcal{V}) \;=\; \mathcal{V}$$
This states that whenever $(x, y) \in (K \times K)\backslash\mathcal{V}$, all solutions to the differential inclusion $(x', y') \in F(x) \times \{0\}$ leave $(K \times K)$ before possibly reaching the target $\mathrm{Graph}(R^{|K}_{|K})$.

The first statement means that whenever (x, y) belongs to $\mathcal{V}$, there exist T and a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ such that $(x(t), y)$ belongs to $\mathcal{V}$ until it reaches $\mathrm{Graph}(R^{|K}_{|K})$. This is equivalent to saying that

$$\forall\, t \in [0, T], \;\; y \in U^K_{(F,R)}(x(t))$$

The second statement means that whenever (x, y) does not belong to $\mathcal{V}$, all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ are such that $(x(t), y)$ do not belong to $\mathcal{V}$ whenever $(x(t), y) \in K \times K$, i.e., whenever $x(\cdot)$ is viable in K on the interval $[0, T]$. This is equivalent to saying that for all solutions to $x' \in F(x)$ viable in K on the interval $[0, T]$,

$$\forall\, t \in [0, T], \;\; y \in K\backslash U^K_{(F,R)}(x(t))$$

Let us consider now $y \in \partial U^K_{(F,R)}(x)$. This means that there exists a sequence $y_n \in K$ such that $y_n \in K\backslash U^K_{(F,R)}(x)$. Hence (x, y_n) does not belong to the capture basin of $\mathrm{Graph}(R^{|K}_{|K})$ viable in $K \times K$. Therefore we know that for any solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0, T]$, for any $t \in [0, T]$, $y_n \in K\backslash U^K_{(F,R)}(x(t))$ and, in particular, that $y_n \in K\backslash U^K_{(F,R)}(x(T)) = R(x(T))$. Taking the limit when $n \to +\infty$, we infer that

$$\forall\, t \in [0, T], \;\; y \in \partial_K U^K_{(F,R)}(x(t))$$

2.4 Hamilton-Jacobi Charaterization of the Substratum

Before stating the general result characterizing the substratum as a solution to a system of first-order partial differential inclusions, let us consider the case when the impulse differential inclusion is actually an impulse differential equation (f, r) where $f : X \mapsto X$ is Lipschitz, $r : X \mapsto X$ is single-valued and continuous, that K is viable under (f, r) and $C := K \cap R^{-1}(K)$ crossable by f. In this case, $\Gamma^K_{(f,r)}$ is single-valued and Lipschitz from K to K.

We shall deduce from Theorem 2.4.3 below the following

Proposition 2.4.1 *Let us assume that $f : X \mapsto X$ is Lipschitz, $r : X \mapsto X$ is single-valued and continuous, that C is crossable by f, that K is viable under (f, r) and $\Gamma^K_{(f,r)}$ is differentiable. Then it is the unique solution to the system of first-order partial differential equations*

$$\forall\, x \in K \backslash C,\ \forall\, j = 1, \ldots, n,\ -\frac{\partial u_j(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial u_j(t,x)}{\partial x_i} f_i(x) \;=\; 0$$

or, in a more compact form,

$$\forall\, x \in K \backslash C,\ \ -\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} f(x) \;=\; 0$$

satisfying the condition

$$\forall\, x \in C,\ \ u(0,x) \;=\; r(x)$$

Actually, thanks to the concepts of contingent derivative, we shall show that the substratum $\Gamma^K_{(F,R)}$ is the unique (set-valued) solution in the "Frankowska sense" to the "Hamilton-Jacobi inclusion"

$$0 \;\in\; -\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} \cdot F(x) \tag{2.2}$$

satisfying the condition

$$\forall\, x \in C,\ \ V(0,x) \;=\; R(x) \cap K$$

For that purpose, we recall that the (graphical contingent) derivative of a set-valued map $V : \mathbf{R}_+ \times K \rightsquigarrow K$ may be defined by the relation

$$\mathrm{Graph}(DV(T,x,y)) \;:=\; T_{\mathrm{Graph}(V)}(T,x,y)$$

Definition 2.4.2 *We shall say that a set-valued map* $V : \mathbf{R}_+ \times K \rightsquigarrow K$ *is a* Frankowska solution *to the Hamilton-Jacobi system of first-order partial differential inclusions (2.2) satisfying the initial condition* $V(0,x) = R(x)$ *if its graph is closed, if*

$$\forall\, t > 0,\ \forall\, y \in V(t,x),\ \exists\, v \in F(x) \quad \textit{such that}\ 0 \in DV(t,x,y)(-1,v)$$

and if for every $v \in F(x)$

$$\forall\, t \geq 0,\ \forall\, y \in V(t,x), \quad 0 \in DV(t,x,y)(1,-v)$$

or

$$\begin{cases} i) & -v \in T_{X\setminus K}(x) \ \text{ if } \ x \in \partial K \\ ii) & -v \in T_K(x) \ \text{ if } \ y \in \partial K \end{cases}$$

Theorem 2.4.3 *Let us assume that* F *is Marchaud, that* $C := K \cap R^{-1}(K)$ *is closed and that the graph of* $R : C \rightsquigarrow K$ *is closed.*

1. *The substratum* $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ *is the* **largest** *set-valued map* $V : \mathbf{R}_+ \times K \rightsquigarrow K$ *with closed graph contained in* $K \times K$ *satisfying*

$$\forall\, t > 0,\ y \in V(t,x),\ \exists\, v \in F(x) \quad \textit{such that}\ 0 \in DV(t,x,y)(-1,v)$$

   and the condition $V(0,x) = R(x) \cap K$,

2. *If furthermore,* F *is assumed to be Lipschitz, the substratum* $\Gamma^K_{(F,R)} : K \rightsquigarrow K$ *is the* **unique** *Frankowska solution* $V : \mathbf{R}_+ \times K \rightsquigarrow K$ *to the Hamilton-Jacobi system of first-order differential inclusions (2.2) satisfying the initial condition* $V(0,x) = R(x)$.

Proof — When F is Marchaud, to say that $\text{Graph}(V)\backslash(\{0\} \times \text{Graph}\left(R^{|K}_{|K}\right))$ is locally viable under $\{-1\} \times F \times \{0\}$ means that

$$\forall\, (t,x,y) \in \text{Graph}(V)\backslash(\{0\}\times\text{Graph}\left(R^{|K}_{|K}\right)), \ \ \{-1\}\times F(x)\times\{0\}\cap T_{\text{Graph}(V)}(t,x,y) \neq \emptyset$$

We observe that $(t,x,y) \in \text{Graph}(V)\backslash(\{0\} \times \text{Graph}(R^{|K}_{|K}))$ whenever $t > 0$ and we recall that

$$T_{\text{Graph}(V)}(t,x,y) \ = \ \text{Graph}(DV(t,x,y))$$

so that the above condition reads

$$\forall\, t > 0, \forall\, y \in \Gamma^K_{(F,R)}(t,x),\ \exists\, v \in F(x) \quad \text{such that} \ \ 0 \ \in \ DV(t,x,y)(-1,v)$$

When F is assumed to be Lipschitz, to say that

$$\text{Capt}^{\mathbf{R}_+\times K\times K}_{\{-1\}\times F\times\{0\}}(\text{Graph}(V)) \;=\; \text{Graph}(V)$$

means that

1. for any $(t,x,y) \in \text{Graph}(V)) \cap \text{Int}(\mathbf{R}_+ \times K \times K)$,

   $$(\{1\} \times -F(x) \times \{0\}) \;\subset\; T_{\text{Graph}(V)}(t,x,y) \;=\; \text{Graph}(DV(t,x,y))$$

   This is equivalent to say that for every $v \in F(x)$,

   $$\forall\, t > 0,\; x \in \text{Int}(K),\; y \in V(t,x) \cap \text{Int}(K),\;\; 0 \;\in\; DV(t,x,y)(1,-v) \quad (2.3)$$

2. and otherwise, for any $(t,x,y) \in \text{Graph}(V)) \cap \partial(\mathbf{R}_+ \times K \times K)$,

   $$(\{1\} \times -F(x) \times \{0\}) \;\subset\; T_{\text{Graph}(V)}(t,x,y) \cup T_{(\mathbf{R}\times X\times X)\backslash(\mathbf{R}_+\times K\times K)}(t,x,y)$$

   This means that for every $v \in F(x)$,

   $$\begin{cases} i) & 0 \;\in\; DV(t,x,y)(1,-v) \text{ if } t = 0, y \in R(x) \\ ii) & 0 \;\in\; D(t,x,y)(1,-v) \text{ or } -v \in T_{X\backslash K} \text{ if } t \geq 0, x \in \partial K,\; y \in R(x) \\ iii) & 0 \;\in\; D(t,x,y)(1,-v) \text{ or } -v \in T_K \text{ if } t \geq 0,\; y \in R(x) \cap \partial K \end{cases}$$

   Indeed,

   $$(\mathbf{R}\times X\times X)\backslash(\mathbf{R}_+\times K\times K) = (\mathbf{R}_-\times K\times K)\cup(\mathbf{R}_+\times(X\backslash K)\times K)\cup(\mathbf{R}_+\times K\times(X\backslash K))$$

   Therefore, condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times K \times K$ at $(0,x,y)$ is impossible, condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times (X\backslash K) \times K$ at (t,x,y) when $x \in \partial K$ means that $-v$ belongs to $T_{X\backslash K}(x)$ and condition $(1,-v,0)$ belongs to the contingent cone to $\mathbf{R}_- \times K \times (X\backslash K)$ at (t,x,y) when $y \in \partial K$ means that $-v$ belongs to $T_K(x)$. $\square$

For the initialization map, we obtain the following Hamilton-Jacobi inclusion :

Theorem 2.4.4 *Let us assume that F is Marchaud, that $C := K \cap R^{-1}(K)$ is closed and that the graph of $R : C \rightsquigarrow K$ is closed.*

1. *The initialization map $U^K_{(F,R)} : K \rightsquigarrow K$ is the* **largest** *set-valued map $V : \mathbf{R}_+ \times K \rightsquigarrow K$ with closed graph contained in $K \times K$ satisfying*

   $$\forall\, y \in V(x),\; \exists\, v \in F(x) \quad \textit{such that } 0 \in DV(x,y)(v)$$

2. *If furthermore, F is assumed to be Lipschitz, the initialization map $U^K_{(F,R)}$: $K \rightsquigarrow K$ is the* **unique** *Frankowska solution $V : \mathbf{R}_+ \times K \rightsquigarrow K$ to the Hamilton-Jacobi system of first-order differential inclusions (2.2) satisfying the condition $\forall\, x \in C,\ V(x) = R(x)$.*

Proof — When F is Marchaud, to say that $\mathrm{Graph}(V)\backslash(\mathrm{Graph}\left(R^{|K}_{|K})\right)$ is locally viable under $F \times \{0\}$ means that

$$\forall\, (x,y) \in \mathrm{Graph}(V)\backslash(\mathrm{Graph}\left(R^{|K}_{|K})\right),\ \ F(x) \times \{0\} \cap T_{\mathrm{Graph}(V)}(x,y) \neq \emptyset$$

We observe that $(x,y) \in \mathrm{Graph}(V)\backslash(\mathrm{Graph}(R^{|K}_{|K}))$ whenever $y \in V(x)\backslash R(x) \cap K$ and we recall that

$$T_{\mathrm{Graph}(V)}(x,y) \ = \ \mathrm{Graph}(DV(x,y))$$

so that the above condition reads

$$\forall\, y \in U^K_{(F,R)}(x),\ \exists\, v \in F(x) \quad \text{such that} \ \ 0 \ \in \ DV(x,y)(v)$$

When F is assumed to be Lipschitz, to say that

$$\mathrm{Capt}^{K \times K}_{F \times \{0\}}(\mathrm{Graph}(V)) \ = \ \mathrm{Graph}(V)$$

means that

1. for any $(x,y) \in \mathrm{Graph}(V)) \cap \mathrm{Int}(K \times K)$,

   $$(-F(x) \times \{0\}) \ \subset \ T_{\mathrm{Graph}(V)}(x,y) \ = \ \mathrm{Graph}(DV(x,y))$$

   This is equivalent to say that for every $v \in F(x)$,

   $$\forall x \in \mathrm{Int}(K),\ y \in V(x) \cap \mathrm{Int}(K),\ \ 0 \ \in \ DV(x,y)(-v) \qquad (2.4)$$

2. and otherwise, for any $(x,y) \in \mathrm{Graph}(V)) \cap \partial(K \times K)$,

   $$-F(x) \times \{0\}) \ \subset \ T_{\mathrm{Graph}(V)}(x,y) \cup T_{(X \times X)\backslash(K \times K)}(x,y)$$

   This means that for every $v \in F(x)$,

   $$\begin{cases} i) \quad 0 \ \in \ D(x,y)(-v) \text{ or } -v \in T_{X\backslash K} \text{ if } x \in \partial K,\ y \in R(x) \\ ii) \quad 0 \ \in \ D(x,y)(-v) \text{ or } -v \in T_K \text{ if } y \in R(x) \cap \partial K \end{cases}$$

Indeed,

$$\overline{(X \times X)\backslash(K \times K)} = ((X\backslash K) \times K) \cup (K \times (X\backslash K))$$

Therefore, condition $(-v, 0)$ belongs to the contingent cone to $(X\backslash K) \times K$ at (x, y) when $x \in \partial K$ means that $-v$ belongs to $T_{X\backslash K}(x)$ and condition $(-v, 0)$ belongs to the contingent cone to $K \times (X\backslash K)$ at (x, y) when $y \in \partial K$ means that $-v$ belongs to $T_K(x)$. □

2.5 Boundary Impulse Differential Inclusions

We consider here the case when the resetting of initial conditions happens only at time t when the state of the system has to leave K, i.e., when $\tau_K^\sharp(x(t)) = 0$.

In this case, the reset map $R : \partial K \rightsquigarrow X$ needs to be defined only on the boundary of K, actually, on the egress set $\mathrm{Egress}_F(K)$ of K. In other words, the reset map can be regarded as a set-valued boundary data and extended to K by setting

$$\forall\, x \notin \partial K, \quad R(x) \ := \ \emptyset \text{ or } S(x) \ := \ \emptyset$$

Therefore, the inverse image $R^{-1}(K)$ is contained in the boundary ∂K of K.

Definition 2.5.1 *An hybrid differential inclusion (F, R) defined on a closed subset K is called a* boundary hybrid differential inclusion *if the domain of the reset map R is contained in the boundary ∂K of K.*

When F is Marchaud and R is upper semicontinuous with compact images, we know that the hybrid system (F, R) is viable if and only if $\mathrm{Egress}_F(K) \subset R^{-1}(K)$, i.e., if and only if for every $x \in \mathrm{Egress}_F(K)$, $R(x) \cap K$ is not empty, or, equivalently, if and only if

$$\forall\, x \in \partial K\backslash\mathrm{Dom}(R), \quad F(x) \cap T_K(x) \ \neq \ \emptyset$$

Therefore, a viable run $x(\cdot)$ is described in the following way: If t_n is an impulse time and if $x_n \in \mathrm{Egress}_F(K)$, then

1. If $t_{n+1} = t_n$, so that, $x_n \in R^{-1}(\mathrm{Egress}_F(K))$, we take and $x_{n+1} \in R(x_n) \cap \mathrm{Egress}_F(K)$
2. if $t_{n+1} > t_n$ (this is the case whenever $x_n \notin R^{-1}(\mathrm{Egress}_F(K))$), we take a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ stating from x_n at time t_n
   (a) which may remain forever viable in K if $x_n \in \mathrm{Viab}_F(K)$

(b) which may remain viable in K until the time $t_{n+1} := \tau_K(x(\cdot)) + t_n$ when the solution is about to leave K at $x(^-t_{n+1}) \in \text{Egress}_F(K)$. Hence, we take any $x_{n+1} \in R(x(^-t_{n+1})) \cap K$ as a new initialized state. $\square$

A boundary impulse differential inclusion is non Zeno, and actually, exhaustive, whenever the reset map R maps ∂K to the interior of K and when K is closed, bounded, since in this case $R(\partial K)$ is a compact subset of the interior of K.

Definition 2.5.2 *We shall say that* a closed subset K is transverse to a set-valued map F *if for any solution* $x(\cdot) \in \mathcal{S}_F^K(x)$, $\omega_{\partial K}(x(\cdot)) = \tau_K(x)$.

The transversality of K to F means that as soon as a solution $x(\cdot)$ reaches the boundary ∂K of K it leaves K, i.e., that it crosses the boundary ∂K of K without "traveling" on it.

Lemma 6.1.24 implies that if a closed subset K is transverse to F, then the exit functional is continuous.

Lemma 2.5.3 *Let us posit assumption*

$$\forall\, c \in \partial K, \;\; \tau_K^{F^\sharp}(c) \;=\; 0 \tag{2.5}$$

Then K is transverse to F.

Proof — Indeed, assume that there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ such that $\omega_{\partial K}(x(\cdot)) < \tau_K(x(\cdot))$. Since $c := x(\omega_{\partial K}(x(\cdot)))$ belongs to the boundary ∂K, we know that $\tau_K^{F^\sharp}(c) = 0$, and thus that there exists a sequence of times t_n converging to 0 such that

$$x(\omega_{\partial K}(x(\cdot)) + t_n) \;\notin\; K$$

This is a contradiction as soon as $\omega_{\partial K}(x(\cdot)) + t_n < \tau_K^{F^\sharp}(x)$. $\square$

We can provide sufficient conditions implying that assumption (2.5) holds true involving the complement of the contingent cone to the complement, which is the Dubovitsky-Miliutin cone defined by

Definition 2.5.4 *The* Dubovitsky-Miliutin tangent cone $D_K(x)$ *to* K *is defined by:*

$$\begin{cases} v \in D_K(x) \;\text{ if and only if} \\ \exists\; \varepsilon > 0,\;\; \exists\, \alpha > 0 \;\text{ such that }\; x+]0,\alpha](v+\varepsilon B) \;\subset K \end{cases}$$

We recall that

$$\forall\, x \in \partial K,\ \ D_K(x) \ = \ X \setminus T_{X\setminus K}(x)$$

We thus observe the following

Lemma 2.5.5 *Let us assume that K is closed and a backward repeller. If*

$$\forall\, x \in \partial K,\ \ F(x) \ \subset \ D_{X\setminus K}(x)$$

then assumption (2.5) is satisfied.

Proof — Indeed, Theorem 4.3.4 of VIABILITY THEORY, [8, Aubin] that whenever for $x \in \partial K$ and $F(x) \ \subset \ D_{X\setminus K}(x)$, there exists $T > 0$ such that $\vartheta_F(t,x) \in X\setminus K$ for all $t \in]0,T]$. Then $\tau_K^{F^\sharp}(x) = 0$. □

For simplicity, let us consider when K is closed, bounded and transverse to a Lipschitz single-valued map f, that K is a repeller backward invariant and that the reset map $r : \partial K \mapsto \mathrm{Int}(K)$ is continuous. Then we know that the initialization map $u := u_{(f,r)}$ is the unique Frankowska solution to the boundary value problem

$$\frac{\partial u(x)}{\partial x} f(x) \ = \ 0$$

satisfying the Dirichlet boundary condition

$$\forall\, x \in \partial K,\ \ u(x) \ = \ r(x)$$

2.6 Threshold Systems

2.6.1 An Example

Let $X := \mathbf{R}^n$. A threshold system is a system defined on $X \times X$ where x_i is regarded as the ith component of the state of the system and y_i as a threshold. They are subjected to constraints of the form

$$\forall\, i = 1, \ldots, n,\ \ x_i \ \geq \ \alpha_i,\ \ y_i \ \leq \ \beta_i \,\&\, h_i(x_i) \ \leq \ y_i \tag{2.6}$$

Let us denote by

$$\mathbf{I}(x,y) \ := \ \{i = 1, \ldots, n \mid h(x)_i = y_i\}$$

the set of active constraints. and set

$$h(x) \ := \ (h_i(x_i))_{i=1,\ldots,n}$$

When a threshold $y_i = h_i(x_i)$ is reached, reset maps r_i and s_i are triggered for mapping a pair (x_i, y_i) to a new pair $(r_i(x_i), s_i(x_i, y_i))$ satisfying

$$r_i(x_i) \geq \alpha_i, \ \ s_i(x_i, y_i) \leq \beta_i \ \& \ h_i(r_i(x_i)) < s_i(x_i, y_i)$$

We shall extend these single-valued maps defined on the set

$$C \ := \ \{(x, y) \mid \mathbf{I}(x, y) \ \neq \ \emptyset\}$$

to a set-valued maps (again denoted by) r_i and s_i defined by

$$(r_i(x_i), s_i(x_i, y_i)) \ := \begin{cases} \emptyset & \text{if} \ \ h(x)_i < y_i \\ (r_i(x_i), s_i(x_i, y_i) & \text{if} \ \ h(x)_i = y_i \end{cases} \tag{2.7}$$

maps taking empty values are simple examples of set-valued maps. We also set

$$r(x) \ := \ (r_i(x_i))_{i=1,\ldots,n} \ \& \ s(x, y) \ := \ (s_i(x_i, y_i))_{i=1,\ldots,n}$$

In order to take into account a possible evolution of the thresholds (in "learning processes", for instance), we shall consider the case when the evolution of the states and thresholds is governed by a system of differential equations of the form

$$\begin{cases} i) & x'(t) \ = \ f(x(t)) \\ ii) & y'(t) \ = \ g(x(t), y(t)) \end{cases}$$

Theorem 2.6.1 *We assume that the maps $f : X \rightsquigarrow X$, $g : X \times X \rightsquigarrow X$ are Lipschitz, that the functions h_i are differentiable, that the reset maps r_i and s_i defined by (2.7) are continuous and that there exists $a > 0$ such that, for every $(x, y) \in K$,*

$$\begin{cases} i) & \forall \ i = 1, \ldots, n, \ \ f_i(x_1, \ldots, \alpha_i, \ldots, x_n) = 0 \\ ii) & \forall \ j = 1, \ldots, n, \ \ g_j(x, y_1, \ldots, \beta_j, \ldots, y_n) = 0 \\ iii) & \forall \ i \notin \mathbf{I}(x, y) \ \ \langle h_i'(x_i), f_i(x) \rangle \leq g_i(x, y) + a(h_i(x_i) - y_i) \end{cases} \tag{2.8}$$

Then the subset K of pairs (x, y) satisfying (2.6) is viable under the impulse differential equation $((f, g), (r, s))$.

If we assume furthermore that

$$\forall\, i \in \mathbf{I}(x,y),\;\; h_i'(x)f_i(x) - g_i(x,y) \;=\; 0 \qquad (2.9)$$

then the initialization map $(x,y) \in K \mapsto u(x,y) \in X \times X$ *is the* **unique** *Frankowska solution to the system of first order partial differential equations*

$$\begin{cases} i) \quad \forall\, i = 1,\ldots,n, \;\; \sum_{l=1}^{n} \frac{\partial u_i}{\partial x_l} f_l(x) + \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} g_k(x,y) \;=\; 0 \\ ii) \;\; \forall\, j = 1,\ldots,n, \;\; \sum_{l=1}^{n} \frac{\partial u_j}{\partial x_l} f_l(x) + \sum_{k=1}^{n} \frac{\partial u_j}{\partial x_k} g_k(x,y) \;=\; 0 \end{cases}$$

satisfying the condition

$$\forall\, i \in \mathbf{I}(x,y),\;\; u(x,y) \;=\; (r(x), s(x,y))$$

A run $(x(\cdot), y(\cdot))$ associated with an (strictly increasing) sequence $\mathcal{T} := \{t_n\}_{n\geq 0}$ is reset at each impulse time t_n according t the rule

$$\forall\, i \in \mathbf{I}(x(^-t_n), y(^-t_n)),\;\; x_i(t_n) = r_i(x_i(^-t_n)) \;\&\; y_i(t_n) = s_i(x_i(^-t_n), y_i(^-t_n))$$

so that the pair $(x(t_n), y(t_n))$ satisfies

$$\forall\, i = 1,\ldots,n,\;\; h_i(x_i(t_n)) \;<\; y_i(t_n)$$

In other words, at each impulse time t_n, only the components $i \in \mathbf{I}(x(^-t_n), y(^-t_n))$ reaching the thresholds at impulse time are reset.

Proof — Threshold systems fit the general framework when the continuous and discrete dynamics are the pairs (f,g) and (r,s) respectively and when the subset K is defined by (2.6). We observe that the discrete egress set is the set of pairs (x,y) satisfying

$$\forall\, i = 1,\ldots,n,\;\; x_i \;\geq\; \alpha_i,\;\; y_i \;\leq\; \beta_i \;\&\; h_i(x_i) \;<\; y_i$$

which is locally viable thanks to assumptions (2.8)i) and ii). Assumption (2.8)iii) implies that K is a repeller since whenever $(x,y) \in K$

$$\frac{d}{dt}(y_i(t) - h_i(x_i(t))) \;=\; g_i(x(t), y(t)) - \langle h_i' f_i(x(t))\rangle \;\geq\; a(y_i(t) - h_i(x_i(t)))$$

so that, whenever $y_i(t) - h_i(x_i(t)) \geq 0$,

$$y_i(t) - h_i(x_i(t)) \ \geq \ y_i(0) - h_i(x_i(0))e^{at}$$

Assumption (2.9) implies that K is backward invariant. Therefore, K being a backward invariant repeller, we deduce that the initialization map $(x, y) \in K \mapsto u(x, y) \in X \times X$ is the unique Frankowska solution to the above system of first order partial differential equations. $\square$

2.6.2 More General Situations

Naturally, this situation can be extended to the more general case when we have a system of the form

$$\begin{cases} i) & x'(t) \ = \ f(x(t)) \\ ii) & y'(t) \ = \ g(x(t), y(t)) \end{cases}$$

and when $h : X \mapsto Y$ can be regarded as a constraint map (such as resource, output-input maps or technological maps in specific contexts) and y a resource or a threshold, $L \subset X$, $M \subset Y$ and $P \subset M$ (most often P is a cone defining an order relation) and

$$K \ := \ \{(x, y) \in L \times M \mid h(x) - y \in P\}$$

where K is a compact backward invariant. We set

$$C \ := \ \{(x, y) \in K \times L \mid h(x) - y \in \partial P\}$$

Lemma 2.6.2 *Let* $X := \mathbf{R}^n$ *and* $P := \mathbf{R}^n_-$ *the negative orthant. Assume that* f *and* g *are Lipschitz, that* $h : X \mapsto Y$ *is differentiable and that the closed subsets* L *and* M *are convex and satisfy the* constraint qualification property*: For every* $(x, y) \in K$*, there exist* $u \in T_L(x)$ *and* $v \in T_M(y)$ *such that*

$$\forall \ i = 1, \ldots, n, \ \ (h'(x)u)_i < v_i$$

If

$$\forall \ (x, y) \in K, \ \ f(x) \in -T_L(x) \ \& \ g(x, y) \ \in \ -T_M(y)$$

and if there exists $a > 0$ *such that*

$$\forall \ (x, y) \in K \backslash C, \ \ h'(x)f(x) - g(x, y) \ \in \ a(h(x) - y) - P$$

then the subset K is a repeller.

If we assume furthermore that

$$\forall\ (x,y) \in C,\ \ h'(x)f(x) - g(x,y)\ \in\ T_P(h(x)-y)$$

then K is backward invariant under (f,g).

Proof — The above condition implies that for every solution $(x(\cdot),y(\cdot)) \in \mathcal{S}(f,g)(x,y)$, we have

$$h(x(t)) - y(t)\ \in\ e^{at}(h(x)-y) - P$$

which implies that K is a repeller under (f,g).

For proving that K is backward invariant, we have to prove that

$$\forall\ (x,y) \in K,\ \ (-f(x), -g(x,y))\ \in\ T_K(x,y)$$

Since the sets L and M are convex, the constraint qualification assumption implies that the contingent cone $T_K(x,y)$ to K at (x,y) is equal to

$$T_K(x,y)\ :=\ \{(u,v) \in T_L(x) \times T_M(y) \mid h'(x)u - v \in -T_P(h(x)-y)\}$$

Let us denote by

$$\mathbf{I}(x,y)\ :=\ \{i = 1,\ldots,n \mid h(x)_i = y_i\}$$

the set of active constraints. This means that for any $i \in \mathbf{I}(x,y)$, we must have $(h'(x)u)_i - vi \geq 0$. Since $(-f(x), -g(x,y))$ belongs to $T_L(x) \times T_M(y)$, we derive from the sufficient condition that $-(h'(x)u)_i - g(x,y)_i \leq 0$ is satisfied. Hence K is backward invariant under (f,g). □

Let us consider a reset map r defined on C and mapping it to the set

$$\{(x,y) \in L \times M \mid h(x) - y \in \mathrm{Int}(P)\}$$

Therefore, the above subset K is viable under the impulse differential equation $((f,g),r)$. The graph of the initialization map $u := u_{((f,g),r)} : K \mapsto X \times Y$ is the unique Frankowska solution to the system of first-order partial differential equations

$$\frac{\partial u(x,y)}{\partial x} f(x) + \frac{\partial u(x,y)}{\partial y} g(x,y)\ =\ 0$$

satisfying the condition

$$\forall\ (x,y) \in C,\ \ u(x,y)\ =\ r(x,y)$$

2.7 Existence of Cadenced Runs

We begin with the following asymptotic property of a run that implies the existence of a cadenced run:

Theorem 2.7.1 *Let us assume that F is Marchaud, that R is upper semicontinuous, that $R(K)$ is compact and that $R(K) \cap \mathrm{Viab}(K) = \emptyset$. Let $x(\cdot)$ be a run associated with a sequence $\mathcal{T}(x(\cdot))$ of impulse times t_n viable in K.*

If the sequence of reinitialized states $x(t_n)$ of the run $x(\cdot)$ converges to some $\overline{x}$, then a subsequence of the motives $x(\cdot + t_n)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ and viable in K.

Proof — We shall set from now on $C := K \cap R^{-1}(K)$, that plays the role of a target.

Let us assume that the sequence $x(t_n)$ converges to some $\overline{x} \in R(K)$. If we denote by $x_n(\cdot) := x(\cdot + t_n)$ the nth motive of the run, we see that $x_n(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ and satisfying $x_n(\tau_{n+1}) \in R^{-1}(x(t_{n+1}))$.

Since $\overline{x}$ does not belong to the viability kernel $\mathrm{Viab}(K)$ of K under F by assumption, Proposition 2.1.3 implies that the cadences $\tau_n := t_n - t_{n-1}$ are bounded by a finite time $\overline{T}$.

By Theorem 6.1.6, a subsequence $x_{n_p}(\cdot)$ converges uniformly on the compact interval $[0, \overline{T}]$ to some solution $\overline{x}(\cdot) \in \mathcal{S}(\overline{x})$ to the differential inclusion $x' \in F(x)$ starting at $\overline{x}$. Another subsequence of cadences $\tau_{n_{pq}+1}$ converges to some $\overline{\tau} \in [0, \overline{T}]$. Hence $x_{n_{pq}}(\tau_{n_{pq}+1})$ converges to $\overline{x}(\overline{\tau})$. Since $x_{n_{pq}}(\tau_{n_{pq}+1})$ belongs to $R^{-1}(x(t_{n_{pq}+1}))$, since $x(t_{n_{pq}+1})$ converges also to $\overline{x}$ by assumption and since the graph of the reset map R is closed, we infer that $\overline{x}(\overline{\tau})$ belongs to $R^{-1}(\overline{x})$. Hence a subsequence of the motives $x_n(\cdot) := x(\cdot + t_n)$ of the run $x(\cdot)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ of rhythm $\overline{\tau}$. □

We next provide a sufficient condition for the existence of a cadenced run when the dynamics f governing the continuous evolution is single-valued and Lipschitz.

Proposition 2.7.2 *Let $f : X \mapsto X$ be a Lipschitz single-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : C \rightsquigarrow K$ be an upper semicontinuous set-valued map with nonempty compact convex images.*

Let us assume that

$$\forall\, x \in K \backslash C, \;\; f(x) \;\in\; T_K(x) \tag{2.10}$$

that $\text{Viab}(K\backslash R^{-1}(K)) = \emptyset$ *and that*

$$\text{the hitting function } \varpi^{\flat}_{(K,R^{-1}(K))} \text{ is continuous} \tag{2.11}$$

Then there exists a cadenced run of the impulse differential equation (f, R) *viable in* K*.*

Proof — The Viability Theorem implies that from every $x \in K$ starts a solution $x(\cdot) =: \vartheta(\cdot, x)$ to the differential equation $x' = f(x)$, that is unique because f is assumed to be Lipschitz.

Since $\text{Viab}(K\backslash R^{-1}(K))$ is empty, the solution $x(\cdot)$ leaves K in finite time whenever $x \in K\backslash C$, and leaves it through C by Theorem 2.1.2 since $K\backslash C$ is locally viable thanks to assumption (2.10).

We thus infer that the hitting function $\varpi^{\flat}_{(K,R^{-1}(K))}$ is finite on K. Since it is assumed to be continuous, the map $x \rightsquigarrow \vartheta(\varpi^{\flat}_{(K,R^{-1}(K))}(x), x)$ is also continuous. Hence the set-valued map $S : x \in K \rightsquigarrow R(\vartheta_f(\varpi^{\flat}_{(K,R^{-1}(K))}(x), x)) \subset K$ is upper semicontinuous with closed convex images since R enjoys these properties.

The Kakutani Fixed-Point Theorem implies that the set-valued map $S : K \rightsquigarrow X$ has a fixed point $\overline{x}$, from which starts a solution $\overline{x}(\cdot)$ to the differential equation $x' = f(x)$ satisfying $\overline{x} \in R(\overline{x}(\overline{\tau}))$, which is then the motive $\overline{x}(\cdot)$ of a cadenced run of rhythm $\overline{\tau} := \varpi^{\flat}_{(K,R^{-1}(K))}(\overline{x})$. □

The problem now is to provide sufficient conditions for the hitting function to be continuous instead of being only lower semicontinuous.

When L is a closed subset with nonempty interior such that $C := K \cap L$, a sufficient condition is that it coincides with the hitting function $\varpi^{\flat}_{\text{Int}(L)}$, which is upper semicontinuous. For instance, when $C := \partial K$ and when K is the closure of its interior — which is the case when K is a closed convex subset with nonempty interior — one can take L to be the closure of the complement of K.

For this purpose, we need to introduce the Dubovitsky-Miliutin and hypertangent cones:

Definition 2.7.3 *The* Dubovitsky-Miliutin tangent cone $D_L(x)$ *to* L *is defined by:*

$$\begin{cases} v \in D_L(x) \text{ if} \\ \exists\ \varepsilon > 0,\ \ \exists\, \alpha > 0 \text{ such that } x+]0,\alpha](v + \varepsilon B) \subset L \end{cases}$$

The hypertangent cone $H_L(x)$ *to* L *at* $x \in \partial L$ *is the set of elements* $v \in X$ *such that there exist* $\varepsilon > 0,\ \delta > 0$ *and* $\eta > 0$ *for which*

$$B(x,\eta) \cap L+]0,\delta](v + \varepsilon B) \subset L$$

We recall that for any x in the boundary of L, the *Dubovitsky-Miliutin cone* $D_L(x)$ to L at x is the complement of the contingent cone $T_{X\backslash L}(x)$ to the complement $X\backslash L$ of L at $x \in \partial L$:

$$\forall\, x \in \partial L,\;\; D_L(x) \;=\; X \setminus T_{X\backslash L}(x)$$

and that the graph of the hypertangent cone is open in $L \times X$ (see Chapter 4 of [31, Aubin & Frankowska] for more details).

Proposition 4.3.5 of [8, Aubin] states that if the set-valued map F is Marchaud and if $F(x) \subset D_L(x)$ at $x \in \partial L$, then there exist $\rho_x > 0$ and $T_x > 0$ such that, for all solutions to the differential inclusion $x' \in F(x)$,

$$\forall\, t \in [0, T_x],\;\; d(x(t), \partial L) \;\geq\; \rho_x t$$

This implies that whenever $f(x) \in D_L(x)$, then $\varpi^\flat_{\mathrm{Int}(L)}(x) = 0$. We thus deduce from Theorem 6.1.24 the following Lemma:

Lemma 2.7.4 *Assume that f is a Lipschitz single-valued map, that L is a closed subset with a nonempty interior and that for every $x \in \partial L$, $f(x) \in D_L(x)$. Then the hitting functions $\varpi^\flat_L$ and $\varpi^\flat_{\mathrm{Int}(L)}$ coincide and, consequently, are continuous on the capture basin* $\mathrm{Capt}(L)$ *of L under f.*

We deduce from Proposition 2.7.2 and the above Lemma that

Proposition 2.7.5 *Let $f : X \mapsto X$ be a Lipschitz single-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : K \rightsquigarrow K$ be an upper semicontinuous set-valued map with closed convex images and that $R(K)$ is compact. Let us assume that L is a closed subset with nonempty interior such that $C := K \cap R^{-1}(K) \cap L$ is not empty and different from K, that*

$$\left\{ \begin{array}{ll} i) & \forall\, x \in K\backslash C,\;\; f(x) \;\in\; T_K(x) \\ ii) & \forall\, x \in K \cap \partial L,\;\; f(x) \;\in\; D_L(x) \end{array} \right. \tag{2.12}$$

and that $\mathrm{Viab}(K\backslash R^{-1}(K))$ *is empty.*

Then there exists a cadenced run to the impulse differential equation (f, R) viable in K.

We now use the techniques of [129, Haddad & Lasry] for extending Proposition 2.7.5 to the case of impulse differential inclusions:

Theorem 2.7.6 *Let $F : X \mapsto X$ be a Marchaud set-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : C \rightsquigarrow K$ be an upper semicontinuous set-valued map with nonempty compact convex images. Let us assume that L is a closed subset with nonempty interior such that $C := K \cap R^{-1}(K) \cap L$ is not empty and different from K, that*

$$\begin{cases} i) & \forall\, x \in K\backslash C, \;\; F(x) \;\cap\; T_K(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in K \cap \partial L, \;\; F(x) \;\subset\; H_L(x) \end{cases} \tag{2.13}$$

that $R(K) \cap \mathrm{Viab}(K)$ and $\mathrm{Viab}(K\backslash R^{-1}(K))$ are empty.

Then there exists a cadenced run of the impulse differential inclusion (F, R) viable in K.

Proof — Following [129, Haddad & Lasry], we use their basic Lemma (see also Theorem 1.6.1 of [26, Aubin & Cellina]) for approximating the Marchaud map F by Lipschitz Marchaud set-valued maps F_n defined by

$$\forall\, x \in K\backslash C, \;\; F_n(x) \;:=\; \sum_{\text{finite}} \psi_i^n(x) C_i^n \tag{2.14}$$

where ψ_i^n are Lipschitz and C_i^n are convex compact subsets contained in the image of F. They satisfy:

$$\begin{cases} i) & \forall\, n \geq 0, \;\; F(x) \subset \cdots \subset F_n(x) \\ ii) & \forall\, \varepsilon > 0, \; \exists\, N(\varepsilon, x) \;\mid\; F_n(x) \subset F(x) + \varepsilon B \end{cases} \tag{2.15}$$

Therefore, assumption (2.12) implies that

$$\forall\, n \geq 0, \; \forall\, x \in K\backslash C, \;\; F_n(x) \cap T_K(x) \neq \emptyset \tag{2.16}$$

We define now the set-valued map $G_n : K \rightsquigarrow X$ by

$$\forall\, n \geq 0, \; \forall\, x \in K\backslash C, \;\; G_n(x) \;:=\; F_n(x) + \frac{1}{n} B \tag{2.17}$$

It is obvious that these set-valued maps G_n are Lipschitz with closed convex values. Moreover, since $\mathrm{Int}(T_K(x))$ is nonempty, then $v + \frac{1}{n} B$ belongs to the interior of $T_K(x)$ for all $v \in T_K(x)$. Therefore (2.15) implies that

$$\forall\, n \geq 0, \; \forall\, x \in K\backslash C, \;\; G_n(x) \cap \mathrm{Int}(T_K(x)) \neq \emptyset \tag{2.18}$$

We also deduce that

$$\forall\, n \geq 0,\ \forall\, x \in K \cap \partial L,\ \ G_n(x)\ \subset\ H_L(x) \tag{2.19}$$

Since G_n is Lipschitz and since the maps $x \rightsquigarrow \mathrm{Int}(T_K(x))$ and $x \rightsquigarrow D_K(x)$ have an open graph, then there exists a Lipschitz selection g_n:

$$\begin{cases} i) & \forall\, x \in K \backslash C,\ \ g_n(x)\ \in\ G_n(x) \cap \mathrm{Int}(T_K(x)) \\ ii) & \forall\, x \in K \cap \partial L,\ \ g_n(x)\ \in\ H_L(x) \end{cases} \tag{2.20}$$

Hence, by Proposition 2.7.5, there exist for each n an initial state $\overline{x}_n$ from which starts a solution $\overline{x}_n(\cdot)$ to the differential equation $x' = g_n(x)$ satisfying $\overline{x}_n \in R(\overline{x}_n(\overline{\tau}_n))$, which is the motive $x_n(\cdot)$ of a cadenced run of rhythm $\overline{\tau}_n := \varpi^{g_n}_{(K,R^{-1}(K))}(\overline{x}_n)$.

Since K is compact, a subsequence of such initial states (again denoted by) $\overline{x}_n$ converges to some $\overline{x} \in K \cap R(K)$. Since $\overline{x}$ does not belong to the viability kernel $\mathrm{Viab}(K)$ of K under F, Proposition 2.1.3 implies that the cadences $\tau_n := t_n - t_{n-1}$ are bounded by a finite time $\overline{T}$.

By Theorem 6.1.6, a subsequence (again denoted by) $\overline{x}_n(\cdot)$ converges uniformly on the compact interval $[0, \overline{T}]$ to some solution $\overline{x}(\cdot) \in \mathcal{S}(\overline{x})$ to the differential inclusion $x' \in F(x)$ starting at $\overline{x}$. Another subsequence of cadences (again denoted by) $\overline{\tau}_n$ converges to some $\overline{\tau} \in [0, \overline{T}]$. Hence $\overline{x}_n(\tau_n)$ converges to $\overline{x}(\overline{\tau})$. Since $\overline{x}_n(\overline{\tau}_n)$ belongs to $R^{-1}(\overline{x}_n)$ and since the graph of the reset map R is closed, we infer that $\overline{x}(\overline{\tau})$ belongs to $R^{-1}(\overline{x})$. Hence a subsequence of the motives $x_n(\cdot) := x(\cdot + t_n)$ of the run $x(\cdot)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ of rhythm $\overline{\tau}$. □

2.8 Reset Kernels under Impulse Differential Inclusions

When K is not viable under (F, R), we introduce the following concepts:

Definition 2.8.1 *Let us consider an impulse differential inclusion* (F, R) *and a subset* K*.*

We shall denote by $\mathrm{Reset}_{(F,R)}(K)$ *the subset of initial states* $x_0 \in K$ *from which starts at least one run viable in* K *and call it the* reset kernel *of* K *under the impulse differential inclusion* (F, R)*.*

For characterizing the reset kernel of K, we need the following lemma:

Lemma 2.8.2 *Let assume that $F : X \rightsquigarrow X$ is Marchaud, that K is a closed repeller under F, that $R : X \rightsquigarrow X$ is upper semicontinuous and that*

$$\forall\, x \in K, \ \ R(x) \cap (K + B) \text{ is compact}$$

Then, for every closed subset $L \subset K$, the subset $\mathrm{Cap}_F^K(R^{-1}(L) \cap K)$ is closed.

Proof — By Lemma 5.2.10, the subset $R^{-1}(L)\cap K$ is closed and by Theorem 6.3.13, its viable-capture basin is also closed. □

Theorem 2.8.3 *Let assume that $F : X \rightsquigarrow X$ is Marchaud, that $R : X \rightsquigarrow X$ is upper semicontinuous and that*

$$\forall\, x \in K, \ \ R(x) \cap (K + B) \text{ is compact}$$

Assume also that K is a closed repeller under both F and R. The reset kernel $\mathrm{Reset}_{(F,R)}(K)$ is the largest closed subset of K viable under the impulse differential inclusion (F, R). It is the largest closed solution to the "fixed set" problem

$$\mathrm{Reset}_{(F,R)}(K) \ := \ \mathrm{Capt}_F^K \left(R^{-1}(\mathrm{Reset}_{F,S}(K)) \cap K \right)$$

contained in K.

Furthermore, setting $K_0 := K$ and recursively

$$K_{n+1} \ := \ K_n \cap \mathrm{Cap}_F^K(R^{-1}(K_n) \cap K)$$

it is equal to

$$\mathrm{Reset}_{(F,C)}(K) \ = \ \bigcap_{n \geq 0} K_n$$

(the reset kernel algorithm*).*

Proof — We first observe that the subsets K_n are closed by Lemma 2.8.2.

Second, we note that any closed subset $L \subset K$ viable under (F, R) is contained in the reset kernel $\mathrm{Reset}_{(F,R)}(K)$, since from any $x_0 \in K$ starts a viable run in L, and thus in K.

The reset kernel $\mathrm{Reset}_{(F,R)}(K)$ is viable under (F, R). Indeed, let $x_0 \in \mathrm{Reset}_{(F,R)}(K)$ and $x(\cdot)$ be a run starting from x_0 viable in K, associated with impulse times

$t_0 = 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and reset initial states x_n. Take any $T > 0$ and show that $x(T)$ belongs to the reset kernel $\mathrm{Reset}_{(F,R)}(K)$. We consider the run $y(\cdot)$ defined by $y(t) := x(t+T)$, starting at time 0 from $x(T)$, with impulse times $s_n := t_n + T$ and reset initial states $y_n := x_n \in K$. Furthermore, for every interval $[t_n, t_{n+1}[$ whenever $t_{n+1} > t_n$, $y(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ viable in K. Hence, $y(\cdot)$ is a run of the impulse differential inclusion starting from $x(T)$ and viable in K, and thus, belongs to the reset kernel $\mathrm{Reset}_{(F,R)}(K)$.

Assume next that x_0 belongs to the reset kernel $\mathrm{Reset}_{(F,R)}(K)$ under (F, R) and show that it belongs to to the intersection

$$K_\propto := \bigcap_{n \geq 0} K_n$$

of the subsets K_n. Indeed, there exists a run $x(\cdot)$ starting from x_0 viable in K accociated with a sequence of impulse times $0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and of states $\xi_n := x(^-t_n) \in R^{-1}(K) \cap K$ and of initial states $x_n \in R(\xi_n)$. Each element x_n of the sequence of initial states belongs to K. Since $\xi_n \in R^{-1}(K) \cap K$ and $x_n \in \mathrm{Capt}_F^K(R^{-1}(K) \cap K)$ by construction, then each element x_n of the sequence of initial states belongs to K_1. Assume that each element x_n of the sequence of initial states belongs to K_j. Therefore, as for the case $j = 0$, we deduce that $x_n \in \mathrm{Capt}_F^K(R^{-1}(K_j) \cap K)$. Since it belongs to K_j, we infer that it also belongs to K_{j+1}. Therefore, the sequence of initial states x_n ranges over the intersection $K_\propto$ of the subsets K_n, so that $\mathrm{Reset}_{(F,R)}(K) \subset K_\propto$.

Let us prove now that from any $x_0 \in K_\propto$ starts a viable run, i.e., that $K_\propto$ is contained in the reset kernel of K. For any $n \geq 0$, one can find a solution $x_n(\cdot)$ to the differential inclusion $x' \in F(x)$ starting from x_0, a time $t_n \leq \tau_K^{F^\sharp}(x_0)$ such that $\xi_n := x_n(^-t_n)$ and $y_n \in R(\xi_n) \cap K_{n-1}$ since $x_0 \in \mathrm{Cap}_F^K(R^{-1}(K_n) \cap K)$. One can prove, as in the proof of Lemma 2.8.2, that subsequences (again denoted by) t_n, $x_n(\cdot)$ and y_n converge to some $t \leq \tau_K^{F^\sharp}(x_0)$, $x(\cdot)$ and y respectively, where $x(\cdot)$ is a solution to the differential inclusion starting from x_0, $\xi = x(t)$ and $y \in R(\xi) \cap K$. Since the elements y_n belong to K_{n-1} and since the sequence K_n is decreasing, this limit y belongs to the intersection $K_\propto$ of the subsets K_n. Hence $\xi = x(t)$ belongs to $R^{-1}(K_\propto) \cap K$ and thus, x_0 belongs to $\mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$.

It remains to check that $\mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$ is contained in $K_\propto$. Indeed, starting form $x_0 \in \mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$, there exists a solution to the differential inclusion $x' \in F(x)$ viable in K until it reaches $R^{-1}(K_\propto) \cap K$ at some ξ, when it can be reset to some $x_1 \in R(\xi) \cap K_\propto$. Therefore, a discrete sequence of initial states starts from x_0 and is viable in K, so that x_0 belongs to the reset kernel of K, which is thus equal to $K_\propto$. $\square$

Theorem 2.8.4 *Let us assume that F is Marchaud and that K is a closed repeller under both F and R. Then the reset kernel* $\text{Reset}_{(F,R)}(K)$ *is the largest closed subset D of K satisfying*

$$\begin{cases} i) & K \cap R^{-1}(\text{Reset}_{(F,R)}(K)) \subset D \subset K \\ ii) & \forall\, x \in D \backslash R^{-1}(\text{Reset}_{(F,R)}(K)), \;\; F(x) \cap T_D(x) \neq \emptyset \end{cases}$$

or, equivalently, in terms of normal cones and support functions, if and only if $\text{Reset}_{(F,R)}(K)$ *is the largest closed subset D satisfying*

$$\begin{cases} i) & K \cap R^{-1}(\text{Reset}_{(F,R)}(K)) \subset D \subset K \\ ii) & \forall\, x \in D \backslash R^{-1}(\text{Reset}_{(F,R)}(K)), \; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \geq 0 \end{cases}$$

When K is backward invariant under F, we infer the following consequence:

Theorem 2.8.5 *Assume that $F : X \rightsquigarrow X$ is Marchaud and Lipschitz and that K is a closed repeller under both F and R and backward invariant under F. Then the reset kernel* $\text{Reset}_{(F,R)}$ *under the impulse differential inclusion (F, R) is the unique Frankowska extension of* $R^{-1}(\text{Reset}_{(F,R)}(K)) \cap K$*, i.e., the unique closed subset D such that*

$$R^{-1}(D) \cap K \subset D \subset K$$

satisfying

$$\begin{cases} i) & \forall\, x \in D \backslash R^{-1}(D), \;\; F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall\, x \in D, \;\; F(x) \subset -T_D(x) \end{cases}$$

or again,

$$\begin{cases} i) & \forall\, x \in D \backslash R^{-1}(D), \; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) = 0 \\ ii) & \forall\, x \in D, \; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \leq 0 \end{cases}$$

Example Let $S \subset X$ be a closed subset (the switching set), with which we associate the reset map $R := \mathbf{1} + S$ defined by

$$\forall\, x \in X, \;\; R(x) := x + S$$

Then a closed subset $L \subset K$ is a viable under $(F, \mathbf{1} + S)$ if and only if

$$L \ = \ \mathrm{Capt}_F^K((L - S) \cap K)$$

If not, assuming that the subsets $(K + B - x) \cap S$ when x ranges over K are compact, the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ of K is the largest closed subset contained in K viable under $(F, \mathbf{1} + S)$, equal to the largest fixed set

$$\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) \ = \ \mathrm{Capt}_F^K((\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S) \cap K)$$

and also equal to the intersection of the closed subsets K_n defined recursively by

$$K_{n+1} \ := \ K_n \cap \mathrm{Capt}_F^K((K_n - S) \cap K)$$

If we assume furthermore that K is backward invariant under F, then the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ is the **unique** closed subset $D \subset K$ containing $(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S) \cap K$ which is backward invariant and such that $D \backslash (\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S)$ is locally viable.

When F is assumed to be also Lipschitz, we deduce from the characterization of invariant subsets that the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ is the **unique** closed subset $D \subset K$ containing $(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S) \cap K$ satisfying

$$\begin{cases} i) & \forall\, x \in D, \ \ F(x) \ \subset \ -T_D(x) \\ ii) & \forall\, x \in D \backslash (\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S), \ \ F(x) \cap T_D(x) \ \neq \ \emptyset \end{cases}$$

or, in normal form,

$$\begin{cases} i) & \forall\, x \in D, \, \forall\, p \in N_D(x), \ \ \sigma(F(x), -p) \ \leq \ 0 \\ ii) & \forall\, x \in D \backslash (\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) - S), \ \ \forall\, p \in N_D(x), \ \ \sigma(F(x), -p) \ = \ 0 \end{cases}$$

Remark: Viability Kernel of the Initialization Map — Since the sequence of initialized states is governed by the initialization map, the question arises to compare the discrete viability kernel of K under the initialization map with the reset kernel of K under the impulse differential inclusion:

Proposition 2.8.6 *The reset kernel* $\mathrm{Reset}_{(F,R)}(K)$ *of* K *under the impulse differential inclusion is equal to the discrete viability kernel* $Viab_{U_{(F,R)}}(K)$ *of* K *under the initialization map* $U_{(F,R)}$.

Proof — The reset kernel $\text{Reset}_{(F,R)}(K)$ is obviously contained in $Viab_{U_{(F,R)}}(K)$. Assume now that x_0 belongs to the discrete viability kernel $Viab_{U_{(F,R)}}(K)$ and let us choose a viable sequence $x_{n+1} \in U_{(F,R)}(x_n) \cap K$ of reinitialized states. By definition of the initialization map, there exists a solution $x(\cdot)$ of the impulse differential inclusion such that for any n, $x_{n+1} \in R(x(^{-}t_n))$ which is viable in K. Then x_0 belongs to the reset kernel $\text{Reset}_{(F,R)}(K)$. □

Chapter 3

Asymptotic Stability

3.1 Lyapunov Functions

3.1.1 Definition of Lyapunov Functions for Impulse Differential Inclusions

Let $F : X \mapsto X$ be a Marchaud map, $R : X \rightsquigarrow X$ is an upper semicontinuous map with compact images and $S := \mathbf{1} - R$ governing the evolution of runs of the impulse differential inclusion described by (F, R) by

$$x'(t) \;\in\; F(x(t)) + \sum_{k=1}^{\infty} S(x(^{-}t_k))\delta(t_k) \tag{3.1}$$

where $t_0 \leq t_1 \cdots \leq \cdots \leq t_n \leq \cdots$ denotes a sequence of “switching” times and $x(^{-}t_k) \in K$ a sequence of elements of K.

We also introduce a time-dependent function $w(\cdot)$ defined as a solution to the differential equation

$$w'(t) \;=\; -\varphi(x(t), w(t)) \tag{3.2}$$

where $\varphi : X \times \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth.

This section is devoted to specific viability constraints — called, dynamical inequalities — which can be written in the form

$$\forall\, t \in [0, T], \;\; \mathbf{v}(x(t)) \;\leq\; w(t)$$

where $\mathbf{v} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is a given nontrivial nonnegative extended function.

We recall that the epigraph of the contingent epiderivative $D_\uparrow \mathbf{u}(x)$ of $\mathbf{u}$ at x is the contingent cone to the epigraph of $\mathbf{u}$ at $(x, \mathbf{u}(x))$:

$$\mathcal{E}p(D_\uparrow \mathbf{u}(x)) \ := \ T_{\mathcal{E}p(\mathbf{u})}(x, \mathbf{u}(x))$$

and that

$$\forall \, u \in X, \ \ D_\uparrow \mathbf{u}(x)(u) \ = \liminf_{h \to 0+, \, u' \in u} \frac{\mathbf{u}(x + hu') - \mathbf{u}(x)}{h}$$

See for instance Chapter 6 of [31, Aubin & Frankowska] for more details.

When F is Marchaud and φ continuous with linear growth, we know that the two following statements are equivalent:

1. From every $x_0 \in \mathrm{Dom}(\mathbf{v})$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ and from $\mathbf{v}(x_0)$ starts a solution $w(\cdot)$ to the differential equation $w' = \varphi(x, w)$ such that $\mathbf{v}(x(t)) \leq w(t)$ for all positive times

2. the dynamics F and the extended function $\mathbf{v}$ are linked by the property[1]

$$\forall \, x \in \mathrm{Dom}(\mathbf{v}), \ \ \inf_{v \in F(x)} D_\uparrow \mathbf{v}(x)(v) + \varphi(x, \mathbf{v}(x)) \ \leq \ 0$$

If these equivalent properties do not hold true, there exists an extended lower semicontinuous function $\mathbf{v}_\propto \geq \mathbf{v}$ which is the smallest of the lower semicontinuous functions $\mathbf{v} \geq \mathbf{v}$ satisfying the above equivalent properties.

3.1.2 Characterization of Lyapunov Functions

These results can be also extended to the case of impulse differential inclusions:

Theorem 3.1.1 *Let $\mathbf{v}$ be a nontrivial nonnegative lower semicontinuous extended function, $F : X \rightsquigarrow X$, be a Marchaud map, $R : X \rightsquigarrow X$ be an upper semicontinuous map with compact images and $\varphi : X \times \mathbf{R}_+ \to \mathbf{R}$ be continuous with linear growth. We set*

---

[1] We recognize the classical definition of one brand of Lyapunov functions because when $\mathbf{v}$ is differentiable, $F \equiv f$ is single-valued, it boils down to

$$\langle \mathbf{v}'(x), f(x) \rangle + \varphi(x, \mathbf{v}(x)) \leq 0$$

1. $\mathbf{v}_R(x) := \inf_{y\in R(x)} \mathbf{v}(y)$, *the* marginal function,

2. $\mathbf{R}_{\mathbf{v}_R}(x) := \{y \in R(x) \mid \mathbf{v}(y) = \mathbf{v}_R(x)\}$, *the* marginal map.

Then the two following conditions are equivalent:

1. *for any initial state* $x_0 \in \mathrm{Dom}(\mathbf{v})$, there exist a run $x(\cdot)$ *to the impulse differential inclusion* (F, R) *and a solution to the differential equation* $w(\cdot)$ *to (3.2) satisfying property*

$$\forall\, t \geq 0,\ \ \mathbf{v}(x(t)) \ \leq\ w(t), \quad w(0) = \mathbf{v}(x(0)) \tag{3.3}$$

2. $\mathbf{v}$ *is a contingent solution to the* Hamilton-Jacobi variational inequalities*:* whenever $\mathbf{v}(x) < \mathbf{v}_R(x)$*, then*

$$\inf_{v\in F(x)} D_\uparrow \mathbf{v}(x)(v) + \varphi(x, \mathbf{v}(x)) \ \leq\ 0 \tag{3.4}$$

Proof — We set $\mathbf{F}(x, w) := F(x) \times \{-\varphi(x, w)\}$ and $\mathbf{R}(x, w) := R(x) \times \{w\}$. Obviously, the impulse differential inclusion $(\mathbf{F}, \mathbf{R})$ has a run satisfying (3.3) if and only if the auxiliary impulse differential inclusion $(\mathbf{F}, \mathbf{R})$ has a run starting at $(x_0, \mathbf{v}(x_0))$ viable in $\mathcal{E}p(\mathbf{v})$.

Let us set $\mathbf{v}_R(x) := \inf_{y\in R(x)} \mathbf{v}(y)$. We then note that $\mathbf{R}^{-1}(\mathcal{E}p(\mathbf{v})) = \mathcal{E}p(\mathbf{v}_R)$ and that $(x, w) \in \mathcal{E}p(\mathbf{v})\backslash\mathbf{R}^{-1}\mathcal{E}p(\mathbf{v})$ if and only if

$$\mathbf{v}(x) \ \leq\ w \ <\ \mathbf{v}_R(x) := \inf_{y\in R(x)} \mathbf{v}(y)$$

Since $\mathbf{F}$ is Marchaud and $\mathbf{R}$ is upper semicontinuous with compact images, this is equivalent to say that for all $(x, w) \in \mathcal{E}p(\mathbf{v})\backslash\mathbf{R}^{-1}\mathcal{E}p(\mathbf{v})$, there exists $u \in F(x)$ such that $(u, -\varphi(x, w)) \in T_{\mathcal{E}p(\mathbf{v})}(x, w)$.

This condition implies (3.4) because by taking $w = \mathbf{v}((x))$, we infer that

$$(v, -\varphi(x, \mathbf{v}(x))) \in T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x)) = \mathcal{E}p(D_\uparrow \mathbf{v}(x))$$

for some $v \in F(x)$. Hence $\mathbf{v}$ is a contingent solution to the Hamilton-Jacobi variational inequality (3.4).

Conversely, since $F(x)$ is compact and $v \mapsto D_\uparrow \mathbf{v}(x)(v)$ is lower semicontinuous, (3.4) implies that there exists $v \in F(x)$ such that the pair $(v, -\varphi(x, \mathbf{v}(x)))$ belongs to $T_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$. If $\mathbf{v}(x) < w < \mathbf{v}_R(x)$, we observe that for every $\mu \in \mathbf{R}$, (v, μ) belongs to $T_{\mathcal{E}p(\mathbf{v})}(x, w)$, and in particular, for $\mu := -\varphi(x, w)$. □

Therefore, a run $(x(\cdot), w(\cdot)$ of the auxiliary impulse differential inclusion $(\mathbf{F}, \mathbf{R})$ is defined in the following way: At impulse time t_n and initial conditions x_n and $w(t_n)$ such that $\mathbf{v}(x_n) \leq w(t_n)$, then

1. if $\mathbf{v}_R(x_n) \leq w(t_n)$, we take $t_{n+1} = t_n$ and $x_{n+1} \in R(x_n)$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x_n)$, so that $\mathbf{v}(x_{n+1}) \leq w(t_{n+1}) = w(t_n)$,

2. if $w(t_n) < \mathbf{v}_R(x_n)$, there exist a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at time t_n from x_n and a solution $w(\cdot)$ to the differential equation $w' = \varphi(x, w)$ starting at time t_n from $w(t_n)$ satisfying $\mathbf{v}(x(t)) \leq w(t)$ until a time t_{n+1} such that $\mathbf{v}_R(x(^{-}t_{n+1})) = w(t_{n+1})$ if

$$w(t_n) < \min(\mathbf{v}_{\propto}(x_n), \mathbf{v}_R(x_n))$$

We thus reset the initialized state by taking $x_{n+1} \in R(x(^{-}t_{n+1}))$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x(^{-}t_{n+1}))$ and $w_{n+1} := w(t_{n+1})$, so that $\mathbf{v}(x_{n+1}) = w(t_{n+1})$. □

Remark — We can reformulate the viability theorem in the following way:

Corollary 3.1.2 *Let $F : X \rightsquigarrow X$ be a Marchaud map and $R : X \rightsquigarrow$ be an upper semicontinuous map with compact images. A closed subset K is viable under (F, R) if and only if its indicator ψ_K is a contingent solution to the Hamilton-Jacobi variational equation*

$$\forall\, x \in K \backslash R^{-1}(K), \quad \inf_{v \in F(x)} D_{\uparrow}\psi_K(x)(v) \;=\; 0$$

Example: Exponential Lyapunov Functions of a Impulse Differential Inclusion Let us assume that the nontrivial lower semicontinuous extended function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is bounded from below: we set

$$v_0 \;:=\; \inf_{x \in X} \mathbf{v}(x)$$

The function $\mathbf{v} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is said *to enjoy the a-Lyapunov property* if and only if for any initial state x_0, there exists a run $x(\cdot)$ of the impulse differential inclusion (F, R) satisfying

$$\forall\, t \geq 0, \;\; \mathbf{v}(x(t)) \;\leq\; w(t) := (\mathbf{v}(x_0) - v_0)e^{-at} + v_0 \tag{3.5}$$

Such inequalities allow us to deduce many properties on the asymptotic behavior of $\mathbf{v}$ along some runs of the impulsive differential inclusion, such as the fact that $\mathbf{v}(x(t))$ converges to v_0.

We deduce from Theorem 3.1.1 with $\varphi(x,w) := a(w - v_0)$ that property (3.5) holds true if and only if whenever $\mathbf{v}(x) < \inf_{y\in R(x)} \mathbf{v}(y)$, then

$$\inf_{v\in F(x)} D_{\uparrow}\mathbf{v}(x)(v) + a(\mathbf{v}(x) - v_0) \quad \leq \quad 0 \tag{3.6}$$

In particular, when $\mathbf{v}(x) := \|x - c\|^2$ and $v_0 := 0$:

$$\inf_{v\in F(x)} \langle x - c, v\rangle + 2a\|x - c\|^2 \leq 0$$

We call them *exponential Lyapunov functions of a impulse differential inclusion (with respect to a).*

3.1.3 Lyapunov Functions for Hybrid Control Systems

Since a hybrid control systems defined by

$$\left\{ \begin{array}{ll} i) & x'(t) = f(e(t), x(t), u(t)) \\ ii) & x(t) \in U(e(t), x(t)) \end{array} \right. \tag{3.7}$$

reset by a set-valued map $R : E \times X \rightsquigarrow E \times X$ is the impulse differential inclusions $\{0\} \times f(K(e), x, U(x))$, we can translate Theorem 3.1.1 to this case for characterizing an "hybrid" Lyapunov function, defined as an extended functions $\mathbf{v} : E \times X \mapsto \mathbf{R} \cup \{+\infty\}$ possibly depending on the locations (for instance $\mathbf{v}(e,x) := \|x - c(e)\|$ where $c(e) \in K(e)$ is an equilibrium, i.e., a solution to the equation $f(e, c(e), u(e)) = 0$ where $u(e) \in U(c(e))$).

Theorem 3.1.3 *Let $\mathbf{v} : E \times X \rightsquigarrow \mathbf{R} \cup \{+\infty\}$ be a nontrivial nonnegative lower semicontinuous extended function, $U : E \times X \rightsquigarrow E \times X$, be a Marchaud map, $f : E \times X \times Y$ be a continuous map with linear growth affine with respect to the controls and $R : E \times X \rightsquigarrow E \times X$ be an upper semicontinuous map with compact images and $\varphi : X \times \mathbf{R}_+ \to \mathbf{R}$ be continuous with linear growth. We set*

(a) $\mathbf{v}_R(e,x) \;:=\; \inf_{(f,y)\in R(e,x)} \mathbf{v}(f,y)$, *the* marginal function*,*

(b) $\mathbf{R}_{\mathbf{v}_R}(e,x) \;:=\; \{(f,y) \in R(e,x) \mid \mathbf{v}(f,y) = \mathbf{v}_R(e,x)\}$, *the* marginal map.

Then the two following conditions are equivalent:

(a) *for any initial state* $x_0 \in \mathrm{Dom}(\mathbf{v})$*,* there exist a run $(e(\cdot), x(\cdot)$ *to the hybrid control system (3.7) and a solution to the differential equation* $w(\cdot)$ *to (3.2) satisfying property:*

$$\forall\, t \in [t_n, t_{n+1}[,\ \ \mathbf{v}(e(t_n), x(t)) \ \leq\ w(t)$$

(b) $\mathbf{v}$ *is a contingent solution to the* Hamilton-Jacobi variational inequalities*:* whenever $\mathbf{v}(e,x) < \mathbf{v}_R(e,x)$*, then*

$$\inf_{v \in F(e,x)} D_{\uparrow}\mathbf{v}(e,x)(v) + \varphi(x, \mathbf{v}(e,x)) \ \leq\ 0$$

3.2 Hybrid Gradient Methods for Global Optimization

Let us consider the minimization problem

$$v_0 \ :=\ \inf_{x \in X} \mathbf{v}(x)$$

where $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is a nontrivial lower semicontinuous extended function assumed to be bounded from below.

A way to introduce the "Gradient Method" is to use the simple differential inclusion

$$\forall\, t \geq 0,\ \ x'(t)\ \in\ B$$

where B denotes the unit ball of the finite dimensional vector space X, leaving open the direction to be chosen by the algorithm. Instead of choosing the velocities at random as in the methods of simulated annealing and being satisfied by convergence in probability, we shall ask whether $\mathbf{v}$ can be an exponential Lyapunov function for the differential inclusion $x' \in B$.

We introduce the function $\mathbf{v}_{\propto} \geq \mathbf{v}$ which is the smallest of the lower semicontinuous a-Lyapunov functions larger than or equal to $\mathbf{v}$, the minima of which coincide with the global minima of $\mathbf{v}$. The gradient algorithm for $\mathbf{v}_{\propto}$ converges to global minimal of $\mathbf{v}$ (see [38, Aubin & Najman] and Chapter 8 of [10, Aubin] about the "Montagnes Russes Algorithm").

Instead of using this (costly but efficient) algorithm, we can use a reset map R as a discrete search algorithm which can be paired with the usual gradient method for jumping over local minima.

Theorem 3.1.1 implies that whenever $\mathbf{v}(x) < \inf_{y\in R(x)} \mathbf{v}(y)$, then

$$\inf_{u\in B} D_{\uparrow}\mathbf{v}(x)(u) + a(\mathbf{v}(x) - v_0) \ \leq\ 0 \tag{3.8}$$

is necessary and sufficient for the existence of one run $x(\cdot)$ to the impulse differential inclusion (B, R) starting from any given initial state $x_0 \in \mathrm{Dom}(\mathbf{v})$.

Therefore, a run $(x(\cdot), (\cdot)$ of the auxiliary impulse differential inclusion $B \times \{-a(\mathbf{v}(x) - v_0)\}, R \times \mathbf{1})$ is defined in the following way: At impulse time t_n and initial conditions x_n and $(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$ such that $\mathbf{v}(x_n) \leq (\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$, then

(a) if $\mathbf{v}_R(x_n) \leq (\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0$, we take $t_{n+1} = t_n$ and $x_{n+1} \in R(x_n)$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x_n)$, so that $\mathbf{v}(x_{n+1}) \leq (\mathbf{v}(x_0) - v_0)e^{-at_{n+1}} + v_0$,

(b) if $(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0 < \mathbf{v}_R(x_n)$, there exist a solution $x(\cdot)$ to the gradient method

$$x'(t) \in \{w \in B | D_{\uparrow}\mathbf{v}(x)(w) + a(\mathbf{v}(x) - v_0) \leq 0\}$$

starting at time t_n from x_n satisfying $\mathbf{v}(x(t)) \leq (\mathbf{v}(x_0) - v_0)e^{-at} + v_0$ until a time t_{n+1} such that $\mathbf{v}_R(x(^{-}t_{n+1})) = w(t_{n+1})$ if

$$(\mathbf{v}(x_0) - v_0)e^{-at_n} + v_0 < \min(\mathbf{v}_{\propto}(x_n), \mathbf{v}_R(x_n))$$

We thus reset the initialized state by taking $x_{n+1} \in R(x(^{-}t_{n+1}))$ such that $\mathbf{v}(x_{n+1}) = \mathbf{v}_R(x(^{-}t_{n+1}))$, so that $\mathbf{v}(x_{n+1}) = (\mathbf{v}(x_0) - v_0)e^{-at_{n+1}} + v_0$. □

Consequently, the algorithm stops at some time T and at a state $\bar{x}$ from which starts a sequence $x_{j+1} \in R(x_j)$ satisfying

$$\forall\, j \geq 0,\ \ \mathbf{u}_R(x_j) \ \leq\ (\mathbf{v}(x_0) - v_0)e^{-aT} + v_0$$

or else, $\mathbf{v}(x(t))$ converges to v_0. □

Remark — Observe also that the Fermat rule states that at every local minimum x of $\mathbf{v}$, $0 \leq \inf_{u\in B} D_{\uparrow}\mathbf{v}(x)(u)$. Therefore, an exponential Lyapunov function satisfying

$$\text{If } v_0 < \mathbf{v}(x) < \inf_{y\in R(x)} \mathbf{v}(y), \text{ then } \inf_{u\in B} D_{\uparrow}\mathbf{v}(x)(u) \ \leq\ -a(\mathbf{v}(x) - v_0) \ <\ 0$$

does not have local minima when $v_0 < \mathbf{v}(x) < \mathbf{u}_R(x) := \inf_{y \in R(x)} \mathbf{v}(y)$. In other words, local minima — if any — lie in the subset

$$\mathbf{v}^{-1}(v_0) \bigcup \{x \quad \text{such that} \quad \mathbf{v}(x) \geq \inf_{y \in R(x)} \mathbf{v}(y)\}$$

Chapter 4

Optimal Impulse Control

4.1 The Auxiliary System

The evolution of a control problem (P, f) with a priori feedback map $P : X \leadsto \mathcal{P}$ from X to some finite dimensional vector space $\mathcal{P}$ is governed by

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), u(t)) \\ ii) & u(t) & \in & P(x(t)) \end{array} \right. \tag{4.1}$$

Denote by $\mathcal{S}_{(P,f)}(x_0)$ the set of pairs $(x(\cdot), u(\cdot))$ solutions to the control problem (4.1) starting from x_0 at time 0, i.e., such that $x(0) = x_0$.

Let us introduce now a nonnegative lower semicontinuous "Lagrangian"

$$l : (x, v) \in \text{Graph}(P) \mapsto l(x, v) \in \mathbf{R}_+$$

assumed to be convex with respect to u and to have linear growth

$$l(x, u) \ \leq \ c(\|x\| + \|u\| + 1)$$

Let us introduce a lower semicontinuous "cost function" $\mathbf{w} : X \mapsto \mathbf{R} \cup \{+\infty\}$ and a Lagrangian $l : \text{Graph}(P) \mapsto \mathbf{R}$.

We shall characterize the value function

$$\left\{ \begin{array}{l} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \ := \\ \displaystyle\inf_{(x(\cdot),u(\cdot)) \in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k} \mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{array} \right.$$

of the control problem (4.1) under the Lagrangian l and the cost function $\mathbf{w}$ by proving that its epigraph is the reset kernel of $K \times \mathbf{R}_+$ under the auxiliary impulse differential inclusion (G, R) we now define.

We associate with the control system (P, f) and the Lagrangian l the set-valued map $G : X \times \mathbf{R}_+ \rightsquigarrow X \times \mathbf{R}$ defined by

$$G(x,y) \ := \ \{\{f(x,u)\} \times (ay + [-c(\|x\| + \|u\| + 1), -l(x,u)])\}_{u \in P(x)}$$

which is a Marchaud map whenever (P, f) is Marchaud and the Lagrangian l satisfies the above assumptions.

We shall also associate with the function $\mathbf{w}$ the auxiliary reset map $R : X \times \mathbf{R} \rightsquigarrow X \times \mathbf{R}$ defined by

$$R(x,y) \ := \ (x,y) + \mathcal{H}yp(\mathbf{w})$$

associated with the constant switching map $\mathcal{H}yp(\mathbf{w})$.

In summary, we asociate with the optimal impulse control system the auxiliary impulse differential inclusion (G, R) on the vector space $X \times \mathbf{R}$. We shall prove that the reset kernel of $X \times \mathbf{R}_+$ under (G, R) is the epigraph of the value function of the optimal impulse control problem and translate the properties of the reset kernel for obtaining corresponding properties of the value function.

Since the reset kernel involves the inverse image $R^{-1}(\mathcal{E}p(\mathbf{v}))$ and the capture basin of the epigraph of a function $\mathbf{u}$, we shall first show that they are respectively the epigraph of an inf-convolution and the epigraph of a stopping time.

We associate with any extended functions[1] $\mathbf{u}$ and $\mathbf{w}$ the extended function $\mathbf{u} * \mathbf{w}$ defined by

$$(\mathbf{u} * \mathbf{w})(x) \ := \ \inf_{y \in X} (\mathbf{u}(y) + \mathbf{w}(y - x)) \ := \ \inf_{y \in X} (\mathbf{u}(x + y) + \mathbf{w}(y))$$

We shall prove that under adequate assumptions, the inverse image $R^{-1}(\mathcal{E}p(\mathbf{u}))$ of the epigraph of a lower semicontinuous nonnegative extended function u is the epigraph of $\mathbf{u} * \mathbf{w}$:

[1] The inf-convolution or episum $\mathbf{u} \oplus \mathbf{w}$ of the functions $\mathbf{u}$ and $\mathbf{w}$ is defined by

$$(\mathbf{u} * \mathbf{w})(x) \ := \ \inf_{y \in X} (\mathbf{u}(y) + \mathbf{w}(x - y)) \ := \ \inf_{y \in X} (\mathbf{u}(x - y) + \mathbf{w}(y))$$

The impulse optimal control problem leads instead to the functions $\mathbf{u}$ and $\mathbf{w}$, which have the same kind of properties.

Theorem 4.1.1 *Let us assume that* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is lower semicontinuous and nonnegative and that* $\mathbf{w} : X \mapsto \mathbf{R} \cup \{\infty\}$ *is lower semicontinuous, nonnegative and coercive in the sense that*

$$c \;:=\; \liminf_{\|y\| \to +\infty} \frac{\mathbf{w}(y)}{\|y\|} \;>\; 0$$

Then the function $\mathbf{u} * \mathbf{w}$ *is lower semicontinuous and*

$$\begin{cases} i) & R^{-1}(\mathcal{E}p(\mathbf{u})) \;=\; \mathcal{E}p(\mathbf{u} * \mathbf{w}) \\ ii) & R^{-1}(\mathcal{E}p(\mathbf{u})) \cap \mathcal{E}p(\mathbf{u}) \;=\; \mathcal{E}p(\max(\mathbf{u}, \mathbf{u} * \mathbf{w})) \\ iii) & \mathcal{E}p(\mathbf{u}) \backslash R^{-1}(\mathcal{E}p(\mathbf{u})) \;=\; \mathcal{E}_0 \;:=\; \{(x, y) \in X \times \mathbf{R} \quad \textit{such that} \\ & \mathbf{u}(x) \;\le\; y \;<\; (\mathbf{u} * \mathbf{w})(x)\} \end{cases}$$

Furthermore, for every $x \in \mathrm{Dom}(\mathbf{u} * \mathbf{w})$*, the set*

$$R_{\mathbf{w}}(x) \;:=\; \{y \in X \mid (\mathbf{u} * \mathbf{w})(x) \;=\; \mathbf{u}(x + y) + \mathbf{w}(y)\}$$

is not empty.

Proof — We first observe that

$$R^{-1}(\mathcal{E}p(\mathbf{u})) \;=\; \mathcal{E}p(\mathbf{u}) - \mathcal{H}yp(-\mathbf{w})$$

and consequently, that

$$R^{-1}(\mathcal{E}p(\mathbf{u})) \;\subset\; \mathcal{E}p(\mathbf{u} * \mathbf{w})$$

Before proving the converse inclusion, let us prove that the function $\mathbf{u} * \mathbf{w}$ is lower semicontinuous. For that purpose, we have to check that its epigraph is closed. Let (x_n, λ_n) be a sequence of the epigraph of $\mathbf{u} * \mathbf{w}$ converging to (x, λ). By definition of the infimum, we can associate y_n such that

$$\forall\, n \ge 0, \;\; \mathbf{u}(y_n + x_n)\mathbf{w}(y_n) \;\le\; (\mathbf{u} * \mathbf{w})(x_n) + \frac{1}{n} \;\le\; \lambda + 1 \;=:\; \alpha$$

Since $\mathbf{u}$ is nonnegative, we infer that

$$\frac{\mathbf{w}(y_n)}{\|y_n\|} \;\le\; \frac{\alpha}{\|y_n\|}$$

and since $\mathbf{w}$ is coercive, that there exists $R > 0$ such that for every $\|y_n\| \geq R$,

$$\frac{c}{2} \leq \frac{\mathbf{w}(y_n)}{\|y_n\|} \leq \frac{\alpha}{\|y_n\|}$$

Hence we infer that

$$\forall\, n \geq 0, \quad \|y_n\| \;\leq\; \max\left(R, \frac{2\alpha}{c}\right)$$

Since X is finite dimensional, this bounded sequence is compact, so that a subsequence (again denoted by) y_n converges to some y. The function $(x, y) \mapsto \mathbf{u}(x + y) + \mathbf{w}(y)$ being lower semicontinuous, we infer by taking the limit that

$$(\mathbf{u} * \mathbf{w})(x) \;\leq\; \mathbf{u}(x + y) + \mathbf{w}(y) \;\leq\; \lambda$$

so that (x, λ) belongs to the epigraph of $\mathbf{u} * \mathbf{w}$, which is then closed.

In particular, taking $x_n := x$ and $\lambda := (\mathbf{u} * \mathbf{w})(x)$, we deduce that

$$(\mathbf{u} * \mathbf{w})(x) \;=\; \mathbf{u}(x + y) + \mathbf{w}(y)$$

so that the limit y achieves the infimum of the minimization problem $(\mathbf{u}*\mathbf{w})(x)$. This also implies that

$$(x, (\mathbf{u} * \mathbf{w})(x)) \;=\; (x + y, \mathbf{u}(x + y)) - (y, -\mathbf{w}(y))$$

and thus, that $R^{-1}(\mathcal{E}p(\mathbf{u}))$ coincides with the epigraph of $\mathbf{u} * \mathbf{w}$.

The other properties are obvious consequences. □

Corollary 4.1.2 *We posit the assumptions of Theorem 4.1.1 and we assume furthermore that* $\inf_{y \in X} \mathbf{w}(y) > 0$, *then the* continuation set *defined by*

$$\mathbf{C} \;:=\; \{x \in X \mid \mathbf{u}(x) \;<\; (\mathbf{u} * \mathbf{w})(x)\}$$

is not empty.

Proof — The continuation set is the projection onto X of the set $\mathcal{E}p(\mathbf{u}) \backslash R^{-1}(\mathcal{E}p(\mathbf{u}))$ of $\mathcal{E}p(\mathbf{u})$ under the reset map R.

Assume that the continuation subset $\mathbf{C}$ is empty. Then for every $x \in X$, we should have

$$\mathbf{u}(x) \;=\; (\mathbf{u} * \mathbf{w})(x) \;\geq\; \inf_{y \in X} \mathbf{w}(y) + \inf_{y \in X} \mathbf{u}(y)$$

so that, by taking the infimum on the left hand-side of this inequality, we obtain $\inf_{y \in X} \mathbf{w}(y) = 0$, a contradiction. □

We also observe that by defining recursively the extended function $\mathbf{u}_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ by the discrete viability algorithm

$$\mathcal{E}p(\mathbf{u}_{n+1}) \;:=\; \mathcal{E}p(\mathbf{u}_n) \cap R^{-1}(\mathcal{E}p(\mathbf{u}_n))$$

we obtain

$$\mathbf{u}_n(x) \;=\; \max\left(\mathbf{u}(x), \inf_{y \in Y}(\mathbf{w}(y) + \mathbf{v}(x+y), \ldots, \inf_{y_j \in X,\, j=1,\ldots n}\left(\sum_{j=1}^{n} \mathbf{w}(y_j) + \mathbf{v}\left(x + \sum_{j=1}^{n} y_j\right)\right)\right)$$

Therefore,

$$\forall\, x \in X, \;\; \mathbf{u}_n(x) \;\geq\; n \inf_{x \in X} \mathbf{w}(x) + \inf_{x \in X} \mathbf{u}(x)$$

If we assume that $\inf_{y \in X} \mathbf{w}(y) > 0$, then

$$\mathbf{u}_\infty(x) \;:=\; \sup_{n \geq 0} \mathbf{u}_n(x)$$

is the constant function $+\infty$. In other words, the epigraph of $\mathbf{u}$ is a repeller under the discrete dynamical system defined by $R(x,y) := (x,y) + \mathcal{H}yp(-\mathbf{w})$, since its discrete viability kernel, being the epigraph of $\mathbf{u}_\infty = +\infty$.

This implies that the set $\mathcal{E}p(\mathbf{u}) \backslash R^{-1}(\mathcal{E}p(\mathbf{u}))$ is not empty.

Actually, the subsets

$$\mathcal{E}_n \;:=\; \{(x,y) \in X \times \mathbf{R} \mid \mathbf{u}_n(x) \;\leq\; y \;<\; \mathbf{u}_{n+1}(x)\}$$

form a partition of the subset

$$\mathcal{E}_0 \;=\; \bigcup_{n \geq 1} \mathcal{E}_n$$

Let us associate with any $x \in X$ the integer $n(x)$ such that $\mathbf{u}_1(x) = \mathbf{u}_{n(x)}(x) < \mathbf{u}_{n(x)+1}(x)$ and the map $\vec{R}_{\mathbf{w}}$ defined by

$$\vec{R}_{\mathbf{w}}(x) \ := \ \left\{ \sum_{j=1}^{n(x)} y_j \mid \mathbf{u}_{n(x)}(x) \ = \ \sum_{j=1}^{n(x)} \mathbf{w}(y_j) + \mathbf{u}\left(x + \sum_{j=1}^{n(x)} y_j \right) \right\}$$

Therefore, for any x belonging to the stopping set $\mathbf{S}$ defined by

$$\mathbf{S} \ := \ \{x \in X \quad \text{such that} \ \ \mathbf{u}(x) \ = \ (\mathbf{u} * \mathbf{w})(x)\}$$

we have

$$\mathbf{u}(\vec{R}_{\mathbf{w}}(x)) \ < \ (\mathbf{u} * \mathbf{w})(\vec{R}_{\mathbf{w}}(x))$$

4.2 Stopping Time Value Function

Let us consider a function[2] $\mathbf{u} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$. We associate with it the stopping time problem

$$\Gamma_{(P,f,l)}(\mathbf{u})(x) \ := \ \inf_{(x(\cdot),u(\cdot)) \in \mathcal{S}_{(P,f)}(x)} \ \inf_{t \geq 0} \left(e^{-at}\mathbf{u}(x(t)) + \int_0^t e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right)$$

The function $\Gamma_{(P,f,l)}(\mathbf{u})$ is called the stopping time function associated with $\mathbf{u}$.

We shall characterize its epigraph:

Proposition 4.2.1 *Let us assume that the control system* (P, f) *is Marchaud, that the Lagrangian* $l : \text{Graph}(P) \rightsquigarrow \mathbf{R}_+ \cup \{+\infty\}$ *is nontrivial, nonnegative, lower semicontinuous, convex with respect to* u *and has linear growth*

$$l(x, u) \ \leq \ c(\|x\| + \|u\| + 1)$$

and that $\mathbf{u} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *is nontrivial, non negative and lower semicontinuous.*

Then the capture basin $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ *of the epigraph of* $\mathbf{u}$ *under* G *is the epigraph of the stopping time function* $\Gamma_{(P,f,l)}(\mathbf{u})$*, which is then lower semicontinuous.*

[2] regarded as an "obstacle" in problems of unilateral mechanics.

Proof — To say that a pair (x, y) belongs to the capture basin $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ means that there exist $t \geq 0$ and a solution $(x(\cdot), u(\cdot)) \in \mathcal{S}_{(P,f)}(x)$ such that

$$\left(x(t), e^{at}y - \int_0^t e^{a(t-\tau)} l(x(\tau), u(\tau))d\tau\right) \in \mathcal{E}p(\mathbf{u})$$

i.e., if and only if

$$e^{-at}\mathbf{u}(x(t)) + \int_0^t e^{-a\tau} l(x(\tau), u(\tau))d\tau \leq y$$

This implies that

$$\begin{cases} \Gamma_{(P,f,l)}(\mathbf{u})(x) \\ := \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \left(e^{-at}\mathbf{u}(x(t)) + \int_0^t le^{-a\tau}(x(\tau), u(\tau))d\tau\right) \leq y \end{cases}$$

and thus, that $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ is contained in $\mathcal{E}p(\Gamma_{(P,f,l)}(\mathbf{u}))$.

Since F is Marchaud, the subset $\widehat{\mathcal{S}}_{(P,f)}(x)$ of solutions $(x(\cdot), u(\cdot))$ to control system (4.1) is compact in the space $\mathcal{C}(0, T, X) \times L^1(0, T, X)$ where $\mathcal{C}(0, T, X)$ supplied with the uniform convergence and $L^1(0, T, X)$ with the weakened topologu. On the other hand, the functional

$$(x(\cdot), u(\cdot)) \mapsto \int_0^t e^{-a\tau} l(x(\tau), u(\tau))d\tau$$

is lower semicontinuous on $\mathcal{C}(0, \infty, X) \times L^1(0, \infty, \mathcal{P})$ when L^1 is supplied with the weakened topology (see for instance Proposition 6.3.4 of [26, Aubin]). Then the infimum

$$\Gamma_{(P,f,l)}(\mathbf{u})(x) := e^{-a\bar{t}}\mathbf{u}(\bar{x}(\bar{t})) + \int_0^{\bar{t}} e^{-a\tau} l(\bar{x}(\tau), \bar{u}(\tau))d\tau$$

is reached by a solution $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{S}_{(P,f)}(x)$ and a time $\bar{t} \geq 0$.

Therefore, if $y \geq \Gamma_{(P,f,l)}(\mathbf{u})(x)$, then the pair

$$\left(\bar{x}(t), \bar{y}(t) := e^{at}y - \int_0^t e^{a(t-s)} l(\bar{x}(s), \bar{u}(s))ds\right)$$

is a solution to the differential inclusion $(x', y') \in G(x, y)$ that reaches $\mathcal{E}p(\mathbf{u})$ at time $\bar{t}$ because

$$\begin{cases} \bar{y}(\bar{t}) \geq e^{at}\Gamma_{(P,f,l)}(\mathbf{u})(T, x) - \int_0^{\bar{t}} e^{a(\bar{t}-s)} l(\bar{x}(s), \bar{u}(s))ds \\ = e^{a\bar{t}}\left(e^{-a\bar{t}}\mathbf{u}(\bar{x}(\bar{t})) + \int_0^{\bar{t}} e^{-a\tau} l(\bar{x}(\tau), \bar{u}(\tau))d\tau = u(\bar{x}(\bar{t}))\right) \end{cases}$$

This states that (x,y) belongs to the capture basin of $\mathbf{R}_+ \times \mathcal{E}p(\mathbf{u})$, which is then equal to the epigraph of $\Gamma_{(P,f,l)}(\mathbf{u})$. Being closed when the system is Marchaud, this means that the value function is lower semicontinuous. $\square$

Theorem 4.2.2 *We posit the assumptions of Propposition 4.2.1 and we assume that*

$$\forall\, x \in X, \quad \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_F(x)} \int_0^{+\infty} e^{-a\tau} l(x(\tau),u(\tau))d\tau \;=\; +\infty$$

Then the stopping time function $\mathbf{v}_\propto := \Gamma_{(P,f,l)}(\mathbf{u})$ *is characterized as the smallest of the nonnegative lower semicontinuous functions* $\mathbf{v} : X \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying for every* x

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; \mathbf{u}(x) \\ ii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \\ & \inf_{u\in P(x)} \left(D_\uparrow \mathbf{v}(x)(f(x,u)) + l(x,u)\right) - a\mathbf{v}(x) \;\leq\; 0 \end{cases}$$

Or, equivalently, in a dual form, denoting by

$$H(x,y,p) \;:=\; \sup_{u\in P(x)} \left(\langle p, f(x,u)\rangle + l(x,u)\right) - ay \;=\; \sigma(G(x,y),(p,-1))$$

the Hamiltonian associated with the optimal control problem, the stopping time function is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : X \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying for every* x

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; \mathbf{u}(x) \\ ii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x),\ \forall\, p \in \partial_-\mathbf{v}(x), \quad H(x,\mathbf{v}(x),p) \;\geq\; 0 \\ & \text{and } \forall\, p \in \partial_-^\infty \mathbf{v}(x), \quad \sigma(f(x,P(x)),p) \;\geq\; 0 \end{cases}$$

If we assume furthermore that P, f *and* l *are Lipschitz, then the stopping time function is the unique Frankowska solution* $\mathbf{v} \geq 0$ *to the system of differential inequalities: for every* $x \in \mathrm{Dom}(\mathbf{v})$,

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; \mathbf{u}(x) \\ ii) & \sup_{u\in P(x)}(D_\uparrow \mathbf{v}(x)(-f(x,u)) - l(x,u)) + a\mathbf{v}(x) \;\leq\; 0 \\ iii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \quad \inf_{u\in P(x)} \left(D_\uparrow \mathbf{v}(x)(f(x,u)) + l(x,u)\right) - a\mathbf{v}(x) \;\leq\; 0 \end{cases}$$

or, equivalently, in dual form,

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x,\mathbf{v}(x),p) \ \leq \ 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \ \sigma(f(x,P(x)),p) \ \leq \ 0 \\ iii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \ \ \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x,\mathbf{v}(x),p) \ = \ 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \ \sigma(f(x,P(x)),p) \ = \ 0 \end{cases}$$

which can reformulated in the form of "variational inequalities"[3]*:*

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x,\mathbf{v}(x),p) \ \leq \ 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \ \sigma(f(x,P(x)),p) \ \leq \ 0 \\ iii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x,\mathbf{v}(x),p)(\mathbf{v}(x)-\mathbf{u}(x)) \ = \ 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \sigma(f(x,P(x)),p)(\mathbf{v}(x)-\mathbf{u}(x)) \ = \ 0 \end{cases}$$

Knowing the stopping time function, an optimal solution is obtained in the following way. Starting from x_0 such that $\mathbf{v}(x_0) < \mathbf{u}(x_0)$, we choose a solution to the control system

$$\begin{cases} i) & x'(t) \ = \ f(x(t),u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}_{(P,f,l)}(x(t)) \end{cases}$$

where

$$\mathbf{R}_{(P,f,l)}(x) \ := \ \{u \in P(x) \mid D_{\uparrow}\mathbf{v}_{(P,f,l)}(x)(f(x,u)) + l(x,u) - a\mathbf{v}(x) \leq 0\}$$

until the first time $\bar{t} \geq 0$ when $\mathbf{v}(x(\bar{t})) = \mathbf{u}(x(\bar{t}))$.

4.3 Value Function of an Optimal Impulse Control

We shall characterize the value function

$$\begin{cases} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \ := \\ \displaystyle\inf_{(x(\cdot),u(\cdot))\in\mathcal{R}^{K}_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

[3]See for instance [55, Bensoussan & Lions J.-L.]. The subset $\{x \in X \mid \mathbf{v}(x) < \mathbf{u}(x)\}$ is called the continuation set and the subset $\{x \in X \mid \mathbf{v}(x) = \mathbf{u}(x)\}$ is called the stopping set.

of the control problem (4.1) under the Lagrangian l and the cost function $\mathbf{w}$ by proving that its epigraph is the reset kernel of $X \times \mathbf{R}_+$ under the auxiliary differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ with the constant switching map $S = \mathcal{H}yp(-\mathbf{w})$:

Theorem 4.3.1 *Let us assume that the control system (P, f) is Marchaud, that the Lagrangian $l : \mathrm{Graph}(P) \rightsquigarrow \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, nonnegative, lower semicontinuous, convex with respect to u and has linear growth*

$$l(x,u) \;\leq\; c(\|x\| + \|u\| + 1)$$

that K is closed and that $\mathbf{w} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, lower semicontinuous and satisifes $\inf_{y \in \mathbf{w}(y)} > 0$.
Then the epigraph of the value function

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \inf_{(x(\cdot),u(\cdot)) \in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k} \mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right)$$

for the optimal impulse control problem (4.1) under the Lagrangian l and the cost function $\mathbf{w}$ is the reset kernel of $K \times \mathbf{R}_+$ under the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$ and the optimal runs are the runs of the impulse differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ viable in $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$.

Proof — Let us consider a pair (x_0, y_0) in the reset kernel of $K \times \mathbf{R}_+$ under the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$. Then there exists a run $(x(\cdot), y(\cdot))$ starting from (x_0, y_0) viable in $K \times \mathbf{R}_+$.

A run to the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$ can be written in the form

$$\forall\, t \in [t_n, t_{n+1}[, \begin{cases} x(t) \;=\; x_0 + \displaystyle\sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau), u(\tau)) d\tau \\ y(t) \;\leq\; e^{at} \left(y_0 + \displaystyle\sum_{k=1}^{n} e^{-at_k} \eta_k - \int_0^t e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

and

$$\begin{cases} i) \quad x(t_{n+1}) \;=\; x_0 + \displaystyle\sum_{k=1}^{n+1} \xi_k + \int_0^{t_{n+1}} f(x(\tau), u(\tau)) d\tau \\ ii) \quad y(t_{n+1}) \;\leq\; e^{at_{n+1}} \left(y_0 - \displaystyle\sum_{k=1}^{n+1} e^{-at_k} \eta_k - \int_0^{t_{n+1}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

where, for every integer k, $\eta_k \leq -\mathbf{w}(\xi_k)$.
Indeed, assuming the above formula to be true, then, for every $t \in [t_{n+1}, t_{n+2}[$,

$$\begin{cases} y(t) \ \leq \ e^{a(t-t_{n+1})}y(t_{n+1}) - \int_{t_{n+1}}^{t} e^{a(t-\tau)}l(x(\tau),u(\tau))d\tau \\ = \ e^{at}\left(y_0 + \sum_{k=1}^{n+1} e^{-at_k}\eta_k - \int_0^{t_{n+1}} e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \\ -\int_{t_{n+1}}^{t} e^{a(t-\tau)}l(x(\tau),u(\tau))d\tau \\ = \ e^{at}\left(y_0 + \sum_{k=1}^{n+1} e^{-at_k}\eta_k - \int_0^{t} e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

and when $t = t_{n+2}$, we take

$$\begin{cases} x(t_{n+2}) \ = \ x(^{-}t_{n+2}) + \xi_{n+2} \\ y(t_{n_2}) \ \leq \ y(^{-}t_{n+2}) + \eta_{n+2} \\ = \ e^{at_{n+2}}\left(y_0 + \sum_{k=1}^{n+1} e^{-at_k}\eta_k - \int_0^{t_{n+2}} e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) + \eta_{n+2} \\ = \ e^{at_{n+2}}\left(y_0 + \sum_{k=1}^{n+2} e^{-at_k}\eta_k - \int_0^{t_{n+2}} e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

Therefore, a run to the impulse differential inclusion $(G, \mathbf{1}+\mathcal{H}yp(-\mathbf{w}))$ viable in $K \times \mathbf{R}_+$ satisfies

$$\forall\, t \in [t_n, t_{n+1}[, \begin{cases} x(t) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau),u(\tau))d\tau \\ e^{-at}y(t) + \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau \ \leq \ y_0 \end{cases}$$

and, at switching times t_{n+1},

$$\begin{cases} i) \quad x(t_{n+1}) \ = \ x_0 + \sum_{k=1}^{n+1} \xi_k + \int_0^{t_{n+1}} f(x(\tau),u(\tau))d\tau \\ ii) \quad e^{-at_{n+1}}y_{n+1}(t) + \sum_{k=1}^{n+1} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{t_{n+1}} e^{-a\tau}l(x(\tau),u(\tau))d\tau \ \leq \ y_0 \end{cases}$$

Since the run $(x(\cdot), y(\cdot))$ is viable in $K \times \mathbf{R}_+$, we infer that $y(t) \geq 0$, and thus, that

$$\forall\, t \in [t_n, t_{n+1}[, \quad \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau \ \leq \ y_0$$

The run cannot stop in finite time t_n where $t_{n+j} = t_n$ for any $j \in \mathbf{N}$. Indeed, in this case, we would obtain

$$\sum_{k=1}^{n-1} e^{-at_k}\mathbf{w}(\xi_k) + e^{-at_n}\sum_{j=1}^{+\infty}\mathbf{w}(\xi_{n+j}) + \int_0^{t_n} e^{-a\tau} l(x(\tau), u(\tau))d\tau \ \leq \ y_0$$

Since $\mathbf{w}(\xi_{n_j}) \geq c := \inf_y \mathbf{w}(y) > 0$, the left hand-side of the above inequality would go to $+\infty$ whereas the right hand-side is bounded.

We denote by $n[t]$ the largest switching time smaller than or equal to t. Since the cost function $\mathbf{w}$ and the Lagrangian l are nonnegative, the function

$$\sum_{k=1}^{n[t]} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{t} e^{-a\tau} l(x(\tau), u(\tau))d\tau$$

is non decreasing, so that, taking the limit when $t \to +\infty$, we infer that the optimal cost function

$$\left\{ \begin{array}{l} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \ := \\ \displaystyle\inf_{(x(\cdot),u(\cdot))\in\mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau \right) \end{array} \right.$$

satisfies

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) \ \leq \ y_0$$

Therefore, the reset kernel of $K \times \mathbf{R}_+$ is contained in the epigraph of the value function $\mathbf{v}_{(P,f,l,\mathbf{w})}$.

Under the assumptions of the theorem, the infimum in the optimal control problem

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\bar{\xi}_k) + \int_0^{+\infty} e^{-a\tau} l(\bar{x}(\tau), \bar{u}(\tau))d\tau$$

is achieved at some run $(\bar{x}(\cdot), \bar{u}(\cdot))$. We set for all $t \in [t_n, t_{n+1}[$,

$$\bar{y}(t) \ := \ e^{at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x) - \sum_{k=1}^{n} e^{a(t-t_k)}\mathbf{w}(\bar{\xi}_k) - \int_0^{t} e^{a(t-\tau)} l(\bar{x}(\tau), \bar{u}(\tau))d\tau$$

and we deduce that the pair $(\bar{x}(\cdot), \bar{y}(\cdot))$ is a run of the impulse differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ viable in $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$. This implies that the graph of the value function $\mathbf{v}_{(P,f,l,\mathbf{w})}$ is contained in the reset kernel of $K \times \mathbf{R}_+$, so that the epigraph of the value function is actually equal to this reset kernel.

We also deduce that the runs viable in the epigraph of the value function are the runs satisfying

(a) for any switching time t_n, we have $y(^-t_n) = \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n))$ and

$$\begin{cases} \mathbf{v}_{(P,f,l,\mathbf{w})}(x(t_n)) \ = \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)+\xi_n) \\ \leq \ y(t_n) \ = \ y(^-t_n)+\eta_n \ \leq \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)) - \mathbf{w}(\xi_n) \end{cases}$$

and thus

$$\begin{cases} (\mathbf{v}_{(P,f,l,\mathbf{w})} * \mathbf{w})(x(^-t_n)) \\ \leq \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)+\xi_n) + \mathbf{w}(\xi_n) \ \leq \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)) \end{cases}$$

(b) for any $t \in [t_n, t_{n+1}[$,

$$\begin{cases} e^{-at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x(t)) \ \leq \ e^{-at}y(t) \ = \\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) - \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\eta_k) - \int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau \end{cases}$$

This amounts to saying that for every $t \in [t_n, t_{n+1}[$,

$$\begin{cases} x(t) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau),u(\tau))d\tau \\ e^{-at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x(t)) + \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau \\ \leq \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) \end{cases}$$

Therefore, the optimal runs are the runs of the of the impulse differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ viable in $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$. □

The epigraph of the value function of an optimal impulse control problem being the reset kernel under an impulse differential inclusion, it enjoys the characterizations and the properties of reset kernels which we now translate in the control framework.

4.4 Characterizations of the Value Function

Since under the assumptions of Theorem 4.3.1, the epigraph of the value function of an optimal impulse control problem is the reset kernel of $K \times \mathbf{R}_+$ under $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$, we can translate the properties of reset kernels into corresponding properties of the value function.

We begin by introducing notations of nonsmooth analysis: We shall denote by $\mathbf{u}|_K$, the restriction of $\mathbf{u}$ to K, the function equal to $\mathbf{u}$ on K and to $+\infty$ outside K. We recall that the epiderivative of the restriction of $\mathbf{u}$ to K at $x \in K$ is the restriction of the epiderivative to the contingent cone

$$\forall\, x \in K, \;\; D_{\uparrow}\mathbf{u}|_K(x) \;=\; D\mathbf{u}(x)|_{T_K(x)}$$

under constraint qualification assumptions when $\mathbf{u}$ and K are convex and under transversality conditions in the nonconvex case. See for instance Chapter 6 of [31, Aubin & Frankowska] for more details. We also recall that the subdifferential of the restriction of $\mathbf{u}$ to K at $x \in K$ is the sum of the subdifferential and of the normal cone

$$\forall\, x \in K, \;\; \partial_{-}\mathbf{u}|_K(x) \;=\; \partial_{-}\mathbf{u}(x) + N_K(x)$$

under constraint qualification assumptions when $\mathbf{u}$ and K are convex and under transversality conditions in the nonconvex case.

We also need to assume that $X \times \mathbf{R}_+$ is a repller under G: This is the topic of the following

Lemma 4.4.1 *Assume that the Lagrangian is nonnegative. The closed subset $X \times \mathbf{R}_+$ is backward invariant under G and the closed subset $K \times \mathbf{R}_+$ is backward invariant under G if and only if K is backward invariant under the control problem (4.1). It is a repeller under G whenever*

$$\forall\, x \in X, \quad \inf_{(x(\cdot),u(\cdot)) \in \mathcal{S}_F(x)} \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \;=\; +\infty$$

One can provide a sufficient condition for $X \times \mathbf{R}_+$ to be a repeller under the auxilairy differential inclusion G:

Lemma 4.4.2 *Let us assume that*

$$\left\{\begin{array}{lll} i) & \inf_{x\in X}\inf_{u\in P(x)}\frac{\langle x,f(x,u)\rangle}{\|x\|} & \geq \ \gamma(\|x\|+1) \\ ii) & \inf_{x\in X}\inf_{u\in P(x)} l(x,u) & \geq \ \delta(\|x\|+1) \end{array}\right. \tag{4.2}$$

If $a<\gamma$, then $X\times\mathbf{R}_+$ is a repeller.

Proof — Let $(x(\cdot),y(\cdot))$ be a solution to the differential inclusion $(x',y')\in G(x,y)$ starting from (x_0,y_0). Therefore

$$\frac{d}{dt}\|x(t)\| \ = \ \left\langle x'(t),\frac{x(t)}{\|x(t)\|}\right\rangle \ = \ \left\langle f(x(t),u(t)),\frac{x(t)}{\|x(t)\|}\right\rangle \ \geq \ \gamma(\|x(t)\|+1)$$

so that

$$\forall\, t\geq 0,\ \ \|x(t)\| \ \geq \ e^{\gamma t}(\|x_0\|+1)-1$$

Furthermore, since

$$l(x(\tau),u(\tau)) \ \geq \ \delta(\|x(\tau)\|+1) \ \geq \ \delta(\|x_0\|+1)e^{\gamma t}$$

and since

$$e^{-at}y(t) \ = \ y_0-\int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau$$

we infer that

$$e^{-at}y(t) \ \leq \ y_0-\delta(\|x_0\|+1)\int_0^t e^{(\gamma-a)\tau} \ = \ y_0-\frac{\delta(\|x_0\|+1)}{\gamma-a}\left(e^{(\gamma-a)t}-1\right)$$

Consequently, if $a<\gamma$

$$e^{-at}y(t) \ \leq \ y_0+\frac{\delta(\|x_0\|+1)}{\gamma-a}-\frac{\delta(\|x_0\|+1)}{\gamma-a}e^{(\gamma-a)t}$$

so that $y(t)$ becomes negative in finite time. □

We now have in our hands all what we need to prove the following

Theorem 4.4.3 *We posit the assumptions of Theorem 4.3.1 and we assume that $\mathbf{w}$ is coercive, that $\inf_{y\in X}\mathbf{w}(y)>0$ and that*

$$\forall\, x\in K,\ \ \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_F(x)}\int_0^{+\infty} e^{-a\tau}l(x(\tau),u(\tau))d\tau \ = \ +\infty$$

Then the value function $\mathbf{v}_\propto := \mathbf{v}_{(P,f,l,\mathbf{w})}$ *of the optimal impulse control problem (4.1) under the Lagrangian* l *and the cost function* $\mathbf{w}$ *defined by*

$$\begin{cases} \mathbf{v}_\propto(x) \; := \; \mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \\ \inf_{(x(\cdot),u(\cdot))\in\mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

is the solution to the problem

$$\mathbf{v}_\propto \; = \; \Gamma_{(P,f,l)}(\mathbf{v}_\propto * \mathbf{w})$$

i.e., for every $x \in \mathrm{Dom}(\mathbf{v}_\propto)$

$$\begin{cases} \mathbf{v}_\propto(x) \; = \; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \\ \left(e^{-at}\left(\mathbf{v}_\propto(x(t)+y) + \mathbf{w}(y)\right) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

It is equal to the epigraphical limit of the increasing sequence of functions $\mathbf{v}_n$ *defined recursively*[4] *by* $\mathbf{v}_0 := 0$ *and*

$$\begin{cases} \mathbf{v}_{n+1}(x) \; := \; \max\Big(\mathbf{v}_n(x), \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \\ \left(e^{-at}\left(\mathbf{v}_n(x(t)+y) + \mathbf{w}(y)\right) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

It is the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : K \mapsto \mathbf{R} \cup \{+\infty\}$ *satisfying the inequalities*

$$\begin{cases} \mathbf{v}(x) \; \geq \; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \\ \left(e^{-at}\left(\mathbf{v}(x(t)+y) + \mathbf{w}(y)\right) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau\right) \end{cases}$$

It is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : K \mapsto \mathbf{R} \cup \{+\infty\}$ *satisfying for every* $x \in K$*:*

$$\begin{cases} i) & 0 \; \leq \; \mathbf{v}(x) \; \leq \; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \text{if } \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x), \\ & \inf_{u\in P(x)} \left(D_\uparrow \mathbf{v}(x)(f(x,u)) + l(x,u)\right) - a\mathbf{v}(x) \; \leq \; 0 \end{cases}$$

[4] Compare with [208, Tartar], reproduced in Chapter 15 of [6, Aubin].

Introducing the Hamiltonian defined by

$$H(x,y,p) \;:=\; \sup_{u\in P(x)} (\langle p, f(x,u)\rangle + l(x,u)) - ay \;=\; \sigma(G(x,y),(p,-1))$$

the value function is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : K \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying for every* $x \in K$

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \text{if } \mathbf{v}(x) < (\mathbf{v}*\mathbf{w})(x),\; \forall\, p \in \partial_- \mathbf{v}(x),\;\; H(x,\mathbf{v}(x),p) \;\geq\; 0 \\ & \text{and } \forall\, p \in \partial_-^{\infty}\mathbf{v}|_K(x),\;\; \sigma(f(x,P(x)),p) \;\geq\; 0 \end{cases}$$

If we assume furthermore that P*,* f *and* l *are Lipschitz and that* K *is backward invariant under the control system (4.1), then the value function is the unique Frankowska solution* $\mathbf{v} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *to the system of "differential inequalities": for every* $x \in K$*,*

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v}*\mathbf{w})(x) \\ ii) & \sup_{u\in P(x)} (D_{\uparrow}\mathbf{v}|_K(x)(-f(x,u)) - l(x,u)) + a(x) \;\leq\; 0 \\ iii) & \text{if } \mathbf{v}(x) < (\mathbf{v}*\mathbf{w})(x), \\ & \inf_{u\in P(x)} (D_{\uparrow}\mathbf{v}|_K(x)(f(x,u)) + l(x,u)) + a\mathbf{v}(x) \;\leq\; 0 \end{cases}$$

Or, equivalently, it is the unique solution to the system

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v}*\mathbf{w})(x) \\ ii) & \forall\, p \in \partial_-\mathbf{v}|_K(x),\;\; H(x,\mathbf{v}(x),p) \;\leq\; 0 \\ & \text{and } \forall\, p \in \partial_-^{\infty}\mathbf{v}(x),\;\; \sigma(f(x,P(x)),p) \;\leq\; 0 \\ iii) & \text{if } \mathbf{v}(x) < (\mathbf{v}*\mathbf{w})(x),\;\; \forall\, p \in \partial_-\mathbf{v}|_K(x),\;\; H(x,\mathbf{v}(x),p) \;=\; 0 \\ & \text{and } \forall\, p \in \partial_-^{\infty}\mathbf{v}|_K(x),\;\; \sigma(f(x,P(x)),p) \;=\; 0 \end{cases}$$

which can reformulated in the form of "quasi variational inequalities":

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v}*\mathbf{w})(x) \\ ii) & \forall\, p \in \partial_-\mathbf{v}|_K(x),\;\; H(x,\mathbf{v}(x),p) \;\leq\; 0 \\ & \text{and } \forall\, p \in \partial_-^{\infty}\mathbf{v}(x),\;\; \sigma(f(x,P(x)),p) \;\leq\; 0 \\ iii) & \forall\, p \in \partial_-\mathbf{v}|_K(x),\;\; H(x,\mathbf{v}(x),p)(\mathbf{v}(x) - (\mathbf{v}*\mathbf{w})(x)) \;=\; 0 \\ & \text{and } \forall\, p \in \partial_-^{\infty}\mathbf{v}|_K(x),\; \sigma(f(x,P(x)),p)(\mathbf{v}(x) - (\mathbf{v}*\mathbf{w})(x)) \;=\; 0 \end{cases}$$

Knowing the value function $\mathbf{v}$*, we introduce the two regulation maps* $\mathbf{R}_{(f,P)}$ *and* $\mathbf{R}_{\mathbf{w}}$ *defined by*

$$\mathbf{R}_{(f,P)}(x) \;:=\; \{u \in P(x) \mid D_{\uparrow}\mathbf{v}(x)(f(x,u)) + l(x,u) - a\mathbf{v}(x) \leq 0\}$$

and

$$\mathbf{R_w}(x) \ := \ \{y \in X \mid \mathbf{v}(y) + \mathbf{w}(y-x) = (\mathbf{v} * \mathbf{w})(x)\}$$

Therefore, an optimal run is obtained in the following way[5]*: Starting from* x_0 *such that* $\mathbf{v}(x_0) < (\mathbf{v} * \mathbf{w})(x_0)$*, we choose a solution to the control system*

$$\begin{cases} i) & x'(t) \ = \ f(x(t), u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}_{(f,P)}(x(t)) \end{cases}$$

until the time $t_1 \geq 0$ *when* $\mathbf{v}(x({}^-t_1)) = (\mathbf{v} * \mathbf{w})(x({}^-t_1))$*.*

At this stage, we take for reinitialized sate $x_1 \in \mathbf{R_w}(x({}^-t_1)$*, and we reiterate the process.*

When the value function is continuous and when P, f and l are Lipschitz, one can deduce from [17, Aubin] that the backward invariance of the epigraph of $\mathbf{v}$ under G is equivalent to the invariance of the hypograph of $\mathbf{v}$ under G. In normal form, this amounts to saying that

$$\forall \, p \in \partial_-\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ \leq \ 0$$

is equivalent to

$$\forall \, p \in \partial_+\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), -p) \ \geq \ 0$$

where $\partial_+\mathbf{v}(x) := -\partial_-(-\mathbf{v})(x)$. This means that the value function is a viscosity solution (in the sense of [89, Crandall & Lions P.-L.]) to the quasi variational inequalities. For proofs of abstract theorems on existence of solutions to quasi variational inequalities, we mention the theorem [139, Joly & Mosco], derived (simply) from the Ky Fan inequality in [6, Aubin], and naturally, the book [54, 55, Bensoussan & Lions J.-L.].

[5] The subset $\{x \in K \mid \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x)\}$ is called the continuation set and the subset $\{x \in K \mid \mathbf{v}(x) = (\mathbf{v} * \mathbf{w})(x)\}$ is called the stopping set.

Chapter 5

Discrete Systems

5.1 Set-Valued Maps

Let X be a finite dimensional vector space whose *unit ball* is denoted by B (or B_X if the space must be mentioned). We denote by

$$d_K(x) := d(x,K) := \inf_{y \in K} \|x - y\|$$

the *distance from x to K,* where we set $d(x,\emptyset) := +\infty$. The *ball of radius $r > 0$ around K in X* is denoted by

$$B_X(K,r) := \{x \in X \mid d(x,K) \le r\} = \overline{K + rB_X}$$

When there is no ambiguity, we set

$$B(K,r) := B_X(K,r)$$

The balls $B(K,r)$ are neighborhoods of K. When K is compact, each neighborhood of K contains such a ball around K.

We set

$$K^c := X \backslash K = \text{the complement of } K \text{ and } K \backslash B := K \cap B^c$$

We denote by $\overline{K}$ the closing of K, by $\overset{\circ}{K}$ or $\mathrm{Int}(K)$ its interior, by

$$\widehat{K} := X \backslash \mathrm{Int}(K) = \overline{X \backslash K}$$

the complement of the interior of K and by

$$\partial K := \overline{K} \setminus \overset{\circ}{K} = K \cap \widehat{K} = \overline{K} \cap \overline{X \setminus K}$$

its boundary.

Definition 5.1.1 *Let X and Y be two spaces. A set-valued map F from X to Y is characterized by its* graph *$Graph(F)$, the subset of the product space $X \times Y$ defined by*

$$\text{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

We say that $F(x)$ is the image *or the* value *of F at x.*

A set-valued map is said to be nontrivial *if its graph is not empty, i.e., if there exists at least an element $x \in X$ such that $F(x)$ is not empty.*

We say that F is strict *if all images $F(x)$ are not empty. The* domain *of F is the subset of elements $x \in X$ such that $F(x)$ is not empty:*

$$\text{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$$

The image *of F is the union of the images (or values) $F(x)$, when x ranges over X:*

$$\text{Im}(F) := \bigcup_{x \in X} F(x)$$

The inverse *F^{-1} of F is the set-valued map from Y to X defined by*

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \text{Graph}(F)$$

If $K \subset X$ and $F : K \rightsquigarrow Y$ is a set-valued map defined on K, we shall always extend it to a map (again denoted by) $F : X \rightsquigarrow Y$ defined on the whole space X by setting $F(x) := \emptyset$ whenever $x \notin K$.

We observe at once that

$$F\left(\bigcup_{i \in I} K_i\right) = \bigcup_{i \in I} F(K_i)$$

We shall emphasize the characterization of a set-valued map (as well as single-valued map) by its graph. This point of view has been coined the graphical approach by R.T. Rockafellar.

The domain of F is thus the image of F^{-1} and coincides with the projection of the graph onto the space X and, in a symmetric way, the image of F is equal to the domain of F^{-1} and to the projection of the graph of F onto the space Y.

Sequences of subsets can be regarded as set-valued maps defined on the set $\mathbf{N}$ of integers.

We associate with any shape $K \subset X$ its indicator Ψ_K, which is the set-valued map from the vector space X to another vector space Y defined by

$$\Psi_K(x) \; = \begin{cases} 0 & \text{if} \;\; x \in K \\ \emptyset & \text{if} \;\; x \notin K \end{cases}$$

the graph of which is equal to $K \times \{0\}$.

Definition 5.1.2 *Let $\mathcal{P}$ be a property of a subset (for instance, closed, convex, etc..) Since we do emphasize the interpretation of a set-valued map as a graph (instead of a map from a set to another one), we shall say as a general rule that* a set-valued map satisfies property $\mathcal{P}$ if and only if its graph satisfies it.

For instance, a set-valued map is said to be closed (respectively convex, closed convex, measurable, etc. if and only if its graph is closed (respectively convex, closed convex, measurable, etc.)

If the images of a set-valued map F are closed, convex, bounded, compact, and so on, we say that F is closed-valued, convex-valued, bounded-valued, compact-valued, *and so on.*

When $\star$ denotes an operation on subsets, we use the same notation for the operation on set-valued maps, which is defined by

$$F_1 \star F_2 : \; x \; \rightsquigarrow \; F_1(x) \star F_2(x)$$

We define in that way $F_1 \cap F_2$, $F_1 \cup F_2$, $F_1 + F_2$ (in vector spaces), etc. Similarly, if α is a power map from the subsets of Y to the subsets of Y, we define

$$\alpha(F) : \; x \; \rightsquigarrow \; \alpha(F(x))$$

For instance, we shall use $\overline{F}$, $\mathrm{co}(F)$, etc., to denote the set-valued maps $x \rightsquigarrow \overline{F}(x)$, $x \rightsquigarrow \mathrm{co}(F(x))$, etc.

We shall write

$$F \subset G \iff \mathrm{Graph}(F) \subset \mathrm{Graph}(G)$$

and say that G is an extension of F.

Let us mention the following elementary properties:

$$\begin{cases} i) & F(K_1 \cup K_2) & = & F(K_1) \cup F(K_2) \\ ii) & F(K_1 \cap K_2) & \subset & F(K_1) \cap F(K_2) \\ iii) & F(X \backslash K) & \supset & \mathrm{Im}(F) \setminus F(K) \\ iv) & K_1 \subset K_2 & \Longrightarrow & F(K_1) \subset F(K_2) \end{cases}$$

5.1.1 Inverse Images and Cores

There are two manners to define the inverse image by a set-valued map F of a subset M:

Definition 5.1.3 *The subset $F^{-1}(M)$ defined by*

$$F^{-1}(M) := \{x \mid F(x) \cap M \neq \emptyset\}$$

is called the inverse image *of M by F and the subset $F^{\ominus 1}(M)$ defined by*

$$F^{\ominus 1}(M) := \{x \mid F(x) \subset M\}$$

is called the core *of M by F.*

They naturally coincide when F is single-valued. So, we can associate with a set-valued map $F : X \rightsquigarrow Y$ two power maps F^{-1} and $F^{\ominus 1}$, the inverse and the core of F.

We observe at once that

$$F^{\ominus 1}(Y \setminus M) = X \setminus F^{-1}(M) \ \& \ F^{-1}(Y \setminus M) = X \setminus F^{\ominus 1}(M)$$

and

$$F^{-1}\left(\bigcup_{i\in I} K_i\right) = \bigcup_{i\in I} F^{-1}(K_i) \ \&\ F^{\ominus 1}\left(\bigcap_{i\in I} K_i\right) = \bigcap_{i\in I} F^{\ominus 1}(K_i)$$

5.1.2 Graphical Restrictions

Definition 5.1.4 *Let us consider a set-valued map $F : X \rightsquigarrow Y$ and a subset $K \subset X$ and $L \subset Y$. We shall associate with F and these two subsets its* graphical restriction $F_{|K}^{|L}$ *to $K \times L$ defined by*

$$F_{|K}^{|L}(x) := \begin{cases} F(x) \cap L & \text{if } \ x \in K \\ \emptyset & \text{if } \ x \notin K \end{cases}$$

the graph of which is equal to

$$\text{Graph}(F_{|K}^{|L}) = \text{Graph}(F) \cap (K \times L)$$

Thereore

$$(F_{|K}^{|L})^{-1} = (F^{-1})_{|L}^{|K}$$

We observe that when $M \subset X$ and $P \subset Y$,

$$F_{|K}^{|L}(M) = F(M \cap K) \cap L \ \&\ (F_{|K}^{|L})^{-1}(P) = K \cap F^{-1}(P \cap L)$$

and in particular, that $\text{Dom}(F_{|K}^{|L}) = K \cap F^{-1}(L)$ and $\text{Im}(F_{|K}^{|L}) = F(K) \cap L$.

We set $F^{|L} := F_{|X}^{|L}$, defined by $F^{|L}(x) := F(x) \cap L$, and $F|_{|K} := F_{|K}^{|Y}$

$$F_{|K}(x) := \begin{cases} F(x) & \text{if } \ x \in K \\ \emptyset & \text{if } \ x \notin K \end{cases}$$

restriction of F to K, so that

$$\begin{cases} i) & F^{|L}(M) := F(M) \cap L \ \&\ F_{|K}(M) := F(K \cap M) \\ ii) & (F^{|L})^{-1}(P) := F^{-1}(P \cap L) \ \&\ (F_{|K})^{-1}(P) := K \cap F^{-1}(P) \end{cases}$$

We thus deduce the following statement:

Lemma 5.1.5 *Let $F : X \rightsquigarrow X$ be a set-valued map and $K \subset X$ a nonempty subset. Then $F_{|K}^{|K}$ is a set-valued map the domain of which is equal to $K \cap F^{-1}(K)$ and the image of which is $F(K) \cap K$.*

5.1.3 Composition of Maps

One can conceive as well two dual ways for defining composition products of set-valued maps (which coincide when the maps are single-valued):

Definition 5.1.6 *Let X, Y, Z be metric spaces and*

$$G : X \leadsto Y \quad \& \quad H : Y \leadsto Z$$

be set-valued maps.

1 — The usual composition product (called simply the product*) $H \circ G : X \leadsto Z$ of H and G at x is defined by*

$$(H \circ G)(x) \;:=\; \bigcup_{y \in G(x)} H(y)$$

2 — The square product *$H \square G : X \leadsto Z$ of H and G at x is defined by*

$$(H \square G)(x) \;:=\; \bigcap_{y \in G(x)} H(y)$$

Let $\mathbf{1}$ denote the identity map from one set to itself. We deduce the following formulas

$$\left\{ \begin{array}{lcl} \text{Graph}(H \circ G) & = & (G \times \mathbf{1})^{-1}(\text{Graph}(H)) \\ & = & (\mathbf{1} \times H)(\text{Graph}(G)) \\ & & \\ \text{Graph}(H \square G) & = & (G \times \mathbf{1})^{+1}(\text{Graph}(H)) \end{array} \right. \tag{5.1}$$

as well as formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$\left\{ \begin{array}{lll} i) & (H \circ G)^{-1}(z) & = \; G^{-1}(H^{-1}(z)) \; = \; \left(G^{-1} \circ H^{-1}\right)(z) \\ & & \\ ii) & (H \square G)^{-1}(z) & = \; G^{+1}(H^{-1}(z)) \end{array} \right.$$

We also observe that

$$\left\{ \begin{array}{lll} i) & x \in (H \square G)^{-1}(z) & \Longleftrightarrow \; G(x) \subset H^{-1}(z) \\ & & \\ ii) & x \in (G^{-1} \square H^{-1})(z) & \Longleftrightarrow \; H^{-1}(z) \subset G(x) \end{array} \right.$$

and thus, that

$$G(x) \ = \ H^{-1}(z) \iff x \ \in \ (G^{-1}\square H^{-1})(z) \cap (H\square G)^{-1}(z)$$

Let us also point out the following relations: When M is a subset of Z, then

$$\begin{cases} i) \quad (H \circ G)^{-1}(M) \ = \ G^{-1}(H^{-1}(M)) \\ ii) \quad (H \circ G)^{+1}(M) \ = \ G^{+1}(H^{+1}(M)) \end{cases}$$

5.2 Limits of Sets

5.2.1 Definitions

Limits of sets have been introduced by Painlevé in 1902 before the formalization of metric spaces by Frchet in 1906, and thus, without the concept of topology. They have been popularized by Kuratowski in his famous book TOPOLOGIE and thus, often called *Kuratowski lower and upper limits* of sequences of sets. They are defined without the concept of a topology on the power space.

Definition 5.2.1 *Let $(K_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space E. We say that the subset*

$$\text{Limsup}_{n\to\infty} K_n \ := \ \left\{ x \in E \ \mid \ \liminf_{n\to\infty} d(x, K_n) = 0 \right\}$$

is the upper limit *of the sequence K_n and that the subset*

$$\text{Liminf}_{n\to\infty} K_n \ := \ \{x \in E \ \mid \ \lim_{n\to\infty} d(x, K_n) = 0\}$$

is its lower limit. *A subset K is said to be the* limit *or the* set limit *of the sequence K_n if*

$$K \ = \ \text{Liminf}_{n\to\infty} K_n \ = \ \text{Limsup}_{n\to\infty} K_n \ =: \ \text{Lim}_{n\to\infty} K_n$$

Lower and upper limits are obviously closed. We also see at once that

$$\text{Liminf}_{n\to\infty} K_n \ \subset \ \text{Limsup}_{n\to\infty} K_n$$

and that the upper limits and lower limits of the subsets K_n and of their closures $\overline{K}_n$ do coincide, since $d(x, K_n) = d(x, \overline{K}_n)$.

Any decreasing sequence of subsets K_n has a limit, which is the intersection of their closures:

$$\text{if } \ K_n \subset K_m \ \text{ when } \ n \geq m, \ \text{ then } \ \text{Lim}_{n\to\infty} K_n \ = \ \bigcap_{n\geq 0} \overline{K_n}$$

An upper limit may be empty (no subsequence of elements $x_n \in K_n$ has a cluster point.)

Concerning sequences of singleta $\{x_n\}$, the set limit, when it exists, is either empty (the sequence of elements x_n is not converging), or is a singleton made of the limit of the sequence.

It is easy to check that:

Proposition 5.2.2 *If $(K_n)_{n\in\mathbf{N}}$ is a sequence of subsets of a metric space, then* $\text{Liminf}_{n\to\infty} K_n$ *is the* set of limits of sequences $x_n \in K_n$ *and* $\text{Limsup}_{n\to\infty} K_n$ *is the set* of cluster points of sequences $x_n \in K_n$*, i.e., of limits of subsequences* $x_{n'} \in K_{n'}$.

In other words, upper limits are "thick" cluster points and lower limits "thick" limits.

5.2.2 Calculus of Limits

Theorem 5.2.3 *Let us consider sequences of subsets L_n and M_n of a metric space and assume that there exists a compact subset M satisfying the following property:*

$$\text{for any neighborhood } \mathcal{W} \text{ of } \ M, \ \ \exists\, N \ \text{ such that } \ \forall\, n \geq N, \quad M_n \subset \mathcal{W}$$

Then, for any neighborhood $\mathcal{U}$ of $M \cap (\text{Limsup}_{n\to\infty} L_n)$, there exists an integer N such that $L_n \cap M_n \subset \mathcal{U}$ whenever $n \geq N$.

Proof — If the neighborhood $\mathcal{U}$ contains M, the result follows from the assumption on M. Otherwise, by taking an open neighborhood $\mathcal{U}$, the subset $K := M \backslash \mathcal{U}$ is not empty, disjoint of $\text{Limsup}_{n\to\infty} L_n$ and is compact by assumption.

Let y belong to K. Since y does not belong to $\text{Limsup}_{n\to\infty}L_n$, there exist $\varepsilon_y > 0$ and N_y such that, for all $n \geq N_y$, y does not belong to $B(L_n, \varepsilon_y)$. The subset K being compact, it can be covered by p balls $B(y_i, \varepsilon_{y_i})$. This implies that for all $n \geq N_0 := \max_{i=1,\ldots,p} N_{y_i}$ and

$$\mathcal{V} := \bigcup_{i=1}^{p} B(y_i, \varepsilon_{y_i})$$

the intersections $L_n \cap \mathcal{V}$ are empty.

On the other hand, $\mathcal{W} := \mathcal{U} \cup \mathcal{V}$ being a neighborhood of M, we deduce from the assumption that there exists N_1 such that

$$\forall n \geq N_1, \quad M_n \subset \mathcal{U} \cup \mathcal{V}$$

Therefore $L_n \cap M_n \subset \mathcal{U}$ for all $n \geq \max(N_0, N_1)$. $\square$

It implies the following result:

Theorem 5.2.4 *Let K be a subset of a metric space E satisfying the following property:*

$$\text{for any neighborhood } \mathcal{U} \text{ of } K, \ \exists\, N \text{ such that } \forall\, n \geq N, \ K_n \subset \mathcal{U}$$

Then $\text{Limsup}_{n\longrightarrow\infty}K_n \subset \overline{K}$.

Conversely, if E is compact*, then the upper limit* $\text{Limsup}_{n\to\infty}K_n$ *enjoys the above property (and thus, is the* smallest closed subset satisfying it.*)*

We point out the following obvious properties:

Proposition 5.2.5 *Let K_n, L_n, K_n^i, $(i = 1, \ldots, p)$ be sequences of subsets of a metric space. Then*

$$\begin{cases} i) & \text{Limsup}_{n\to\infty}(K_n \cap L_n) \subset \text{Limsup}_{n\to\infty}K_n \cap \text{Limsup}_{n\to\infty}L_n \\ ii) & \text{Liminf}_{n\to\infty}(K_n \cap L_n) \subset \text{Liminf}_{n\to\infty}K_n \cap \text{Liminf}_{n\to\infty}L_n \\ iii) & \text{Limsup}_{n\to\infty}(K_n \cup L_n) = \text{Limsup}_{n\to\infty}K_n \cup \text{Limsup}_{n\to\infty}L_n \\ iv) & \text{Liminf}_{n\to\infty}(K_n \cup L_n) \supset \text{Liminf}_{n\to\infty}K_n \cup \text{Liminf}_{n\to\infty}L_n \\ v) & \text{Limsup}_{n\to\infty} \prod_{i=1}^p K_n^i \subset \prod_{i=1}^p \text{Limsup}_{n\to\infty}K_n^i \\ vi) & \text{Liminf}_{n\to\infty} \prod_{i=1}^p K_n^i = \prod_{i=1}^p \text{Liminf}_{n\to\infty}K_n^i \end{cases}$$

We need also to relate direct and inverse images of upper and lower limits of a sequence of subsets to the upper and lower limits of their direct and inverse images. We mention now the obvious relations and refer to SET-VALUED ANALYSIS, [31, Aubin & Frankowska] for the proofs of criteria which transform some of the following inclusions to equalities.

Proposition 5.2.6 *Let K_n be a sequence of subsets of a metric space E, M_n be a sequence of subsets of a metric space Y and $f : E \mapsto Y$ be a (single-valued) continuous map. Then*

$$\begin{cases} i) & f(\text{Limsup}_{n\to\infty} K_n) & \subset & \text{Limsup}_{n\to\infty} f(K_n) \\ ii) & f(\text{Liminf}_{n\to\infty} K_n) & \subset & \text{Liminf}_{n\to\infty} f(K_n) \\ iii) & \text{Limsup}_{n\to\infty} f^{-1}(M_n) & \subset & f^{-1}\left(\text{Limsup}_{n\to\infty} M_n\right) \\ iv) & \text{Liminf}_{n\to\infty} f^{-1}(M_n) & \subset & f^{-1}\left(\text{Liminf}_{n\to\infty} M_n\right) \end{cases}$$

Converse results hold true under adequate assumptions (see Chapter 1 of SET-VALUED ANALYSIS, [31, Aubin & Frankowska,]).

5.2.3 Semi-Continuous Maps

We first need to adapt to the set-valued case the concept of continuity. There are two equivalent definitions of a continuous map f at x, the "$\varepsilon - \eta$" definition and the fact that f maps every sequence x_n converging to x to a sequence $f(x_n)$ converging to $f(x)$. Unfortunately, the natural generalizations of these statements to set-valued maps are no longer equivalent.

First, let us introduce these statements:

Definition 5.2.7 *A set-valued map $F : X \rightsquigarrow Y$ is called*

— upper semicontinuous *at $x \in X$ if and only if for any neighborhood $\mathcal{U}$ of $F(x)$,*

$$\exists\, \eta > 0 \quad \text{such that } \forall\, x' \in B_X(x, \eta), \;\; F(x') \subset \mathcal{U}.$$

It is said to be upper semicontinuous *if and only if it is upper semicontinuous at any point of X.*

— lower semicontinuous *at* $x \in \mathrm{Dom}(F)$ *if and only if for any* $y \in F(x)$ *and for any sequence of elements* $x_n \in \mathrm{Dom}(F)$ *converging to* x*, there exists a sequence of elements* $y_n \in F(x_n)$ *converging to* y*. It is said to be* lower semicontinuous *if it is lower semicontinuous at every point* $x \in \mathrm{Dom}(F)$*.*

— continuous *at* $x \in \mathrm{Dom}(F)$ *if it is* both *upper semicontinuous and lower semicontinuous at* x*, and that it is* continuous *if and only if it is continuous at every point of* $\mathrm{Dom}(F)$*.*

Indeed, there exist set-valued maps which enjoy one property without satisfying the other.

Examples — The set-valued map F_1 defined by

$$F_1(x) := \begin{cases} [-1,+1] & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}$$

is lower semicontinuous at zero but not upper semicontinuous at zero.

The set-valued map $F_2 : \mathbf{R} \leadsto \mathbf{R}$ defined by

$$F_2(x) := \begin{cases} \{0\} & \text{if} \quad x \neq 0 \\ [-1,+1] & \text{if} \quad x = 0 \end{cases}$$

is upper semicontinuous at zero but not lower semicontinuous at zero. □

Remark — Let us point out that $F : X \leadsto Y$ *is upper semicontinuous if and only if* $\mathrm{Dom}(F)$ *is closed and if the restriction* $F : \mathrm{Dom}(F) \leadsto Y$ *is upper semicontinuous.*

Indeed, if F is upper semicontinuous and $F(x_0)$ is empty, we take two disjoint neighborhoods of $F(x_0)$, so that the upper semicontinuity of F at x_0 implies the existence of a neighborhood of x_0 which is mapped by F into this empty intersection of neighborhoods. This shows that the complement of the domain of F is open. The restriction of F to its domain is then obviously upper semicontinuous.

The converse statement is easy. □

The connections between semi-continuity of set-valued maps and set limits are given by

Proposition 5.2.8 *A point* (x, y) *belongs to the* closure of the graph *of a set-valued map* $F : X \rightsquigarrow Y$ *if and only if*

$$y \in \text{Limsup}_{x' \to x} F(x')$$

and F *is* lower semicontinuous *at* $x \in \text{Dom}(F)$ *if and only if*

$$F(x) \subset \text{Liminf}_{x' \to x} F(x')$$

Then we can measure the lack of closedness (of the graph) or the lack of lower semicontinuity by the discrepancy between the sets

$$F(x), \ \text{Liminf}_{x' \to x} F(x') \ \text{and} \ \text{Limsup}_{x' \to x} F(x')$$

Remark — This proposition led several authors to call upper semicontinuous maps the ones which are closed in our terminology. Naturally, these two concepts coincide for compact-valued maps, since Theorem 5.2.4 can be easily adapted to the case of set-valued maps:

We can use the concepts of inverse images and cores to characterize upper and lower semicontinuous maps:

Proposition 5.2.9 *A set-valued map* $F : X \rightsquigarrow Y$ *is upper semicontinuous at* x *if the core of any neighborhood of* $F(x)$ *is a neighborhood of* x *and a set-valued map is lower semicontinuous at* x *if the inverse image of any open subset intersecting* $F(x)$ *is a neighborhood of* x.

Hence, F *is upper semicontinuous if and only if the core of any open subset is open and it is lower semicontinuous if and only if the inverse image of any open subset is open.*

If $\text{Dom}(F)$ *is closed, then* F *is lower semicontinuous if and only if the core of any closed subset is closed and* F *is upper semicontinuous if and only if the inverse image of any closed subset is closed.*

Lemma 5.2.10 *Let assume that* K *is a closed subset, that* $F : X \rightsquigarrow Y$ *is upper semicontinuous and that*

$$\forall\, x \in K, \ \ F(x) \cap (L + B) \text{ is compact}$$

where B *is the unit ball.*

Then, for every closed subset $M \subset L$, *the subset* $K \cap F^{-1}(M)$ *is closed.*

Proof — Indeed, let us consider a sequence $x_n \in (F_{|K}^{|L})^{-1}(M) = K \cap F^{-1}(M)$ converging to x, with which we associate a sequence $y_n \in M \cap F(x_n)$. Since F is upper semicontinuous, we know that for n large enough, $F(x_n) \subset F(x) + B$, so that y_n belongs to the set $F(x) \cap (L + B)$, which is compact by assumption. Hence a subsequence (again denoted by) y_n converges to some $y \in F(x) \cap M$. Therefore y belongs to $(F_{|K}^{|L})^{-1}(M)$, which is thus closed. □

We know that the graph of a continuous (single-valued) map is closed and that the converse is true under further assumptions (when we assume that the image of f is relatively compact, for instance.)

This result can be extended to upper semicontinuous set-valued maps. Closed set-valued maps almost characterize upper semicontinuous set-valued maps, as the following result shows.

Proposition 5.2.11 *The graph of an upper semicontinuous set-valued map $F : X \rightsquigarrow Y$ with closed values is closed.*

The converse is true if we assume that the domain of F is closed and that Y is compact.

This will be particularly useful since it provides an easy way to construct upper semicontinuous set-valued maps, by intersecting closed set-valued maps with closed balls, the radii of which are upper semicontinuous (real-valued) functions:

Corollary 5.2.12 *Let $F : X \rightsquigarrow Y$ be a* closed *set-valued map and $r : X \mapsto R$ be an upper semicontinuous function. If the* dimension of Y is finite , *then the cut set-valued map $F_r : X \rightsquigarrow Y$ defined by*

$$F_r(x) := F(x) \cap r(x)B \tag{5.2}$$

is upper semicontinuous.

It follows from Proposition 5.2.11 and the remark that the upper semicontinuity of $r : X \mapsto \mathbf{R}$ implies the upper semicontinuity of $x \rightsquigarrow r(x)B$.

5.2.4 The Marginal Selection

Theorem 5.2.13 (Maximum Theorem) *Let us consider metric spaces E, F and a function $U : E \times F \mapsto \mathbf{R}$, the marginal function*

$$V^\sharp(K, y) \; := \; \sup_{x \in K} U(x, y)$$

and the marginal map $M_U^\sharp$ defined by

$$M_U^\sharp(K, y) \; := \; \{x \in K \; | U(x, y) = V^\sharp(K, y) \; \}$$

Let $K_n \subset E$ be given and a sequence of y_n converging to some $y \in F$.

(a) If U is lower semicontinuous and if $K^\flat := \mathrm{Liminf}_{n \to \infty} K_n$, then

$$V^\sharp(K^\flat, y) \; \leq \; \liminf_{n \to \infty} V^\sharp(K_n, y_n)$$

(b) If U is upper semicontinuous, if $K^\sharp := \mathrm{Limsup}_{n \to \infty} K_n$ and if E is compact, then

$$\limsup_{n \to \infty} V^\sharp(K_n, y_n) \; \leq \; V^\sharp(K^\sharp, y)$$

Furthermore, if U is continuous, E is compact, then

$$\mathrm{Limsup}_{n \to \infty} M_U^\sharp(K_n, y_n) \; \subset \; M_U^\sharp(\mathrm{Limsup}_{n \to \infty} K_n, \lim_{n \to \infty} y_n)$$

Proof — Let us consider a sequence y_n converging to y, fix $\lambda < V^\sharp(K^\flat, y)$ and choose $x \in K^\flat$ such that $\lambda \leq U(x, y)$. Then there exist elements $x_n \in K_n$ converging to x and we know that $U(x_n, y_n) \leq V^\sharp(K_n, y_n)$. Since U is lower semicontinuous, we infer that

$$\lambda \; \leq \; U(x, y) \; \leq \; \liminf_{n \to \infty} U(x_n, y_n) \; \leq \; \liminf_{n \to \infty} V^\sharp(K_n, y_n)$$

By letting λ converge to $V^\sharp(K^\flat, y)$, the claim ensues.

For proving the second statement, pick $y \in F$ and fix $\varepsilon > 0$. Since U is upper semicontinuous, we can associate with any $x \in K^\sharp$ open neighborhoods $\mathcal{V}(x)$ of x and $\mathcal{U}_x(y)$ of y such that

$$\forall \, y_n \; \in \; \mathcal{U}_x(y) \text{ and } x_n \in \mathcal{V}(x), \;\; U(x_n, y_n) \; \leq \; U(x, y) + \varepsilon \qquad (5.3)$$

Since $K^\sharp$ is compact, it can be covered by p neighborhoods $\mathcal{V}(x_i)$, $i = 1, \cdots, p$, the union of which makes up a neighborhood of $K^\sharp$. Then there exists an integer $N > 0$ such that

$$\forall\, n \geq N, \;\; K_n \;\subset\; \bigcup_{i=1}^{p} \mathcal{V}(x_i)$$

by Theorem 5.2.4. By taking y_n in the neighborhood

$$\mathcal{U}(y) \;:=\; \bigcap_{i=1}^{p} \mathcal{U}_{x_i}(y)$$

we observe that

$$\forall\, y_n \in \mathcal{U}(y), \;\; \forall\, x_n \in K_n, \;\; U(x_n, y_n) \leq \sup_{i=1,\cdots,p} U(x_i, y) + \varepsilon \;\leq\; V^\sharp(K^\sharp, y) + \varepsilon$$

(thanks to (5.3)) and we deduce that

$$\forall\, y_n \in \mathcal{U}(y), \;\; V^\sharp(K_n, y_n) \;\leq\; V^\sharp(K^\sharp, y) + \varepsilon$$

Finally, $M_U^\sharp(K_n, y_n) = K_n \cap M_n$ where

$$M_n \;:=\; \left\{x \in E \mid U(x, y_n) \leq V^\sharp(K_n, y_n)\right\}$$

If U is continuous and if E is compact, we infer that

$$\text{Limsup}_{n\to\infty} M_n \;\subset\; M^\sharp \;:=\; \left\{x \in E \mid U(x, y) \leq V^\sharp(K^\sharp, y)\right\}$$

Theorem 5.2.3 implies that

$$\text{Limsup}_{n\to\infty}(K_n \cap M_n) \;\subset\; K^\sharp \cap M^\sharp \;=\; M^\sharp(K^\sharp, y) \;\; \square$$

It may be useful to introduce the following definition:

We deduce the continuity properties of the marginal maps.

Definition 5.2.14 (Marginal Functions) *Consider a set-valued map* $F : X \rightsquigarrow Y$ *and a function* $U : \text{Graph}(F) \mapsto \text{R}$. *We associate with them the* marginal function $V^\sharp : X \mapsto \text{R}$ *defined by*

$$V^\sharp(x) \;:=\; \sup_{y\in F(x)} U(x, y)$$

Theorem 5.2.15 (Maximum Theorem) *Let metric spaces X, Y, a set-valued map $F : X \rightsquigarrow Y$ and a function $U : \mathrm{Graph}(F) \mapsto \mathrm{R}$ be given.*

— *If U and F are* lower semicontinuous*, so is the marginal function $V^\sharp$.*

— *If U and F are* upper semicontinuous *and if the* values of F are compact*, so is the marginal function $V^\sharp$.*

The proof, analogous to the proof of Theorem 5.2.13, is an exercise of topology which is found in many books.

We will use the following corollary quite often:

Corollary 5.2.16 *If a set-valued map F is lower semicontinuous (*resp. *upper semicontinuous with compact values), then the function $(x, y) \mapsto d(y, F(x))$ is upper semicontinuous (*resp. *lower semicontinuous.)*

5.2.5 Contingent Cones

Definition 5.2.17 *Let $K \subset X$ be a subset of a normed vector space X and $x \in K$. Since the contingent cone $T_K(x)$ is the set of elements v such that there exists a sequence of elements $h_n > 0$ converging to 0 and a sequence of $v_n \in X$ converging to v satisfying*

$$\forall\, n \geq 0, \quad x + h_n v_n \;\in\; K$$

we deduce that the contingent cone $T_K(x)$ is the upper limit of the subsets $(K - x)/h$ *(regarded as "set differential quotients")*

$$T_K(x) \;:=\; \mathrm{Limsup}_{h \to 0+} \frac{K - x}{h}$$

Therefore $T_K(x)$ is always a closed cone of "tangent directions" (which is convex when K is convex or, more generally, when the contingent cone is lower semicontinuous[1] at this point, a vector space when K is a smooth manifold).

[1]See SET-VALUED ANALYSIS, [31, Aubin & Frankowska]. A subset K is said to be sleek at $x \in K$ is $T_K(\cdot)$ is lower semicontinuous at this point. A convex subset K is sleek at each of its points.

We introduce the following notation[2]:

$$D_\uparrow d_K(x)(v) \ := \ \liminf_{h\to 0+} \frac{d_K(x+hv)-d_K(x)}{h}$$

We observe that when $x \in K$, a direction v is contingent to K at x if and only if $D_\uparrow d_K(x)(v) \le 0$.

Lemma 5.2.18 *Let K be a closed subset of a finite dimensional vector-space and $\Pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\|y-z\| = d_K(y)$. Then the following inequalities:*

$$D_\uparrow d_K(y)(v) \ \le \ d(v, \overline{\mathrm{co}}(T_K(\Pi_K(y))))$$

hold true.

Proof — Choose $z \in \Pi_K(y)$. Then

$$\frac{d_K(y+hv)-d_K(y)}{h} \ \le \ \frac{\|y-z+hw\|-\|y-z\|}{h}$$

from which we deduce, letting h go to 0,

$$D_\uparrow d_K(y)(v) \ \le \ \left\langle \frac{y-z}{\|y-z\|}, v \right\rangle$$

Taking $w \in T_K(z)$ and recalling that $\langle y-z, w\rangle \le 0$, we infer that

$$\forall\, w \in T_K(z), \ \ D_\uparrow d_K(y)(v) \ \le \ \left\langle \frac{y-z}{\|y-z\|}, v-w \right\rangle$$

This inequality is also true for any $w \in \overline{\mathrm{co}}(T_K(z))$, so that

$$D_\uparrow d_K(y)(v) \ \le \ d(v, \overline{\mathrm{co}}(T_K(z))) \ \square$$

[2]this is the contingent epiderivative of the distance functions d_K. See Set-Valued Analysis, [31, Aubin & Frankowska]) for further details.

5.2.6 Graphical Convergence of Maps

Definition 5.2.19 *Let us consider a sequence of set-valued maps* $F_n : X \rightsquigarrow Y$*. The set-valued map* $F^\sharp := \mathrm{Lim}^\sharp{}_{n\to\infty} F_n$ *from* X *to* Y *defined by*

$$\mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty} F_n) \ := \ \mathrm{Limsup}_{n\to\infty} \mathrm{Graph}(F_n)$$

is called the (graphical) upper limit *of the set-valued maps* F_n*.*

Even for single-valued maps, this is a weaker convergence than the pointwise convergence:

Proposition 5.2.20

(a) If $f_n : X \mapsto Y$ *converges pointwise to* f*, then, for every* $x \in X$*,* $f(x) \in f^\sharp(x)$*. If the sequence is equicontinuous, then* $f^\sharp(x) = \{f(x)\}$*.*

(b) Let $\Omega \subset \mathbf{R}^n$ *be an open subset. If a sequence* $f_n \in L^p(\Omega)$ *converges to* f *in* $L^p(\Omega)$*, then*

$$\text{for almost all } x \in \Omega, \ \ f(x) \in f^\sharp(x)$$

(c) If a sequence $f_n \in L^p(\Omega)$ *converges* weakly *to* f *in* $L^p(\Omega)$*, then*

$$\text{for almost all } x \in \omega, \ \ f(x) \in \overline{\mathrm{co}} f^\sharp(x)$$

5.2.7 Contingent Derivatives of Set-Valued Maps

Let $F : X \rightsquigarrow Y$ be a set-valued map. We introduce the *differential quotients*

$$u \ \rightsquigarrow \ \nabla_h F(x,y)(u) \ := \ \frac{F(x+hu) - y}{h}$$

of a set-valued map $F : X \rightsquigarrow Y$ at $(x,y) \in \mathrm{Graph}(F)$.

Definition 5.2.21 *The contingent derivative* $DF(x,y)$ *of* F *at* $(x,y) \in \mathrm{Graph}(F)$ *is the graphical upper limit of differential quotients:*

$$DF(x,y) \ := \ \mathrm{Lim}^\sharp{}_{h\to 0+} \nabla_h F(x,y)$$

In other words, v belongs to $DF(x,y)(u)$ if and only if there exist sequences $h_n \to 0^+$, $u_n \to u$ and $v_n \to v$ such that $\forall n \geq 0, \quad y + h_n v_n \in F(x + h_n u_n)$.

In particular, if $f : X \mapsto Y$ is a single valued function, we set $Df(x) = Df(x, f(x))$.

We deduce the fundamental formula on the graph of the contingent derivative:

Proposition 5.2.22 *The graph of the contingent derivative of a set-valued map is the contingent cone to its graph: for all* $(x,y) \in \text{Graph}(F)$*,*

$$\text{Graph}(DF(x,y)) = T_{\text{Graph}(F)}(x,y)$$

Proof — Indeed, we know that the contingent cone

$$T_{\text{Graph}(F)}(x,y) \;=\; \text{Limsup}_{h \to 0+} \frac{\text{Graph}(F) - (x,y)}{h}$$

is the upper limit of the differential quotients $\frac{\text{Graph}(F)-(x,y)}{h}$ when $h \to 0+$. It is enough to observe that

$$\text{Graph}(\nabla_h F(x,y)) = \frac{\text{Graph}(F) - (x,y)}{h}$$

and to take the upper limit to conclude. □

We can easily compute the derivative of the inverse of a set-valued map F (or even of a noninjective single-valued map): *The contingent derivative of the inverse of a set-valued map F is the inverse of the contingent derivative*:

$$D(F^{-1})(y,x) \;=\; DF(x,y)^{-1}$$

If K is a subset of X and f is a single-valued map which is Fréchet differentiable around a point $x \in K$, then *the contingent derivative of the restriction of f to K is the restriction of the derivative to the contingent cone*:

$$D(f|_K)(x) = D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)}$$

5.2.8 Extended Functions and their Epigraphs

For reasons motivated by optimization theory, Lyapunov stability, control theory, Hamilton-Jacobi equations and variational inequalities and mathematical morphology, the order relation on $\mathbf{R}$ is involved. This leads us to associate with an extended functions $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ its epigraph instead of its graph. It actually happens that the properties of the extended functions $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ are actually properties of their *epigraphs*. This "epigraphical point of view" is the key to "variational analysis" and to our treatment of Hamilton-Jacobi inequalities.

A function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which $\mathbf{v}$ is finite:

$$\mathrm{Dom}(\mathbf{v}) \ := \ \{x \in X \ \mid \ \mathbf{v}(x) < +\infty\}$$

A function is said to be *nontrivial* if its domain is not empty. Any function $\mathbf{v}$ defined on a subset $K \subset X$ can be regarded as the extended function $\mathbf{v}_K$ equal to $\mathbf{v}$ on K and to $+\infty$ outside of K, whose domain is K.

Since the order relation on the real numbers is involved in the definition of the Lyapunov property (as well as in minimization problems and other dynamical inequalities), we no longer characterize a real-valued function by its graph, but rather by its *epigraph* indexepigraph

$$\mathcal{E}p(\mathbf{v}) \ := \ \{(x,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \le \lambda\}$$

The *hypograph* of a function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{-\infty\}$ is defined in a symmetric way by

$$\mathcal{H}yp(\mathbf{v}) \ := \ \{(x,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \ge \lambda\} \ = \ -\mathcal{E}p(-\mathbf{v})$$

The graph of a real-valued (finite) function is then the intersection of its epigraph and its hypograph.

We also remark that some properties of a function are actually properties of their epigraphs. For instance, *an extended function* $\mathbf{v}$ *is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone).* The epigraph of $\mathbf{v}$ is closed if and only if $\mathbf{v}$ is lower semicontinuous:

$$\forall\, x \in X, \ \ \mathbf{v}(x) \ = \ \liminf_{y \to x} \mathbf{v}(y)$$

We recall the convention $\inf(\emptyset) := +\infty$.

Lemma 5.2.23 *Consider a function* $\mathbf{v} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$*. Its epigraph is closed if and only if*

$$\forall\, x \in X, \quad \mathbf{v}(x) = \liminf_{x' \to x} \mathbf{v}(x')$$

Assume that the epigraph of $\mathbf{v}$ *is a closed cone. Then the following conditions are equivalent:*

$$\begin{cases} i) & \forall\, x \in X, \quad \mathbf{v}(x) > -\infty \\ ii) & \mathbf{v}(0) = 0 \\ iii) & (0, -1) \notin \mathcal{E}p(\mathbf{v}) \end{cases}$$

Proof — Assume that the epigraph of $\mathbf{v}$ is closed and pick $x \in X$. There exists a sequence of elements x_n converging to x such that

$$\lim_{n \to \infty} \mathbf{v}(x_n) = \liminf_{x' \to x} \mathbf{v}(x')$$

Hence, for any $\lambda > \liminf_{x' \to x} \mathbf{v}(x')$, there exist N such that, for all $n \geq N$, $\mathbf{v}(x_n) \leq \lambda$, i.e., such that $(x_n, \lambda) \in \mathcal{E}p(\mathbf{v})$. By taking the limit, we infer that $\mathbf{v}(x) \leq \lambda$, and thus, that $\mathbf{v}(x) \leq \liminf_{x' \to x} \mathbf{v}(x')$. The converse statement is obvious.

Suppose next that the epigraph of $\mathbf{v}$ is a cone. Then it contains $(0, 0)$ and $\mathbf{v}(0) \leq 0$. The statements *ii)* and *iii)* are clearly equivalent.

If *i)* holds true and $\mathbf{v}(0) < 0$, then

$$(0, -1) = \frac{1}{-\mathbf{v}(0)}(0, \mathbf{v}(0))$$

belongs to the epigraph of $\mathbf{v}$, as well as all $(0, -\lambda)$, and (by letting $\lambda \to +\infty$) we deduce that $\mathbf{v}(0) = -\infty$, so that *i)* implies *ii)*.

To end the proof, assume that $\mathbf{v}(0) = 0$ and that for some x, $\mathbf{v}(x) = -\infty$. Then, for any $\varepsilon > 0$, the pair $(x, -1/\varepsilon)$ belongs to the epigraph of $\mathbf{v}$, as well as the pairs $(\varepsilon x, -1)$. By letting ε converge to 0, we infer that $(0, -1)$ belongs also to the epigraph, since it is closed. Hence $\mathbf{v}(0) < 0$, a contradiction. □

Indicators ψ_K *of subsets* K are cost functions defined by

$$\psi_K(x) := 0 \ \text{ if } \ x \in K \ \text{ and } \ +\infty \ \text{ if not}$$

which characterize subsets (as *characteristic functions* do for other purposes) provide important examples of extended functions. It can be regarded as a *membership cost*[3] to K: it costs nothing to belong to K, and $+\infty$ to step outside of K.

Since

$$\mathcal{E}p(\psi K) \;=\; K \times \mathbf{R}_+$$

we deduce that the indicator ψ_K is lower semicontinuous if and only if K is closed and that ψ_K is convex if and only if K is convex. One can regard the sum $\mathbf{v} + \psi_K$ as the restriction of $\mathbf{v}$ to K.

We recall the convention $\inf(\emptyset) := +\infty$.

5.2.9 Epilimits

Definition 5.2.24 *The epigraph of the* lower epilimit $\lim^{\sharp}_{\uparrow n\to\infty}\mathbf{u}_n$ *of a sequence of extended functions* $\mathbf{u}_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is the upper limit of the epigraphs:*

$$\mathcal{E}p(\lim^{\sharp}_{\uparrow n\to\infty}\mathbf{u}_n) \;:=\; \mathrm{Limsup}_{n\to\infty}\mathcal{E}p(\mathbf{u}_n)$$

The function $\lim^{\flat}_{\uparrow n\to\infty}\mathbf{u}_n$ *whose epigraph is the lower limit of the epigraphs of the functions* $\mathbf{u}_n$

$$\mathcal{E}p(\lim^{\flat}_{\uparrow n\to\infty}\mathbf{u}_n) \;:=\; \mathrm{Liminf}_{n\to\infty}\mathcal{E}p(\mathbf{u}_n)$$

is the upper epilimit *of the functions* $\mathbf{u}_n$

One can check that

$$\lim^{\sharp}_{\uparrow n\to\infty}\mathbf{u}_n(x_0) \;=\; \liminf_{n\to\infty, x\to x_0} \mathbf{u}_n(x)$$

5.2.10 Contingent Epiderivatives

When $\mathbf{u}$ is an extended function, we associate with it its epigraph and the contingent cones to this epigraph. This leads to the concept of epiderivatives of extended functions.

[3]Functions $\mathbf{v} : X \mapsto [0, +\infty]$ can be regarded as some kind of *fuzzy sets*, called *toll sets*.

Definition 5.2.25 *Let* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain.*

We associate with it the differential quotients

$$u \rightsquigarrow \nabla_h \mathbf{u}(x)(u) \ := \ \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h}$$

The contingent epiderivative $D_\uparrow \mathbf{u}(x)$ *of* $\mathbf{u}$ *at* $x \in \mathrm{Dom}(\mathbf{u})$ *is the lower epilimit of its differential quotients:*

$$D_\uparrow \mathbf{u}(x) \ = \ \lim{}^\sharp_{\uparrow h \to 0+} \nabla_h \mathbf{u}(x)$$

We shall say that the function $\mathbf{u}$ *is* contingently epidifferentiable *at* x *if for any* $u \in X$, $D_\uparrow \mathbf{u}(x)(u) > -\infty$ *(or, equivalently, if* $D_\uparrow \mathbf{u}(x)(0) = 0$*).*

Proposition 5.2.26 *Let* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain. Then the contingent epiderivative* $D_\uparrow \mathbf{u}(x)$ *satisfies*

$$\forall\, u \in X, \ \ D_\uparrow \mathbf{u}(x)(u) \ = \ \liminf_{h \to 0+, u' \to u} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h}$$

and the epigraph of the contingent epiderivative $D_\uparrow \mathbf{u}(\cdot)$ *is equal to the contingent cone to the epigraph of* $\mathbf{u}$ *at* $(x, \mathbf{u}(x))$ *is*

$$\mathcal{E}p(D_\uparrow \mathbf{u}(x)) \ = \ T_{\mathcal{E}p(\mathbf{u})}(x, \mathbf{u}(x))$$

Proof — The first statement is obvious. For proving the second one, we recall that the contingent cone

$$T_{\mathcal{E}p(\mathbf{u})}(x, \mathbf{u}(x)) \ = \ \mathrm{Limsup}_{h \to 0+} \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$$

is the upper limit of the differential quotients $\dfrac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathcal{E}p(D_\uparrow \mathbf{u}(x)) := T_{\mathcal{E}p(\mathbf{u})}(x, y) \ \ \& \ \ \mathcal{E}p(\nabla_h \mathbf{u}(x)) = \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h}$$

to conclude. □

Consequently, *the epigraph of the contingent epiderivative at x is a closed cone. It is then lower semicontinuous and positively homogeneous whenever* $\mathbf{u}$ *is contingently epidifferentiable at x.*

We observe that the contingent epiderivative of the indicator function ψ_K at $x \in K$ is the indicator of the contingent cone to K at x:

$$D_{\uparrow}\psi_K(x) \ = \ \psi_{T_K(x)}$$

making precise the intuition stating that the contingent cone $T_K(x)$ plays the role of a "derivative of a set", as the limit of differential quotients $\dfrac{K-x}{h}$ of sets.

The hypoderivatives of an extended function are defined in a analogous way: The contingent hypoderivative $D_{\downarrow}\mathbf{u}(x)$ of $\mathbf{u}$ at $x \in \mathrm{Dom}(\mathbf{u})$ is the upper hypolimit of its differential quotients:

$$D_{\downarrow}\mathbf{u}(x) \ = \ \lim{}^{\sharp}_{\downarrow}{}_{h\to 0+}\nabla_h\mathbf{u}(x)$$

We observe that it is equal to

$$\forall\, u \in X, \ \ D_{\downarrow}\mathbf{u}(x)(u) \ = \ \limsup_{h\to 0+, u'\to u} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h}$$

and that *the hypograph of the* contingent hypoderivative $D_{\downarrow}\mathbf{u}(x)$ *of* $\mathbf{u}$ *at x is the contingent cone to the hypograph of* $\mathbf{u}$ *at* $(x, \mathbf{u}(x))$:

$$\mathcal{E}p(D_{\downarrow}\mathbf{u}(x)) = T_{\mathcal{H}yp(\mathbf{u})}(x, \mathbf{u}(x))$$

Definition 5.2.27 *We shall say that* $\mathbf{u} : X \mapsto W$ *is* differentiable from the right *at x if the contingent epiderivative and hypoderivative coincide:*

$$\forall\, v \in X, \ \ D_{\uparrow}\mathbf{u}(x)(v) \ = \ D_{\downarrow}\mathbf{u}(x)(v)$$

Lemma 5.2.28 *Let $K \subset X$ be a closed subset and* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *be an extended function. We denote by* $\mathbf{u}|_K := f + \psi_K$ *the restriction to* $\mathbf{u}$ *at K. Inequality*

$$D_{\uparrow}\mathbf{u}(x)|_{T_K}(x) \ \le \ D_{\uparrow}\mathbf{u}|_K(x)$$

always holds true. It is an equality when $\mathbf{u}$ *is differentiable from the right:* the contingent derivative of the restriction of $\mathbf{u}$ to K is the restriction of the derivative to the contingent cone.

Proof — Indeed, let $x \in K \cap \mathrm{Dom}(\mathbf{u})$. If u belongs to $T_K(x)$, there exist $h_n \to 0+$, $\varepsilon_n \to 0+$ and $x_n := x + h_n u_n \in K$ such that

$$D_\uparrow \mathbf{u}(x)(u) \;\le\; \liminf_{n\to+\infty} \frac{\mathbf{u}(x_n) - \mathbf{u}(x)}{h_n} \;=\; \liminf_{n\to+\infty} \frac{\mathbf{u}|_K(x_n) - \mathbf{u}|_K(x)}{h_n}$$

which implies the inequality. If $\mathbf{u}$ is differentiable from the right, the differential quotient converges to the common value $D_\uparrow \mathbf{u}(x) = D_\uparrow \mathbf{u}|_K(x) = D_\downarrow \mathbf{u}|_K(x)$. □

For locally Lipschitz functions, the contingent epiderivatives are finite:

Proposition 5.2.29 *If* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is Lipschitz around* $x \in \mathrm{Int}(\mathrm{Dom}(\mathbf{u}))$, *then the contingent epiderivative* $D_\uparrow \mathbf{u}(x)$ *is Lipschitz: there exists* $\lambda > 0$ *such that*

$$\forall\, u \in X,\;\; D_\uparrow \mathbf{u}(x)(u) \;=\; \liminf_{h\to 0+} \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h} \;\le\; \lambda \|u\|$$

Proof — Since $\mathbf{u}$ is Lipschitz on some ball $B(x,\eta)$, the above inequality follows immediately from

$$\forall\, u \in \eta B,\;\; \frac{\mathbf{u}(x+hu) - \mathbf{u}(x)}{h} \;\le\; \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h} + \lambda(\|u\| + \|u' - u\|)$$

by taking the liminf when $h \to 0+$ and $u' \to u$. □

For convex functions, we obtain:

Proposition 5.2.30 *When the function* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is convex, the contingent epiderivative is equal to*

$$D_\uparrow \mathbf{u}(x)(u) \;=\; \liminf_{u'\to u} \left(\inf_{h>0} \frac{\mathbf{u}(x+hu') - \mathbf{u}(x)}{h} \right)$$

Proof — Indeed, convexity implies that if $0 < h_1 \le h_2$,

$$\mathcal{E}p(\nabla_{h_2} \mathbf{u}(x)) \;=\; \frac{\mathcal{E}p(\mathbf{u}) - (x, \mathbf{u}(x))}{h_2} \;\subset\; \mathcal{E}p(\nabla_{h_1} \mathbf{u}(x))$$

i.e.,

$$\forall\, u \in X,\;\; \nabla_{h_1}\mathbf{u}(x)(u) \;\leq\; \nabla_{h_2}\mathbf{u}(x)(u)$$

Therefore,

$$\forall\, u \in X,\;\; D\mathbf{u}(x)(u) \;:=\; \lim_{h\to 0+} \frac{\mathbf{u}(x+hu)-\mathbf{u}(x)}{h} \;=\; \inf_{h>0} \frac{\mathbf{u}(x+hu)-\mathbf{u}(x)}{h}$$

and this function $D\mathbf{u}(x)$ is convex with respect to u. Since the epigraph of $D\mathbf{u}(x)$ is the increasing union of the epigraphs of the differential quotients $\nabla_h\mathbf{u}(x)$, we infer that

$$D_{\uparrow}\mathbf{u}(x)(u) \;:=\; \liminf_{u'\to u} D\mathbf{u}(x)(u')$$

We recall the following important property of convex functions defined on finite dimensional vector spaces:

Theorem 5.2.31 *An extended convex function* $\mathbf{u}$ *defined on a finite dimensional vector-space is locally Lipschitz and subdifferentiable on the interior of its domain. Therefore, when* x *belongs to the interior of the domain of* $\mathbf{u}$*, there exists a constant* λ_x *such that*

$$\forall\, u \;\in\; X,\;\; D_{\uparrow}\mathbf{u}(x)(u) \;=\; \inf_{h>0} \frac{\mathbf{u}(x+hu)-\mathbf{u}(x)}{h} \;\leq\; \lambda_x \|u\|$$

The second statement follows from Proposition 5.2.29. □

5.2.11 Generalized Gradients

Definition 5.2.32 *Let* $\mathbf{u} : X \mapsto \mathbf{R}\cup\{+\infty\}$ *be a nontrivial extended function. The continuous linear functionals* $p \in X^\star$ *satisfying*

$$\forall\, v \in X,\;\; \langle p, v\rangle \;\leq\; D_{\uparrow}\mathbf{u}(x)(v)$$

are called the (regular) subgradients *of* $\mathbf{u}$ *at* x*, which constitute the (possibly empty) closed convex subset*

$$\partial_{-}\mathbf{u}(x) \;:=\; \{p \in X^\star \mid \forall\, v \in X,\;\; \langle p, v\rangle \;\;\leq\; D_{\uparrow}\mathbf{u}(x)(v)\}$$

called the (regular) subdifferential *of* $\mathbf{u}$ *at* x_0.

In a symmetric way, the superdifferential $\partial_+\mathbf{u}(x)$ *of* $\mathbf{u}$ *at* x *is defined by*

$$\partial_+\mathbf{u}(x) \ := \ -\partial_-(-\mathbf{u})(x)$$

Naturally, when $\mathbf{u}$ is Fréchet differentiable at x, then

$$D_\uparrow\mathbf{u}(x)(v) \ = \ \langle f'(x), v\rangle$$

so that *the subdifferential* $\partial_-\mathbf{u}(x)$ *is reduced to the gradient* $\mathbf{u}'(x)$.

We observe that

$$\partial_-\mathbf{u}(x) + N_K(x) \ \subset \ \partial(\mathbf{u}|_K)(x)$$

If $\mathbf{u}$ is differentiable at a point $x \in K$, then the *subdifferential of the restriction is the sum of the gradient and the normal cone*:

$$\partial_-(\mathbf{u}|_K)(x) \ = \ \mathbf{u}'(x) + N_K(x)$$

We also note that the subdifferential of the indicator of a subset is the normal cone:

$$\partial_-\psi_K(x) \ = \ N_K(x)$$

that

$$\begin{cases} i) & (p,-1) \in N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \text{ if and only if } p \in \partial_-\mathbf{u}(x) \\ ii) & (p,0) \in N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \text{ if and only if } p \in \mathrm{Dom}(D_\uparrow\mathbf{u}(x))^- \end{cases}$$

so that We also deduce that

$$N_{\mathcal{E}p(\mathbf{u})}(x,\mathbf{u}(x)) \ = \ \{\lambda(q,-1)\}_{q\in\partial_-\mathbf{u}(x),\,\lambda>0} \bigcup \{(q,0)\}_{q\in\mathrm{Dom}(D_\uparrow\mathbf{u}(x))^-}$$

The subset $\mathrm{Dom}(D_\uparrow\mathbf{u}(x))^- \ = \ \{0\}$ whenever the domain of the contingent epiderivative $D_\uparrow\mathbf{u}(x)$ is dense in X. This happens when $\mathbf{u}$ is locally Lipschitz and when the dimension of X is finite:

Proposition 5.2.33 *Let* X *be a finite dimensional vector space,* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* $x_0 \in \mathrm{Dom}(\mathbf{u})$*. Then the subdifferential* $\partial_-\mathbf{u}(x)$ *is the set of elements* $p \in X^\star$ *satisfying*

$$\liminf_{x\to x_0} \frac{\mathbf{u}(x) - \mathbf{u}(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \ \geq \ 0 \qquad (5.4)$$

is the local subdifferential of $\mathbf{u}$ at x_0.

In a symmetric way, the superdifferential $\partial_+\mathbf{u}(x_0)$ *of* $\mathbf{u}$ *at* x_0 *is the subset of elements* $p \in X^\star$ *satisfying*

$$\limsup_{x \to x_0} \frac{\mathbf{u}(x) - \mathbf{u}(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \; \leq \; 0$$

The equivalent formulation (5.4) of the concept of subdifferential has been introduced by Crandall & P.-L. Lions for defining *viscosity solutions* to Hamilton-Jacobi equations.

5.2.12 Moreau-Rockafellar Subdifferentials

When $\mathbf{u}$ is convex, the generalized gradient coincides with the subdifferential introduced by Moreau and Rockafellar for convex functions in the early 60's:

Definition 5.2.34 *Consider a nontrivial function* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *and* $x \in \mathrm{Dom}(\mathbf{u})$. *The closed convex subset* $\partial\mathbf{u}(x)$ *defined by*

$$\partial\mathbf{u}(x) \; = \; \{p \; \in \; X^\star \mid \; \forall \, y \; \in \; X, \; \langle p, y - x \rangle \; \leq \; \mathbf{u}(y) - \mathbf{u}(x)\}$$

(which may be empty) is called the Moreau-Rockafellar subdifferential *of* $\mathbf{u}$ *at* x. *We say that* $\mathbf{u}$ *is* subdifferentiable at x *if* $\partial\mathbf{u}(x) \neq \emptyset$.

Proposition 5.2.35 *Let* $\mathbf{u} : X \mapsto \mathbf{R}_+$ *be a nontrivial extended convex function. Then the subdifferential* $\partial_-\mathbf{u}(x)$ *coincides with Moreau-Rockafellar subdifferential* $\partial\mathbf{u}(x)$.

Furthermore, the graph of the subdifferential map $x \rightsquigarrow \partial\mathbf{u}(x)$ *is closed.*

Let us mention the following simple — but useful — remark:

Proposition 5.2.36 *Assume that* $\mathbf{u} := \mathbf{v} + \mathbf{w}$ *is the sum of a differentiable function* $\mathbf{v}$ *and a convex function* $\mathbf{w}$. *If* $\bar{x}$ *minimizes* $\mathbf{u}$, *then*

$$-\mathbf{v}'(\bar{x}) \; \in \; \partial\mathbf{w}(\bar{x})$$

Proof — Indeed, for $h > 0$ small enough, $\bar{x} + h(y - \bar{x}) = (1 - h)\bar{x} + hy$ so that

$$0 \;\leq\; \frac{\mathbf{u}(\bar{x} + h(y - \bar{x})) - \mathbf{u}(\bar{x})}{h} \;\leq\; \frac{\mathbf{u}(\bar{x} + h(y - \bar{x})) - \mathbf{u}(\bar{x})}{h} + \mathbf{w}(y) - \mathbf{w}(\bar{x})$$

thanks to the convexity of $\mathbf{w}$. Letting h converge to 0 yields

$$0 \;\leq\; \langle \mathbf{v}'(\bar{x}), y - \bar{x} \rangle + \mathbf{w}(y) - \mathbf{w}(\bar{x})$$

so that $-\mathbf{v}'(\bar{x})$ belongs to $\partial \mathbf{w}(\bar{x})$. □

5.3 Discrete Dynamical Systems

Definition 5.3.1 *Let us consider a sequence $\vec{K}$ of subsets $K_n \subset X$, regarded as a* discrete tube $n \in \mathbf{N} \rightsquigarrow K_n \subset X$.

A sequence $\vec{x} := (x_0, \ldots, x_n, \ldots)$ is said to be viable in the tube $\vec{K}$ (on $[0, N]$ or up to time N) if

$$\forall\, 0 \leq n \leq N, \;\; x_n \;\in\; K_n$$

For $N = +\infty$, $\vec{x}$ is said to be viable in $\vec{K}$.

Let us consider a sequence $\vec{R}$ of nontrivial set-valued maps $R_n : X \rightsquigarrow X$. We shall denote by $\vec{S}_{\vec{R}}(x_0)$ the subset of sequences $\vec{x} := \{x_n\}_{n \geq 0}$ solutions to the dynamical system

$$\forall\, n \geq 0, \;\; x_{n+1} \;\in\; R_n(x_n)$$

The product

$$\prod_{n \geq 0} K_n \;:=\; \{\vec{x} \in X^{\mathbf{N}} \mid \forall\, n \geq 0, \;\; x_n \in K_n\}$$

of the subsets K_n is the subset of sequences viable in the tube $n \rightsquigarrow K_n$.

Definition 5.3.2 *We shall say the growth of $\vec{R}$ is linear if there exists $c > 0$ such that*

$$\forall\, n \geq 0, \;\; \forall\, x \in X, \;\; \|R_n(x)\| \;:=\; \sup_{y \in R_n(x)} \|y\| \;\leq\; c\|x\|$$

We deduce at once from the definition the

Lemma 5.3.3 *Let us consider any solution* $\vec{x} \in \vec{S}_{\vec{R}}(x_0)$ *of the dynamical system* $x_{n+1} \in R_n(x_n)$ *starting at* x_0*. If the growth of* $\vec{R}$ *is linear, then*

$$\forall\, n \geq 0, \ \ \|x_n\| \ \leq \ (1+c)^n \|x_0\|$$

We supply the space $X^{\mathbf{N}}$ of sequences $\vec{x}$ with the product topology, i.e., the pointwise convergence of this space regarded as the space of maps from $\mathbf{N}$ to X. The Tychonov Theorem states that if for every $n \geq 0$, the subsets $K_n \subset X$ are compact, then the subset

$$\prod_{n\geq 0} K_n \ := \ \{\vec{x} \in X^{\mathbf{N}} \mid \forall\, n \geq 0, \ \ x_n \in K_n\}$$

of sequences viable in the tube $\vec{K} : n \mapsto K_n$ is compact in $X^{\mathbf{N}}$.

Theorem 5.3.4 *Let* $\vec{R}$ *be a sequence of set-valued maps with closed graph. Then the graph of the solution map* $\vec{S}_{\vec{R}}$ *is closed. Assume furthermore that the growth of* $\vec{R}$ *is linear. Then the sets* $\vec{S}_{\vec{R}}(x_0)$ *of solutions starting at* x_0 *are compact in* $X^{\mathbf{N}}$*. Actually, the graph of restriction to compact subsets* $L \subset X$ *of the solution map* $\vec{S}_{\vec{R}|_L}$ *is compact in* $X \times X^{\mathbf{N}}$*.*

Proof — Indeed, one can write the graph of the solution map in the form

$$\mathrm{Graph}(\vec{S}_R) \ = \ \{\vec{x} \mid \forall\, n \geq 0, \ \ (x_n, x_{n+1}) \in \mathrm{Graph}(R_n)\}$$

We deduce at once that the graph of $\vec{S}_{\vec{R}}$ is closed whenever the graph of $\vec{R}$ is closed.

Let $L \subset X$ be a compact subset of X. Since the growth of $\vec{R}$ is linear, we know that for any $n \geq 0$, the solutions remain in the product of the balls $B(a, (1+c)^n \|L\|)$, which are compact. Therefore, $\vec{S}_{\vec{R}}(L)$ is contained in the product

$$\prod_{n\geq 0} B(a, (1+c)^n \|L\|)$$

which is compact. $\square$

Definition 5.3.5 *Let $C \subset X$ be a subset.*

The functional $\varpi_C : X^{\mathbf{N}} \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $\vec{x}$ its discrete hitting time *$\varpi_C(\vec{x})$ defined by*

$$\varpi_C(\vec{x}) \ := \ \inf \{n \in \mathbf{N} \mid x_n \in C\}$$

is called the discrete hitting functional *of $\vec{x}$ to C.*

The functional $\tau_C := \varpi_{X \setminus C} : X^{\mathbf{N}} \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $\vec{x}$ its discrete exit time *$\tau_C(\vec{x})$ defined by*

$$\tau_C(\vec{x}) \ := \ \inf \{n \in \mathbf{N} \mid x_n \notin C\}$$

is called the discrete exit functional *of $\vec{x}$ from C.*

The functions $\varpi_C^\flat : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and $\varpi_C^\sharp : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\varpi_C^\flat(x) \ := \ \inf_{\vec{x} \in \vec{\mathcal{S}}_R(x)} \varpi_C(\vec{x}) \quad \& \quad \varpi_C^\sharp(x) \ := \ \sup_{\vec{x} \in \vec{\mathcal{S}}_R(x)} \varpi_C(\vec{x})$$

are called the lower and upper hitting function *and the functions $\tau_C^\flat : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and $\tau_C^\sharp : C \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\tau_C^\flat(x) \ := \ \inf_{\vec{x} \in \vec{\mathcal{S}}_R(x)} \tau_C(\vec{x}) \quad \& \quad \tau_C^\sharp(x) \ := \ \sup_{\vec{x} \in \vec{\mathcal{S}}_R(x)} \tau_C(\vec{x})$$

the lower and upper exit function*.*

5.4 Discrete Viability Kernels

In order to characterize closed subsets viable under an impulse differential inclusion (F, R), we need to introduce the definitions of the discrete viability kernel under a discrete system $x_{n+1} \in R(x_n)$ and of the viability kernel of a closed subset under a differential inclusion $x' \in F(x)$.

We say that a subset K is discretely viable under a set-valued map $R : K \rightsquigarrow X$ if from any $x_0 \in K$ starts a sequence $x_{n+1} \in R(x_n)$ viable in K in the sense that for all $n \geq 0$, $x_n \in K$.

Definition 5.4.1 *The subset* $Viab_R^N(K)$ *of initial states* $x_0 \in K$ *such that one solution* $\vec{x}$ *to discrete system* $x_{n+1} \in R(x_n)$ *starting at* x_0 *is viable in* K *for all* $n \in [0, N]$ *is called the* discrete N-viability kernel *and the subset*

$$Viab_R(K) \; := \; Viab_R^{+\infty}(K)$$

is called the discrete viability kernel *of* K *under* R. *A subset* K *is a* discrete repeller *if its discrete viability kernel is empty.*

We say that

$$Egress_R(K) \; := \; K \backslash Viab_R^1(K) \tag{5.5}$$

is the discrete egress set *of* K *under* R *because for any* $x \in Egress_R(K)$, $R^{|K}(x) := R(x) \cap K = \emptyset$.

One can observe that

$$Viab_R^1(K) \; = \; K \cap R^{-1}(K) \;\; \& \;\; Egress_R(K) \; := \; K \backslash R^{-1}(K) \tag{5.6}$$

They are related to subsets called respectively stopping sets and continuation sets in [53, 54, 55, Bensoussan & Lions J.-L.].

By Lemma 5.1.5, denoting by $R_{|K}^{|K}$ the graphical restriction of R to $K \times K$ defined by

$$R_{|K}^{|K}(x) \; := \left\{ \begin{array}{ll} R(x) \cap K & \text{if} \;\; x \in K \\ \emptyset & \text{if} \;\; x \notin K \end{array} \right.$$

we observe that its domain is the 1-discrete viability kernel of R:

$$Viab_R^1(K) \; = \; \text{Dom}(R_{|K}^{|K})$$

We first observe the following property:

Proposition 5.4.2 *The discrete* N*-viability kernel is obtained through the* Viability Kernel Algorithm*:*

$$\forall \, N \geq 1, \;\; Viab_R^N(K) \; = \; Viab_R^1(Viab_R^{N-1}(K)) \tag{5.7}$$

In particular, setting $Egress_R^0(K) := Egress_R(K)$ *and, for* $N \geq 1$,

$$Egress_R^N(K) \; := \; Egress_R(Viab_R^N(K)) \; = \; Viab_R^N(K) \backslash Viab_R^{N+1}(K) \tag{5.8}$$

we deduce that

$$x \in Egress_R^N(K) \text{ if and only if } R(x) \cap Viab_R^N(K) \; = \; \emptyset$$

Proof — We indeed observe that

$$Viab_R^1(K) \; = \; K \cap R^{-1}(K)$$

Next, we note that

$$Viab_R^N(K) \; = \; Viab_R^1(Viab_R^{N-1}(K))$$

Indeed, $x_0 \in Viab_R^N(K)$ if and only if there exists a sequence $x_l \in K$ such that $x_{l+1} \in R(x_l) \cap K$ for $l = 0, \ldots, N-1$. Therefore, $x_1 \in Viab_R^{N-1}(K) \cap R(x_0)$. Since x_0 also belongs to $Viab_R^{N-1}(K)$, we infer that it belongs to $Viab_R^1(Viab_R^{N-1}(K))$.

Conversely, let x_0 belong to $Viab_R^1(Viab_R^{N-1}(K))$. Then x_0 belongs to K and there exists $x_1 \in R(x_0) \cap Viab_R^{N-1}(K))$, so that $x_l \in R(x_{l-1}) \cap K$ for $l = 2, \ldots, N$. Since x_1 belongs to K, we infer that $x_0 \in Viab_R^N(K)$. □

For simplicity, let us set

$$K_N \; := \; Viab_R^N(K), \;\; R_N \; := \; R_{|K_N}^{|K_N} \;\; \& \;\; E_N \; := \; Egress_R^N(K)$$

Therefore, each N-discrete viability kernel can be written $K_N = K_{N+1} \cup E_N$, and R_{N-1} maps elements $x \in K_N$ to nonempty subsets $R_{N-1}(x) \subset K_{N-1}$.

Furthermore, for every $x \in E_N$, $R_{N-1}(x) \subset E_{N-1}$. Therefore, we introduce the maps ${}^N R : E_N \rightsquigarrow E_0 = K \backslash R^{-1}(K)$ defined by

$$\forall \, x \in Egress_R^N(K), \;\; {}^N R(x) \; = \; R_0(R_1(\cdots(R_{N-2}(R_{N-1}(x))))) \; \subset \; Egress_R(K)$$

In other words, the set-valued map ${}^N R$ associates with any $x \in E_N$ the subset of elements $x_N \in K \backslash R^{-1}(K)$ obtained recursively through a sequence $x_1 \in R(x) \cap E_{N-1}$, $x_2 \in R(x_1) \cap E_{N-2}$, …, $x_N \in R(x_{N-1}) \cap E_0$, so that $R(x_N) \cap K = \emptyset$.

We now prove that *the discrete viability kernel* is the largest subset $L \subset K$ of K discretely viable under R:

Proposition 5.4.3 *The discrete viability kernel $Viab_R(K)$ of K under R is the largest subset of K viable under R.*

Proof — Every subset $L \subset K$ viable under R is obviously contained in the discrete viability kernel $Viab_R(K)$.

On the other hand, if $\vec{x}$ is a solution to the discrete system $x_{n+1} \in R(x_n)$ viable in K, then for all $n > 0$, the sequence $\vec{y}^n$ defined by $(\vec{y}^n)_m := \vec{x}_{n+m}$ is also a solution to the discrete system, starting at x_n, viable in K.

Therefore, for any element $x_0 \in Viab_R(K)$, there exists a viable solution $\vec{x}$ to the discrete system starting from x_0, and thus, for all $n \geq 0$, x_n is the initial state of the solution $\vec{y}^n$ viable in K, so that x_n belongs to $Viab_R(K)$. Hence $Viab_R(K)$ is viable under R. □

Proposition 5.4.4 *Let $K \subset X$ be a closed subset. If R is upper semicontinuous with compact images, then for any $N \geq 0$, the discrete N-viability kernels and the N-capture basins are characterized by:*

$$\left\{ \begin{array}{lcl} Viab_R^N(K) & = & \left\{x \in K \mid \tau_K^\sharp(x) \ \geq \ N\right\} \\ \\ \mathrm{Capt}_R^N(K) & = & \left\{x \in X \mid \varpi_K^\flat(x) \ \leq \ N\right\} \end{array} \right. \tag{5.9}$$

Consequently, the discrete exit time function $\tau_K^\sharp$ is the step function defined by

$$\forall\, l \geq 0,\ \ \tau_K^\sharp(x) \ = \ l \ \text{ if } \ x \in Egress_R^l(K) \ := \ Viab_R^{l-1}(K) \backslash Viab_R^l(K)$$

and the discrete hitting time function $\varpi_K^\flat$ is the step function defined by

$$\forall\, l \geq 0,\ \ \varpi_K^\flat(x) \ = \ l \ \text{ if } \ x \in Capt_R^l(K) \backslash Capt_R^{l-1}(K)$$

If R is lower semicontinuous, then for any $N \geq 0$, the discrete N-invariance kernels and N-absorption basins are characterized by:

$$\left\{ \begin{array}{lcl} \mathrm{Inv}_R^N(K) & = & \left\{x \in K \mid \tau_K^\flat(x) \ \geq \ N\right\} \\ \\ \mathrm{Abs}_R^N(K) & = & \left\{x \in X \mid \varpi_K^\sharp(x) \ \leq \ N\right\} \end{array} \right. \tag{5.10}$$

Proposition 5.4.5 *Let assume that K is a closed subset, that $R : X \rightsquigarrow X$ is upper semicontinuous and that*

$$\forall\, x \in K, \quad R(x) \cap (K + B) \text{ is compact}$$

where B is the unit ball. Then the discrete N-viability kernels $Viab_R^N(K)$ are closed. If furthermore, one of them is compact and nonempty, then

(a) either the discrete viability kernel $Viab_R(K)$ is not empty and equal to

$$Viab_R(K) \ := \ \bigcap_{N>0} Viab_R^N(K)$$

(b) or K is a discrete repeller under R and there exists a smaller nonempty discrete N-viability kernel $Viab_R^N(K)$ (called the discrete viability core*).*

Assume now that K is a repeller under R, so that we can partition K as the union

$$K \ = \ \bigcup_{N=0}^{J} Egress_R^N(K)$$

We thus can associate with the set-valued map $R : K \rightsquigarrow X$ the set-valued map $\vec{R} : K \rightsquigarrow X$ defined by

$$\forall\, x \in Egress_R^N(K), \ \ \vec{R}(x) \ := \ {}^N R(x)$$

We thus deduce that $\vec{R}$ maps $Viab_R^1(K) \ = \ K \cap R^{-1}(K)$ to $Egress_R(K) \ := K \backslash R^{-1}(K)$:

$$\forall\, x \in K \cap R^{-1}(K), \ \ \vec{R}(x) \cap R^{-1}(K) \ \subset \ K \backslash R^{-1}(K)$$

In other words, any $x \in K \cap R^{-1}(K)$ belongs to some $Egress_R^N(K)$ with $N = 1, \ldots, J$. Therefore, the set-valued map $\vec{R}$ maps x to a nonempty subset $\vec{R}(x) =^N R(x)$ contained in $K \backslash R^{-1}(K)$.

Corollary 5.4.6 *We posit the assumptions of Proposition 5.4.5. If K is a repeller under R, then the set-valued map $\vec{R}$ is upper semicontinuous.*

Proof — Let $x \in K$, which belongs to some subset $Egress_R^N(K)$ since K is a discrete repeller under R. Hence x belongs to $Viab_R^N(K)$ and $R(x) \cap Viab_R^N(K+1)$ is empty. The subset $R(x) \cap Viab_R^N(K)$ being compact and non empty, there exists $\delta > 0$ such that

$$B(R(x) \cap Viab_R^N(K), \delta) \cap Viab_R^N(K+1) \; = \; \emptyset$$

Since x belongs to $Viab_R^N(K) \backslash Viab_R^{N+1}(K)$, there exists $\gamma > 0$ such that $B(x, \gamma) \cap Viab_R^{N+1}(K) = \emptyset$. The reset map R being upper semicontinuous, we can associate with any $\varepsilon \leq \delta$ an $\eta \leq \gamma$ such that $R(y) \cap Viab_R^N(K)$ is contained in $B(R(x) \cap Viab_R^N(K), \varepsilon)$ whenever $y \in B(x, \eta)$. But for such an y, $R_N(y) = R(y)$ so that we deduce that $R_N(x) = R(x)$ is upper semicontinuous from $Egress_R^N(K)$ to $Viab_R^N(K)$.

We therefore infer that ${}^N R$ is upper semicontinuous from $Egress_R^N(K)$ to K, and thus, that $\vec{R}$ is upper semicontinuous from $Viab_R^1(K)$ to $Egress_R(K)$ with nonempty values. □

Chapter 6

Viability Theory at a Glance

6.1 Viability and Invariance under Differential Inclusions

6.1.1 The Basic Viability Theorems

Definition 6.1.1 *A function* $t \in [0,T] \mapsto x(t) \in X$ *is said to be* viable *in* K *on* $[0,T]$ *if*

$$\forall\, t \in [0,T],\ \ x(t)\ \in\ K$$

and viable in K *if* $T = +\infty$. *When* K *be a subset of* X, *we shall say that* K is locally viable under F *if for any initial state* x_0 *of* K, there exist $T_{x_0} > 0$ *and a solution to differential inclusion* $x' \in F(x)$ *starting at* x_0 viable *in* K *on* $[0, T_{x_0}]$. *It is said to be (globally) viable under* F *if we can always take* $T_{x_0} = \infty$.

Definition 6.1.2 *The contingent cone* $T_K(x)$ *to* $K \subset X$ *at* $x \in K$ *is the set of directions* $v \in X$ *such that there exist sequences* $h_n > 0$ *converging to* 0 *and* v_n *converging to* v *satisfying* $x + h_n v_n \in K$ *for every* n. *The (regular) normal cone* $N_K(x) := T_K(x)^-$ *is the polar cone to the contingent cone* $T_K(x)$.

See for instance [31, Aubin & Frankowska]) or [182, Rockafellar & Wets] for more details.

We denote by

$$\forall\, p \in X^\star,\ \ \sigma(K,p)\ :=\ \sup_{x\in K}\langle p, x\rangle$$

the support function of K. Once can characterize this viability property for Marchaud maps:

Definition 6.1.3 (Marchaud Map) *We shall say that F is a* Marchaud map[1] *if*

$$\begin{cases} i) & \text{the graph and the domain of } F \text{ are nonempty and closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear:} \\ & \exists\, c>0 \mid \forall\, x \in X,\ \ \|F(x)\| := \sup_{v\in F(x)}\ \leq\ c(\|x\|+1) \end{cases}$$

We recall the following viability theorems:

Theorem 6.1.4 *Assume that F is Marchaud. The two following statements hold true:*

(a) If K is closed, then K is (globally) viable under F if and only if

$$\forall\, x \in K,\ \ F(x)\cap T_K(x) \neq \emptyset$$

or, equivalently, in dual form, if and only if

$$\forall\, x\in K,\ \forall\, p \in N_K(x),\ \ \sigma(F(x),-p)\ \geq\ 0$$

(b) If $C\subset K$ is closed, then $K\backslash C$ is locally viable under F if and only if

$$\forall\, x \in K\backslash C,\ \ F(x)\cap T_K(x) \neq \emptyset$$

or, equivalently, in dual form, if and only if

$$\forall\, x\in K\backslash C,\ \forall\, p \in N_K(x),\ \ \sigma(F(x),-p)\ \geq\ 0$$

[1] We can replace the linear growth by the more general condition:

$$\exists\, c>0 \mid \forall\, x\in X,\ \ \sup_{v\in F(x)}\frac{\langle v,x\rangle}{(\|x\|+1)^2}\ \leq\ c$$

Definition 6.1.5 *Let $F : X \rightsquigarrow X$ be a set-valued map and $C \subset X$ be a subset. The* solution map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty, X)$ *associated with the differential inclusion $x' \in F(x)$ maps initial states $x \in X$ to the set $\mathcal{S}_F(x)$ of solutions to the differential inclusion $x' \in F(x)$ starting at x at time $t = 0$.*

We recall the following version of the important Theorem 3.5.2 of Viability Theory, [8, Aubin]:

Theorem 6.1.6 *Assume that $F : X \mapsto X$ is Marchaud. Let $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}_F(x_n)$ be solutions to the differential inclusion $x' \in F(x)$ starting at x_n. Then there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}_F(x)$ uniformly on compact intervals.*

We derive the following properties:

Proposition 6.1.7 *Let $F : X \rightsquigarrow X$ be Marchaud and C be viable under F. Then the closure $\overline{C}$ of C is also viable under F.*

Definition 6.1.8 *Let $F : X \rightsquigarrow X$ be a set-valued map and $C \subset X$ be a subset. The* reachable map $\vartheta_F(\cdot, x)$ *is defined by*

$$\forall\, x \in X, \quad \vartheta_F(t, x) \ := \ \{x(t)\}_{x(\cdot) \in \mathcal{S}_F(x)}$$

We associate with it the reachable tube $t \rightsquigarrow \vartheta_F(t, C)$ *defined by*

$$\vartheta_F(t, C) \ := \ \{x(t)\}_{x(\cdot) \in \mathcal{S}_F(C)}$$

Therefore, the subset $\vartheta_{-F}(t, C)$ is the subset of elements $x \in X$ which reach the subset at the prescribed time t.

We obtain the following properties:

Proposition 6.1.9 *The reachable map $t \rightsquigarrow \vartheta_F(t,x)$ enjoys the semi-group property:* $\forall t,s \geq 0,\ \ \vartheta_F(t+s,x) = \vartheta_F(t,\vartheta_F(s,x))$.

Furthermore,

$$(\vartheta_F(t,\cdot))^{-1} \ := \ \vartheta_{-F}(t,\cdot)$$

If F is Marchaud and $K \subset X$ is closed, the graph of the reachable map $(x,t) \rightsquigarrow \vartheta_F(t,K)$ is closed.

Proof — The semi-group property is obvious. Let us prove the second one: If $y \in \vartheta_F(t,x)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at x such that $y = x(t)$. We set $y(s) := x(t-s)$ if $s \in [0,t]$ and we choose any solution $y(\cdot)$ to the differential inclusion $y' \in -F(y)$ starting at x at time t for $s \geq t$. Then such a function $y(\cdot)$ is a solution to the differential inclusion $y' \in -F(y)$ starting at y and satisfying $y(t) = x$. This shows that $x \in \vartheta_{-F}(t,y)$.

We deduce from Theorem 6.1.6 that if F is Marchaud and $K \subset X$ is closed, the graph of the reachable map $(x,t) \rightsquigarrow \vartheta_F(t,K)$ is closed. $\square$

6.1.2 The Filippov Theorem

Actually, Filippov's Theorem is much more than a mere existence theorem. It also provides an estimate of the distance between a function $y(\cdot)$ and the set $\mathcal{S}_F(x_0)$ of solutions starting at some initial state x_0.

For stating this theorem and characterizing invariance properties, we need Lipschitz maps:

Definition 6.1.10 *The set-valued map F is said to be Lipschitz if there exists a constant $\lambda > 0$ such that*

$$\forall\, x,y \in X,\ \ F(x) \ \subset \ F(y) + \lambda\|x-y\|B$$

Theorem 6.1.11 (Filippov) *Assume that $F : X \rightsquigarrow X$ is λ-Lipschitz with nonempty closed values on the interior of its domain. Let $y(\cdot) : [0,T] \mapsto X$*

be a given absolutely continuous function such that $t \to d(y'(t), F(y(t)))$ *is integrable (for the measure* $e^{-\lambda s}ds$ *when* $T = +\infty$*).*

Then there exists a solution $x(\cdot)$ *to differential inclusion* $x' \in F(x)$ *such that, for all* $t \in [0, T]$,

$$\|x(t) - y(t)\| \leq e^{\lambda t}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \tag{6.1}$$

and for almost all $t \in [0, T[$,

$$\begin{cases} \|x'(t) \ - \ y'(t)\| \ \leq \ d(y'(t), F(y(t))) \\ +\lambda e^{\lambda t}\left(\|x_0 - y(0)\| \ + \ \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \end{cases}$$

It implies the existence of a solution starting from any initial state with any initial velocity:

Corollary 6.1.12 *Assume that* F *is Lipschitz on a neighborhood of* x_0. *Then, for any* $x_0 \in \mathrm{Int}(\mathrm{Dom}(F))$ *and* $v_0 \in F(x_0)$, *there exist* $T > 0$ *and a solution* $x(\cdot)$ *to differential inclusion* $x' \in F(x)$ *defined on* $[0, T]$ *and satisfying* $x(0) = x_0$ *and* $x'(0) = v_0$.

Proof — We apply Filippov's Theorem with $y(t) := x_0 + tv_0$ and $x_0 := y(0)$. Then $d(y'(t), F(y(t))) \leq \lambda t\|v_0\|$ and

$$\begin{cases} \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds \\ \leq \ e^{\lambda t}\int_0^t \lambda\tau\|v_0\|e^{-\lambda\tau}d\tau \ \leq \ \frac{\|v_0\|}{\lambda}\left(e^{\lambda t} - 1 - \lambda t\right) \end{cases}$$

Filippov's Theorem implies the existence of a solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 and satisfying

$$\|x(t) - x_0 - tv_0\| \ \leq \ \frac{\|v_0\|}{\lambda}\left(e^{\lambda t} - 1 - \lambda t\right)$$

Dividing by $t > 0$ and letting t converge to $0+$, we infer $x'(0) = v_0$. □

Filippov's Theorem also implies the Lipschitz dependence of the solution map on the initial condition:

Corollary 6.1.13 *Let $y(\cdot) \in \mathcal{S}_F(y_0)$ and assume that F, $y(\cdot)$ satisfy the assumptions of Filippov's Theorem 6.1.11. Then*

$$d\left(y(t),\ \mathcal{S}_F(x_0)(t)\right) \ \leq\ \|x_0 - y_0\|\, e^{\lambda t}$$

so that the solution map $\mathcal{S}_F$ is lower semicontinuous.

6.1.3 Characterization of Local Invariance

We are ready to prove the characterization of invariant domains under a Lipschitz map:

Theorem 6.1.14 *Let us assume that F is locally Lipschitz and has closed values. There exists an (open) neighborhood $\mathcal{V}(x)$ of $x \in K$ such that the two following properties are equivalent:*

(a) for every $y \in \mathcal{V}(x)$, one can associate with every solution $y(\cdot) \in \mathcal{S}_F(y)$ a sequence $t_n \to 0+$ such that $x(t_n) \in K$,

(b) for all $y \in \mathcal{V}(x)$, $F(y) \subset T_K(y)$.

Consequently, a closed subset K is locally invariant under F if and only if K is an invariance domain.

Proof — Let us assume that $F(y) \subset T_K(y)$ on a neighborhood $\mathcal{V}(x)$ and let $x(\cdot)$ be any solution to the differential inclusion $x' \in F(x)$ starting at $x_0 \in \mathcal{V}(x)$ and defined on some interval $[0, T]$. Let us set $g(t) := d_K(x(t))$, which is absolutely continuous on $[0, T]$.

Let t be a point such that both $x'(t)$ and $g'(t)$ exist and $x'(t)$ belongs to $F(x(t))$. Then there exists $\varepsilon(h)$ converging to 0 with h such that

$$x(t+h) \ =\ x(t) + hx'(t) + h\varepsilon(h)$$

and

$$D_{\uparrow} g(t)(1) \ =\ \lim_{h \to 0+} \frac{d_K(x(t) + hx'(t) + h\varepsilon(h)) - d_K(x(t))}{h}$$

First, we observe that

$$D_{\uparrow} d_K(x)(x'(t)) \ \leq\ d(x'(t), T_K(\Pi_K(x(t))))$$

Indeed, choose $z \in \Pi_K(y)$ and $w \in T_K(z)$. Then

$$\left\{ \begin{array}{l} \frac{d_K(y+hv) - d_K(y)}{h} \;\leq\; \frac{\|y-z\| + d_K(z+hv) - d_K(y)}{h} \\ \\ = \frac{d_K(z+hv)}{h} \;\leq\; \frac{d_K(z+hw)}{h} + \|v-w\| \end{array} \right.$$

Since z belongs to K and $w \in T_K(z)$, the above inequality implies that

$$D_\uparrow d_K(y)(v) \;\leq\; d(v, T_K(z))$$

Let us denote by $\lambda > 0$ the Lipschitz constant of F on $\mathcal{V}(x)$ and choose any y in $\Pi_K(x(t))$. We deduce that:

$$d(x'(t), T_K(\Pi_K(x(t)))) \leq d(x'(t), T_K(y)) \leq d(x'(t), F(y))$$
(because of the tangential condition)

$$\leq d(x'(t), F(x(t))) + \lambda\|y - x(t)\| \;\; \text{(since } F \text{ is Lipschitz)}$$

$$= \; 0 + \lambda d_K(x(t)) \; = \; \lambda g(t)$$

Then g is a solution to

$$\text{for almost all } t \in [0,T], \;\; g'(t) \;\leq\; \lambda g(t) \;\;\&\;\; g(0) = 0$$

We deduce that $g(t) = 0$ for all $t \in [0,T]$, and therefore, that $x(t)$ is viable in K on $[0,T]$, and thus, $d_K(x(t)) = 0$ for all $t \in [0,T]$.

— Let $x_0 \in \mathcal{V}(x)$. We have to prove that any $u_0 \in F(x_0)$ is contingent to K at x_0. Corollary 6.1.12 implies that for all x_0 and $u_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ satisfying $x(0) = x_0$ and $x'(0) = u_0$. By assumption, there exists a sequence $t_n \to 0+$ such that $x(t_n) \in K$. Hence v_0, being the limit of $\frac{x(t_n) - x_0}{t_n}$, belongs to belongs to $T_K(x_0)$. It follows that $F(x_0)$ is contained in $T_K(x_0)$. □

We thus deduce the Invariance Theorem:

Theorem 6.1.15 *Assume that $F : X \rightsquigarrow X$ is Lipschitz. Then a closed subset K is locally invariant under F if and only if*

$$\forall\, x \in K, \;\; F(x) \;\subset\; T_K(x)$$

or, equivalently, in dual form, if and only if

$$\forall\, x \in K, \; \forall\, p \in N_K(x), \; \sigma(F(x), p) \;\leq\; 0$$

We derive the following properties:

Proposition 6.1.16 *Let $F : X \rightsquigarrow X$ be Lipschitz and C be invariant under F. Then the closure $\overline{C}$ of C is also invariant under F.*

6.1.4 Duality between Tangential and Normal Conditions

We provide a recent and very simple proof due to Hlne Frankowska of Theorem 6.1.17:

Theorem 6.1.17 *Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the three following properties are equivalent:*

$$\begin{cases} i) & \forall\, x \in K, \;\; F(x) \cap T_K(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in K, \;\; F(x) \cap \overline{co}\,(T_K(x)) \;\neq\; \emptyset \\ iii) & \forall\, x \in K, \;\; \forall\, p \in N_K(x), \;\; \sigma(F(x), -p) \;:=\; \sup_{v \in F(x)} \langle -p, v \rangle \;\geq\; 0 \end{cases} \tag{6.2}$$

Proof — Clearly (6.2)$i) \Rightarrow$ (6.2)$ii) \Rightarrow$ (6.2)$iii)$. We recall that if $x \in \Pi_K(y)$, then $p := y - x \in N_K(x)$.

Assume that (6.2)iii) holds true and fix $x \in K$. Let us set

$$\varphi(t) \;:=\; d^2(x + tF(x), K) \;=\; \|x + ty_t - x_t\|^2$$

where $y_t \in F(x)$, $x_t \in \Pi_K(x+ty_t)$ and where $\|y_t\| \leq M$. We deduce that φ is lower semicontinuous, that

$$p_t \ := \ x+ty_t-x_t \ \in \ N_K(x_t)$$

and that $\|p_t\| \leq \|x+ty_t-x\| = t\|y_t\| \leq Mt$.
Since for any $u \in F(x)$,

$$\frac{\varphi(t+h)-\varphi(t)}{h} \ \leq \ 2\langle p_t, u\rangle + h\|u\|^2$$

we deduce that

$$D_\uparrow \varphi(t)(1) \ \leq \ 2\sigma^\flat(F(x),p_t) \ := \ 2 \inf_{u\in F(x)} \langle p_t, u\rangle$$

Furthermore, F being upper semicontinuous, we can associate with any $\varepsilon > 0$ an $\eta(\varepsilon) > 0$ such that

$$\forall\, y \in B(x,\eta(\varepsilon)), \ \ F(y) \ \subset \ F(x)+\varepsilon B$$

so that

$$\forall\, y \in B(x,\eta(\varepsilon)), \ \ \sigma^\flat(F(x),p_t) \ \leq \ \sigma^\flat(F(y),p_t)+\varepsilon\|p_t\|$$

Assumption (6.2)iv) implies that $\sigma^\flat(F(x_t),p_t) \leq 0$ because p_t is a normal to K at x_t. Since $\|x-x_t\| \leq \|p_t\|+t\|y_t\| \leq 2tM \leq \eta(\varepsilon)$ whenever $t \leq \frac{\eta(\varepsilon)}{2M}$, we deduce that

$$D_\uparrow \varphi(t)(1) \ \leq \ 2\sigma^\flat(F(x),p_t) \ \leq \ 2\varepsilon\|p_t\| \ \leq \ 2\varepsilon Mt$$

whenever $t \leq \frac{\eta(\varepsilon)}{2M}$. Since φ is lower semicontinuous, we deduce that

$$\forall\, t \leq \frac{\eta(\varepsilon)}{2M}, \ \ \varphi(t) \ \leq \ \varepsilon Mt^2$$

or, equivalently, that

$$\forall\, t \leq \frac{\eta(\varepsilon)}{2M}, \ \ \left\|y_t - \frac{x_t-x}{t}\right\| \ \leq \sqrt{\varepsilon M}$$

Hence $\left\|y_t - \frac{x_t-x}{t}\right\|$ converges to 0 with t. Since $\|y_t\| \leq M$, a subsequence y_{t_n} converges to some $y \in F(x)$. Hence the subsequence $\frac{x_{t_n}-x}{t_n} \in \frac{K-x}{t_n}$ converges to y, so that y belongs also to the contingent cone $T_K(x)$. $\square$

We now provide a characterization of invariance domains which allows to extend to the nonconvex case the dual characterization of the tangential condition involved in the definition of the invariance domains:

Theorem 6.1.18 *Assume that the set-valued map $F : K \rightsquigarrow X$ is lower semicontinuous. Then the three following properties are equivalent:*

$$\begin{cases} i) & \forall\, x \in K, \;\; F(x) \;\subset\; T_K(x) \\ ii) & \forall\, x \in K, \;\; F(x) \;\subset\; \overline{co}\,(T_K(x)) \;\neq\; \emptyset \\ iii) & \forall\, x \in K, \;\; \forall\, p \in N_K(x), \;\; \sigma(F(x),p) \;:=\; \sup_{v \in F(x)} \langle p, v \rangle \;\leq\; 0 \end{cases} \tag{6.3}$$

Proof — Clearly (6.3)i) yields (6.3)ii). The equivalence between (6.3)ii) and (6.3)iii) follows obviously from the Separation Theorem.

For proving that (6.3)iii) implies (6.3)i), let us choose $v_0 \in F(x_0)$ and prove that it belongs to $T_K(x_0)$. Thanks to the Michael Selection Theorem, there exists a continuous selection $f : x \in K \mapsto f(x) \in F(x) \subset X$ such that $f(x_0) = v_0$. Theorem 6.1.17 implies that $f(x) \in T_K(x)$, and thus, that $v_0 = f(x_0)$ belongs to $T_K(x_0)$. □

Remark — The dual characterization of viability domains of a Marchaud map was first noticed in a different context in [125, Guseinov, Subbotin & Ushakov]. A simpler proof of this fact was given by Hélène Frankowska and appeared in [32, Aubin & Frankowska] and in Theorem 3.2.4 of [8, Aubin]. Another proof was provided in [77, Cardaliaguet] and [216, Veliov]. The proof given here is taken from [, Dal Maso & Frankowska].

□

6.1.5 Invariance and Backward Invariance

Definition 6.1.19 *Let $F : X \mapsto X$ be given. A subset K is*

(a) backward invariant *under F if for every $t_0 \in]0, +\infty[$, $x \in K$, all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ arriving at x at time t_0 are viable in K on the interval $[0, t_0]$,*

(b) locally backward invariant *under* F *if for every* $t_0 \in]0,+\infty[$, $x \in K$, *for all solutions* $x(\cdot)$ *to the differential inclusion* $x' \in F(x)$ *arriving at* x *at time* t_0, *there exists* $s \in [0, t_0[$ *such that* $x(\cdot)$ *is viable in* K *on the interval* $[s, t_0]$,

We now compare the invariance of a subset and the backward invariance of its complement:

Theorem 6.1.20 *A subset* K *is invariant under a set-valued map* F *if and only if its complement* $K^c := X\backslash K$ *is backward invariant under* F.

Proof — To say that K is not invariant under F amounts to saying that there exist a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ and $T > 0$ such that

$$x(0) \in K \;\&\; x(T) \in X\backslash K$$

and to say that $X\backslash K$ is not backward invariant amounts to saying that there exist a solution $y(\cdot)$ to the differential inclusion $x' \in F(x)$, $T > 0$ and $S \in [0, T[$ such that

$$y(S) \in K \;\&\; y(T) \in X\backslash K$$

It is obvious that the first statement implies the second one by taking $y(\cdot) = x(\cdot)$ and $S = 0$. Conversely, the second statement implies the first one by taking $x(t) := y(t + S)$ and replacing T by $T - S > 0$ since $x(0) = y(S)$ belongs to K and $x(T - S) = y(T)$ belongs to $X\backslash K$. □

It is also useful to relate backward viability and invariance under F to viability and invariance under $-F$:

Lemma 6.1.21 *Assume that from any* $x \in K$, *there exists a solution to the forward differential inclusion* $x' \in F(x)$ *and a solution to the backward differential inclusion* $x' \in -F(x)$. *Then* K *is locally backward viable (resp. invariant) under* F *if and only if* K *is locally viable (resp. invariant) under* $-F$.

Proof — Let us check this statement for local viability. Assume that K is locally backward viable and infer that K is locally invariant under $-F$. Indeed, let $x \in K$. Then, for any $T > 0$, there exists $S \in [0, T[$ and a solution

$x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on the interval $[S,T]$ and satisfying $x(T) = x$. Let $y(\cdot)$ be a solution to the differential inclusion $y' \in -F(y)$ starting at $y(0) = x(S)$. Then the function $z(\cdot)$ defined by

$$z(t) \ = \begin{cases} x(T-t) & \text{if} \quad t \in [0, T-S] \\ y(t+T-S) & \text{if} \quad t \geq T-S \end{cases}$$

is a solution to the differential inclusion $z' \in -F(z)$ starting at $z(0) = x(T) = x$ and viable in K on the interval $[0, T-S]$.

Conversely, assume that K is locally viable under $-F$ and check that K is locally backward invariant. Let $x \in K$, $T > 0$ and one solution $x(\cdot)$ to differential inclusion $x' \in -F(x)$ viable in K on $[0,R]$ where $R > 0$. Let be any solution $y(\cdot)$ to $y'(t) \in F(y(t))$ starting at x and set

$$z(t) \ = \begin{cases} x(T-t) & \text{if} \quad t \in [0, T] \\ y(t-T) & \text{if} \quad t \geq T \end{cases}$$

Hence $z(\cdot)$ to differential inclusion $z' \in F(z)$ satisfying $x(T) = x \in K$ and viable in K on the interval $[S,T]$ where $S := \max(T-R, 0)$. □

6.1.6 Exit Time Function and Viability Kernels

We shall relate the viability kernels and capture basins to upper exit and lower hitting functions that we are about to define:

Definition 6.1.22 *Let $C \subset X$ be a subset. We denote by*

$$\varpi_C : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$$

the hitting functional *(or* minimal time functional*) associating with $x(\cdot)$ its* hitting time $\varpi_C(x(\cdot))$ *defined by*

$$\varpi_C(x(\cdot)) \ := \ \inf\{t \in [0,+\infty[\ | \ x(t) \in C\}$$

The functional $\tau_K : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its exit time $\tau_K(x(\cdot))$ *defined by*

$$\tau_K(x(\cdot)) \ := \ \inf\{t \in [0,\infty[\ | \ x(t) \notin K\} \ := \ \varpi_{X \setminus K}(x(\cdot))$$

is called the exit functional*.*

If $K \supset C$, we introduce more generally the (constrained) hitting functional $\varpi_{(K,C)}$ *defined by*

$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \ \& \ \forall s \in [0,t], \ x(s) \in K \ \}$$

introduced in [78, Cardaliaguet, Quincampoix & Saint-Pierre]).

We use the convention $\inf\{\emptyset\} := +\infty$ and we observe that

$$\varpi_C(x(\cdot)) \ = \ \varpi_{(X,C)}(x(\cdot)) \ \leq \ \tau_C(x(\cdot))$$

that

$$\forall \ s \in [0, \varpi_C(x(\cdot))], \ \ \varpi_C(x(\cdot + s)) \ = \ \varpi_C(x(\cdot)) - s \tag{6.4}$$

and that if $K_1 \subset K_2$ and $D_1 \supset D_2$, then $\varpi_{(K_1,D_1)}(x(\cdot)) \leq \varpi_{(K_2,D_2)}(x(\cdot))$. Let us point out that

$$\varpi_{\bigcup_{i=1}^n D_i}(x(\cdot)) \ = \ \min_{i=1,\ldots,n} \varpi_{D_i}(x(\cdot))$$

Therefore,

$$\forall \ x \in K \backslash C, \ \ \tau_{K \backslash C}(x(\cdot)) = \min(\varpi_C(x(\cdot), \tau_K(x(\cdot)))$$

since

$$\tau_{K \backslash C}(x) \ = \ \varpi_{X \backslash (K \backslash C)}(x(\cdot)) \ = \ \varpi_{C \cup (X \backslash K)} \ = \ \min(\varpi_C(x(\cdot)), \varpi_{X \backslash K}(x(\cdot)))$$

Consider now a set-valued map $F : X \rightsquigarrow X$ and denote by $\mathcal{S}_F(x)$ the set of solutions $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at the initial state x.

Definition 6.1.23 *Let C and K be two closed subsets such that $C \subset K$. The functions $\varpi^\flat_{(K,C)} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and $\varpi^\sharp_{(K,C)} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\varpi^\flat_{(K,C)}(x) \ := \ \inf_{x(\cdot) \in \mathcal{S}_F(x)} \varpi_{(K,C)}(x(\cdot)) \ \ \& \ \ \varpi^\sharp_{(K,C)}(x) \ := \ \sup_{x(\cdot) \in \mathcal{S}_F(x)} \varpi_{(K,C)}(x(\cdot))$$

are called the lower and upper constrained hitting function *respectively and the functions*

$$\varpi_C^\flat(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_F(x)} \varpi_C(x(\cdot)) \;\&\; \varpi_C^\sharp(x) \;:=\; \sup_{x(\cdot)\in\mathcal{S}_F(x)} \varpi_C(x(\cdot))$$

are called the lower constrained hitting function *and* hitting function *respectively. The functions* $\tau_K^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *and* $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *defined by*

$$\tau_K^\flat(x) \;:=\; \inf_{x(\cdot)\in\mathcal{S}_F(x)} \tau_K(x(\cdot)) \quad \& \quad \tau_K^\sharp(x) \;:=\; \sup_{x(\cdot)\in\mathcal{S}_F(x)} \tau_K(x(\cdot))$$

the lower and upper exit function.

To say that K is a repeller under F amounts to saying that the exit function $\tau_K^\sharp$ is finite on K and to say that $K\backslash C$ is a repeller amounts to saying that all solutions $x(\cdot) \in \mathcal{S}_F(x)$ starting from $x \in K\backslash C$ reach C or leave K in finite time, i.e., satisfy $\tau_{K\backslash C}(x(\cdot)) = \min(\varpi_C(x(\cdot), \tau_K(x(\cdot)))) < +\infty$.

Theorem 6.1.24 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and C and K be two closed subsets such that $C \subset K$. Then the hitting function $\varpi_{(K,C)}^\flat$ is lower semicontinuous and the exit function $\tau_K^\sharp$ is upper semicontinuous. Furthermore, for any $x \in \mathrm{Dom}(\varpi_{(K,C)}^\flat)$, there exists one solution $x^\flat(\cdot) \in \mathcal{S}_F(x)$ which hits C as soon as possible before possibly leaving K*

$$\varpi_{(K,C)}^\flat(x) \;=\; \varpi_{(K,C)}(x^\flat(\cdot))$$

and for any $x \in \mathrm{Dom}(\tau_K^\sharp)$, there exists one solution $x^\sharp(\cdot) \in \mathcal{S}_F(x)$ which remains viable in K as long as possible:

$$\tau_K^\sharp(x) \;=\; \tau_K(x^\sharp(\cdot))$$

This statement is a consequence of the more general Theorem 6.1.26 dealing with upper hypolimits of upper exit functions and epilimits of lower constrained epifunctions of subsets that is proved below.

Consider two sequences of subsets $C_n \subset C$ and $K_n \subset X$ and their Painlev-Kuratowski upper limits

$$C^\sharp \ := \ \text{Limsup}_{n\rightarrow +\infty} C_n \ \& \ K^\sharp \ := \ \text{Limsup}_{n\rightarrow +\infty} K_n$$

Recall that the upper limit of a sequence of constant subsets C is the closure of C.

Definition 6.1.25 *We define the hypolimit* $\lim^\sharp_{\downarrow n\rightarrow\infty}\tau^\sharp_{K_n}$ *of upper exit functions* $\tau^\sharp_{K_n}$ *whose hypograph is the upper limit of the hypographs of the functions* $\tau^\sharp_{K_n}$

$$\mathcal{H}p(\lim{}^\sharp_{\downarrow n\rightarrow\infty}(\tau^\sharp_{K_n})) \ := \ \text{Limsup}_{n\rightarrow\infty}\mathcal{H}p(\tau^\sharp_{K_n})$$

is the upper hypolimit *of the functions* $\tau^\sharp_{K_n}$*, equal to*

$$\left(\lim{}^\sharp_{\downarrow n\rightarrow\infty}\tau^\sharp_{K_n}\right)(x_0) \ = \ \limsup_{n\rightarrow\infty,\, x_n \ \rightarrow_{K_n} \ x_0} \tau^\sharp_{K_n}(x_n)$$

In the same way, we define the upper epilimit $\lim^\sharp_{\uparrow n\rightarrow\infty}\varpi^\flat_{(K_n,C_n)}$ *whose epigraph is the upper limit of the epigraphs of the functions* $\varpi^\flat_{(K_n,C_n)}$

$$\mathcal{E}p(\lim{}^\sharp_{\uparrow n\rightarrow\infty}\varpi^\flat_{(K_n,C_n)}) \ := \ \text{Limsup}_{n\rightarrow\infty}\mathcal{E}p(\varpi^\flat_{(K_n,C_n)})$$

is the upper epilimit *of the functions* $\varpi^\flat_{(K_n,C_n)}$*, equal to*

$$\left(\lim{}^\sharp_{\uparrow n\rightarrow\infty}\varpi^\flat_{(K_n,C_n)}\right)(x_0) \ = \ \liminf_{n\rightarrow\infty,\, x_n \ \rightarrow_{K_n} \ x_0} \varpi^\flat_{(K_n,C_n)}(x_n)$$

We the prove this very useful stability result:

Theorem 6.1.26 *Let* $F : X \rightsquigarrow X$ *be a strict Marchaud map. Consider two sequences of subsets* $C_n \subset C$ *and* $K_n \subset X$ *and their Painlev-Kuratowski upper limits*

$$C^\sharp \ := \ \text{Limsup}_{n\rightarrow +\infty} C_n \ \& \ K^\sharp \ := \ \text{Limsup}_{n\rightarrow +\infty} K_n$$

Then

(a) the upper hypolimit of the upper exit functions of a sequence of subsets K_n *is smaller than or equal to the upper exit function of their upper limit:*

$$\left(\lim{}^\sharp_{\downarrow n\rightarrow\infty}\tau^\sharp_{K_n}\right)(x) \ \leq \ \tau^\sharp_{K^\sharp}(x)$$

(b) the upper epilimit of the lower constrained hitting functions of a sequence of subsets $C_n \subset K_n$ is larger than or equal to the lower constrained hitting function of their upper limit:

$$\left(\lim^{\sharp}_{\uparrow n\to\infty}\varpi^{\flat}_{(K_n,C_n)}\right)(x) \;\geq\; \varpi^{\flat}_{(K^\sharp,C^\sharp)}(x)$$

Proof — Let us begin by proving the first inequality, that can be translated in the form of the inclusion

$$\text{Limsup}_{n\to\infty}\mathcal{H}p(\tau^{\sharp}_{K_n}) \;\subset\; \mathcal{H}p(\tau^{\sharp}_{K^\sharp})$$

For that purpose, let us take a sequence $(T_n, x_n) \in \mathcal{H}p(\tau^{\sharp}_{K_n})$ converging to (T, x) and check that this limit belongs to the hypograph of $\tau^{\sharp}_{K^\sharp}$. By definition, there exists a solution $x_n(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at x_n such that, for every $t \in [0, T_n[$, $x_n(t)$ belongs to K_n. By Theorem 6.1.6, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some solution $x(\cdot)$ to the differential inclusion starting at x. Take $t < T$ and n large enough for having $t < T_n$. In this case, $x_n(t)$ belongs to K_n and passing to the limit, $x(t)$ belongs to $K^\sharp$. This implies that $T \leq \tau^{\sharp}_{K^\sharp}(x)$.

Let us prove now the second inequality, that can be translated in the form of the inclusion

$$\text{Limsup}_{n\to\infty}\mathcal{E}p(\varpi^{\flat}_{(K_n,C_n)}) \;\subset\; \mathcal{E}p(\varpi^{\flat}_{(K_n,C_n)})$$

For that purpose, let us take sequences $(T_n, x_n) \in \mathcal{E}p(\varpi^{\flat}_{(K_n,C_n)})$ converging to (T, x) and check that this limit belongs to the epigraph of $\varpi_{(K^\sharp,C^\sharp)}$.

For every $\varepsilon > 0$, there exist N such that for $n \geq N$, there exists a solution $x_n(\cdot) \in \mathcal{S}_F(x_n)$ and $t_n \leq T_n + \frac{\varepsilon}{2} \leq T + \varepsilon$ such that $x_n(t_n) \in C_n$ and for every $s < t_n$, $x_n(s) \in K_n$. By Theorem 6.1.6, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some solution $x(\cdot) \in \mathcal{S}_F(x)$. Let us consider also a subsequence (again denoted by) t_n converging to some $T^\star \leq T+\varepsilon$. By passing to the limit, we infer that $x(T^\star)$ belongs to $C^\sharp$ and that, for any $s < T^\star$, $x(s)$ belongs to $K^\sharp$. This implies that $\varpi^{\flat}_{(K^\sharp,C^\sharp)}(x) \leq T^\star \leq T+\varepsilon$. We conclude by letting ε converge to 0. $\square$

6.2 Viability Kernels and Capture Basins

We shall answer in this section questions such as:

- starting from K, is it possible to remain viable in K as long as possible,
- starting outside K, to reach as fast as possible the subset K being regarded in this case as a target.

These two very natural questions lead to the introduction of the following concepts: the viability kernels, the capture basins and the viable-capture basins of a subset under a set-valued map.

Definition 6.2.1 *Let $F : X \rightsquigarrow X$ be a set-valued map and $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set.*

(a) The subset $\text{Viab}_F(K)$ *of initial states $x_0 \in K$ such that one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is viable in K for all $t \geq 0$ is called the* viability kernel *of K under F.*

A subset K is a repeller *under F if its viability kernel is empty.*

(b) The subset $\text{Capt}_F^K(C)$ *of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is called the* viable-capture basin *of C in K and*

$$\text{Capt}_F(C) \;:=\; \text{Capt}_F^X(C)$$

is said to be the capture basin *of C.*

A subset $C \subset K$ is said to be isolated in K by F *if it coincides with its viable-capture basin in K:*

$$\text{Capt}_F^K(C) \;=\; C$$

(c) The subset

$$\text{Viab}_F(K, C) \;:=\; \text{Viab}_F(K) \cup \text{Capt}_F^K(C)$$

of initial states $x_0 \in K$ such that one solution $x(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the viability kernel of K with target C *under F.*

We observe that

$$\mathrm{Capt}_F(C) \;=\; \bigcup_{t\geq 0} \vartheta_{-F}(t,C) \;\subset\; K \tag{6.5}$$

The subset $\mathrm{Env}_F(C) := \mathrm{Capt}_{-F}(C)$ is known under various names such as invariance envelope or accessibility map or controlled map of C (See [179, Quincampoix] for properties of invariance envelopes under Lipschitz maps and [17, Aubin] for Marchaud maps).

Henri Poincar introduced the concept of shadow (in French, ombre) of K, which if the set of initial points of K from which (all) solutions leave K in finite time. It is thus equal to the complement $K\backslash\mathrm{Viab}_F(K)$ of the viability kernel of K, which has been introduced in the context of differential inclusions in [7, Aubin]. The concept of viability kernel with a target by a Lipschitz set-valued map has been introduced and studied in [180, Quincampoix & Veliov], where the viability kernel algorithm designed in [183, Saint-Pierre] (see also the survey [80, Cardaliaguet, Quincampoix & Saint-Pierre]) has been extended for approximating the viability kernel with a target.

One could regard the viability kernel $\mathrm{Viab}_F(K)$ of K as the viability kernel $\mathrm{Viab}_F(K,\emptyset)$ of K with target equal to the empty set:

$$\mathrm{Viab}_F(K) \;=\; \mathrm{Viab}_F(K,\emptyset) \;\;\&\;\; \mathrm{Capt}_F^K(\emptyset) \;=\; \emptyset$$

Naturally, the viability kernel $\mathrm{Viab}_F(K,C)$ of K with target C coincides with the capture basin $\mathrm{Capt}_F^K(C)$ of C viable in K whenever the viability kernel $\mathrm{Viab}_F(K)$ is contained in the viable-capture basin $\mathrm{Capt}_F^K(C)$: This is the case when K is a repeller, or, by Lemma 6.3.5 below, when the viability kernel $\mathrm{Viab}_F(K)$ of K is contained in the target C, or even more generally, when $K\backslash C$ is a repeller:

$$\mathrm{Viab}_F(K\backslash C) \;=\; \emptyset \;\Rightarrow\; \mathrm{Viab}_F(K,C) \;=\; \mathrm{Capt}_F^K(C)$$

Therefore, the concept of viability kernel with a target allows us to study both the viability kernel of a closed subset and the viable-capture basin of a target.

We observe that the viability kernel is characterized by

$$\mathrm{Viab}_F(K) \;:=\; \{x\in K \mid \tau_K^{\sharp}(x) \;=\; +\infty\}$$

and that the viable-capture basin

$$\mathrm{Capt}_F^K(C) \;:=\; \{x \in K \mid \varpi^\flat_{(K,C)}(x) \;<\; +\infty\}$$

is the domain of the constrained hitting function $\varpi_{(K,C)}$.

Definition 6.2.2 *The subset*

$$\mathrm{Entry}_F(K) \;:=\; \{x \in \partial K \mid \tau_K^\sharp(x) > 0\}$$

of the upper exit function is called the entry *of K under F and its complement in K*

$$\mathrm{Egress}_F(K) \;:=\; \{x \in \partial K \mid \tau_K^\sharp(x) = 0\}$$

is called the egress subset *of K.*

The entry $\mathrm{Entry}_F(K)$ of K under F is obviously the largest subset of K which is locally viable under F.

The egress set $\mathrm{Egress}_F(K)$ of K is contained in the boundary ∂K of K. It is the set of initial states of the boundary from which all solutions leave K instantaneously in the sense that for all solutions $x(\cdot) \in \mathcal{S}_F(x)$, there exists a sequence $t_n > 0$ converging to 0 such that

$$\forall\, n \geq 0, \;\; x(t_n) \;\notin\; K$$

Let us point out the following useful topological properties:

Proposition 6.2.3 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. If $M \subset X \backslash \mathrm{Viab}_F(K)$ is compact, there exists $T \geq 0$ such that, for every $x \in M$ and every solution $x(\cdot) \in \mathcal{S}_F(x)$, there exists $t \in [0, T]$ such that $x(t) \notin K$.*

Proof — Indeed, M being compact and the exit function being upper semicontinuous, then $\sup_{x\in M} \tau_K^\sharp(x)$ is finite because, for each $x \in M$, $\tau_K^\sharp(x)$ is finite. □

Proposition 6.2.4 *Let us assume that K is a compact and that $F : X \rightsquigarrow X$ is Marchaud.*

Then either the viability kernel of K is not empty or K is a repeller, and in this case, there exists $\overline{T} \in [0, +\infty[$ such that

(a) there exists a solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on the interval $[0, \overline{T}[$,

(b) for any $T > \overline{T}$, for any solution $x(\cdot) \in \mathcal{S}_F(x)$, there exist $t \in]\overline{T}, T]$ such that $x(t) \notin K$.

Proof — When K is a repeller, the exit function is finite. Being compact, $\overline{T} := \sup_{x\in K} \tau_K^\sharp(x)$ is thus finite and achieves its maximum at some $\bar{x}$. By Theorem 6.1.24, there exists a solution $\bar{x}(\cdot) \in \mathcal{S}_F(\bar{x})$ such that $\tau_K(\bar{x}(\cdot)) = \tau_K^\sharp(\bar{x}) = \overline{T}$. $\square$

In other words, when K is a compact repeller, the subset

$$\{x \in K \mid \tau_K^\sharp(x) \ = \ \overline{T}\}$$

can be regarded as the "viability core", so to speak, because it is the subset of initial states from which one solution enjoys the longest "life expectation" $\overline{T}$ in K. The viability kernel, when it is nonempty, is the viability core with infinite life expectation.

6.3 Characterization of Viable Kernels and Capture Basins

This section is devoted to properties and characterizations of viability kernels and viable-capture basins respectively, such as:

Theorem 6.3.1 *Let us assume that F is Marchaud and that K is closed. The viability kernel* $\text{Viab}_F(K)$ *of the subset K is*

(a) the **largest** *closed subset of K viable under F,*

(b) if K is assumed to be backward invariant under F, the **unique** *closed subset $D \subset K$ satisfying*

$$\begin{cases} i) & D \text{ is viable under } F \\ ii) & D \text{ is backward invariant under } F \\ iii) & K \backslash D \text{ is a repeller} \end{cases} \tag{6.6}$$

and

Theorem 6.3.2 *Let us assume that F is Marchaud and let C and K be two closed subsets such that $\mathrm{Viab}_F(K) \subset C \subset K$. Then the viable-capture basin $\mathrm{Capt}_F^K(C)$ of C is*

(a) the **largest** *closed subset D satisfying*

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D \backslash C \text{ is locally viable under } F \end{cases}$$

(b) if K is assumed to be backward invariant under F, the **unique** *closed subset D satisfying*

$$\begin{cases} i) & C \subset D \subset K \\ ii) & D \backslash C \text{ is locally viable under } F \\ iii) & D \text{ is backward invariant under } F \end{cases}$$

The uniqueness properties of the viability kernel and the viable-capture basins are obtained thanks to the Frankowska method[2] consisting in introducing (local) backward invariance together with (local) forward viability of subsets.

[2]Hélène Frankowska did point out in ([121, 122, Frankowska]) the backward invariance and local forward viability properties of the epigraph of the value function of an optimal control problem: She proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is invariant and backward viable under a (natural) auxiliary system. It allowed her to characterize the value functions as unique solutions of contingent inequalities, and, by duality, to obtain lower semicontinuous (or bilateral) solutions to Hamilton-Jacobi partial differential equations, obtained by other methods in [50, Barron & Jensen] (See also [42, Bardi & Capuzzo-Dolcetta] for more details on this point of view). Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant ([117, 118, 119, 120, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions in [89, Crandall and Lions P.-L.]. Such concepts have been extended to solutions of systems of first-order partial differential equations without boundary conditions by Hélène Frankowska and the author (see chapter 8 of [8, Aubin]).

Actually, we can spare the assumption that K is backward invariant in both these theorems if we are ready to trade the property that D is backward invariant with the weaker property that D is isolated in K by F in the sense that $\mathrm{Capt}_F^K(D) = D$. This property happens to be crucial. Indeed, we shall derive these theorems from Theorem 6.3.11 and 6.3.13 below.

6.3.1 Isolated Subsets by a Set-Valued Map

We first characterize the isolation of a target C in K by F:

Proposition 6.3.3 *Let D and K be two closed subsets such that $D \subset K$. Then the following properties are equivalent:*

(a) D is isolated in K by F: $Capt_F^K(D) = D$,

(b) for all $x \in K \backslash D$, no solution can reach D before (possibly) leaving K, or, equivalently, are viable in $K \backslash D$ before leaving K,

(c) property

$$\forall\, y \in D, \forall\, y(\cdot) \in \mathcal{S}_{-F}(x),\ \exists\ t_n \to 0+\ \mid y(t_n)\ \in\ D \cup (X \backslash K) \qquad (6.7)$$

holds true.

Proof — The equivalence between the two first statements is obvious. For proving the equivalence between the first and the third statements, we observe that to say that there exists $x \in \mathrm{Capt}_F^K(D) \backslash D$ amounts to saying that there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at x and $T := \varpi_D(x(\cdot)) \in]0, \infty[$ such that, for every $t \in [0, T[$, $x(t) \in K \backslash D$ and $x(T) \in D$, because D is closed. Let $z(\cdot) \in \mathcal{S}_{-F}(x)$ and

$$y(t)\ := \begin{cases} x(T-t) & \text{if}\ \ t \in [0, T] \\ z(t-T) & \text{if}\ \ t \geq T \end{cases}$$

Then this is equivalent to say that $y(\cdot) \in \mathcal{S}_{-F}(x(T))$ where $y(0) = x(T) \in D$ satisfies

$$\forall\, t \in]0, T],\ \ y(t)\ \in\ K \backslash D$$

i.e., that D satisfies

$$\exists\, y \in D,\ \exists\ \ y(\cdot) \in \mathcal{S}_{-F}(x),\ \exists\, T > 0 \mid \forall\, t \in [0,T[,\ \ y(t) \in K\backslash D \tag{6.8}$$

which is the negation of property (6.7). □

Lemma 6.3.4 below provides some properties that are needed later:

Lemma 6.3.4 *If $D \subset K$ is locally backward invariant under F, then D is isolated in K by F:* $\mathrm{Capt}_F^K(D) = D$.

Conversely, if D is isolated in K by F, then

(a) $D\backslash\partial K = D \cap \mathrm{Int}(K)$ *is locally backward invariant,*

(b) if K is itself locally backward invariant, then the closed subset D is locally backward invariant.

Proof — The first statement being obvious because local backward invariance of D implies property (6.7), we now posit this property (6.7).

Assume that $D\backslash\partial K$ is not locally backward invariant. This means that there exists a solution $y(\cdot) \in \mathcal{S}_{-F}(x)$ and a sequence $t_n > 0$ converging to 0 such that $y(t_n) \notin D \cap \mathrm{Int}(K)$. By (6.7), $y(t_n)$ belongs to $D \cup (X\backslash K)$, and thus, $y(t_n) \notin K$ for n large enough, a contradiction.

Assume now that D is not locally backward invariant. This means that there exists a solution $y(\cdot) \in \mathcal{S}_{-F}(x)$ and a sequence $t_n > 0$ converging to 0 such that $y(t_n) \notin D$. By (6.7), $y(t_n)$ belongs to $D \cup (X\backslash K)$, and thus, $y(t_n) \notin K$. Assuming further that K is locally backward invariant, we know that there exists $T > 0$ such that $y(t) \in K$ for every $t \in [0,T]$. Hence $y(t_n)$ belongs to K, still a contradiction. □

6.3.2 Repellers under a Set-Valued Map

We first observe that a subset K is a repeller if and only if all solutions to the differential inclusion $x' \in F(x)$ starting in K leave K in finite time.

We single out the following

Lemma 6.3.5 *Let $D \subset K$ and K be two subsets: If $K \backslash D$ is a repeller, then the viability kernel $\mathrm{Viab}_F(K)$ is contained in the viable-capture basin $\mathrm{Capt}_F^K(D)$:*

$$\mathrm{Viab}_F(K \backslash D) \; = \; \emptyset \; \Rightarrow \; \mathrm{Viab}_F(K) \; \subset \; \mathrm{Capt}_F^K(D) \; = \; \mathrm{Viab}_F(K, D)$$

Proof — Assume that $\mathrm{Viab}_F(K)$ is not contained in $\mathrm{Capt}_F^K(D)$ and let x belong to $\mathrm{Viab}_F(K) \backslash \mathrm{Capt}_F^K(D)$ and $x(\cdot)$ a solution to the differential inclusion $x' \in F(x)$ starting at x and viable in K. Since x does not belong to $\mathrm{Capt}_F^K(D)$, we deduce that $x(\cdot)$ is viable in $K \backslash D$. This contradicts the assumption that $K \backslash D$ is a repeller. □

6.3.3 Characterization of the Viability Kernel With a Target

Proposition 6.3.6 *Let us consider two subsets C and K of X satisfying $C \subset K$. The viability kernel $\mathrm{Viab}_F(K, C)$ of a subset K with target C under F satisfies the following properties:*

$$\left\{ \begin{array}{ll} i) & \mathrm{Viab}_F(K,C) \backslash C \text{ is locally viable under } F \\ ii) & \mathrm{Viab}_F(K,C) \text{ is isolated in } K \text{ by } F \; (\mathrm{Capt}_F^K(\mathrm{Viab}_F(K,C)) \; = \; \mathrm{Viab}_F(K,C)) \\ iii) & K \backslash \mathrm{Viab}_F(K,C) \text{ is a repeller } (\mathrm{Viab}_F(K \backslash \mathrm{Viab}_F(K,C)) \; = \; \emptyset) \end{array} \right.$$

Proof — For proving the first statement, take $x_0 \in \mathrm{Viab}_F(K, C) \backslash C$ and prove that there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at x_0 viable in $\mathrm{Viab}_F(K, C) \backslash C$ on a nonempty interval. Indeed, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ starting at x_0 viable in K forever or until a finite time $T > 0$ when $x(T) \in C$. Then for all $t \in [0, T[$, the function $y(\cdot)$ defined by $y(\tau) := x(t + \tau)$ is also a solution to the differential inclusion starting at $x(t)$, viable in K forever or until the finite time $T - t > 0$ when $y(T - t) = x(T) \in C$. Hence $x(t)$ does belong to $\mathrm{Viab}_F(K, C) \backslash C$ for every $t \in [0, T[$.

For proving that $\mathrm{Viab}_F(K, C)$ is isolated in K, assume that

$$\forall \, x_0 \; \in \; \mathrm{Capt}_F^K(\mathrm{Viab}_F(K,C)) \backslash \mathrm{Viab}_F(K,C)$$

and derive a contradiction. Starting from x_0, there exists a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ viable in K until it reaches the viability kernel $\mathrm{Viab}_F(K,C)$ of K at some finite time $T > 0$ at $x(T) \in \mathrm{Viab}_F(K,C)$.

Either $x(T)$ belongs to the viability kernel $\mathrm{Viab}_F(K)$, and we concatenate $x(\cdot)$ with one solution to the differential inclusion $x' \in F(x)$ starting at time T from $x(T)$ viable in K, which exists. Hence x_0 would belong to the viability kernel $\mathrm{Viab}_F(K)$, a contradiction.

Or $x(T)$ belongs to the viable-capture basin $\mathrm{Capt}_F^K(C)$, and we concatenate $x(\cdot)$ with one solution to the differential inclusion $x' \in F(x)$ starting at time T from $x(T)$ viable in K until it reaches C in finite time. Hence this concatenated solution is viable in K until it reaches C, and x_0 would belong to the viable-capture basin $\mathrm{Capt}_F^K(C)$, a contradiction.

Finally, the subset $K \backslash \mathrm{Viab}_F(K,C)$ is obviously a repeller: Every solution starting from $x \in K \backslash \mathrm{Viab}_F(K,C)$ must leave K in finite time by definition of the viability kernel of K with the target C. $\square$

We now prove that the viability kernel with a target is closed whenever F is Marchaud.

Theorem 6.3.7 *Assume that F is Marchaud and that $C \subset K$ are closed. Then the viability kernel* $\mathrm{Viab}_F(K,C)$ *of K with target C by F is closed.*

Proof — Let $x_n \in \mathrm{Viab}_F(K,C)$ converge to x and prove that x belongs to the viability kernel $\mathrm{Viab}_F(K,C)$ of K with target C. Since $C \subset \mathrm{Viab}_F(K,C)$, we need only to check this when x does not belong to C.

Assume that x does not belong to the viability kernel $\mathrm{Viab}_F(K,C)$ of K with target C and derive a contradiction. Hence x does neither belong to the viability kernel $\mathrm{Viab}_F(K)$ nor to the viable-capture basin $\mathrm{Capt}_F^K(C)$ of C. Since x does not belong to the viability kernel $\mathrm{Viab}_F(K)$ of K, there exists $\eta > 0$ such that the compact subset $B(x,\eta) \cap K$ is disjoint from the viability kernel $\mathrm{Viab}_F(K)$. Hence

$$\overline{T} \;:=\; \sup_{y \in B(x,\eta)\cap K} \tau_K^{\sharp}(y)$$

is finite, since the upper exit function is finite and upper semicontinuous on a compact set thanks to Theorem 6.1.24.

Therefore, for any n large enough,

$$\varpi^{\flat}_{(K,C)}(x_n) \;\leq\; \tau^{\sharp}_{K}(x_n) \;\leq\; \overline{T}$$

By the lower semicontinuity of the minimal hitting function $\varpi^{\flat}_{(K,C)}$ provided by Theorem 6.1.24, we infer that

$$\varpi^{\flat}_{(K,C)}(x) \;\leq\; \overline{T} \;<\; +\infty$$

so that there exists a solution $x(\cdot) \in \mathcal{S}_F(x)$ such that

$$\varpi_{(K,C)}(x(\cdot)) \;=\; \varpi^{\flat}_{(K,C)}(x) \;\leq\; \overline{T}$$

This implies that x belongs to the viable-capture basin $\mathrm{Capt}^{K}_{F}(C)$, the contradiction we were looking for. □

Theorem 6.3.8 *Assume that F is Marchaud. Let $C \subset D \subset K$ be closed subsets.*

Then the viability kernel $\mathrm{Viab}_F(K, C)$ of K with target C under F is the largest closed subset *$D \subset K$ such that $D \backslash C$ is locally viable.*

Proof — Theorem 6.3.7 and Proposition 6.3.6 imply that the viability kernel $\mathrm{Viab}_F(K, C)$ of K with target C under F is the a closed subset such that $\mathrm{Viab}_F(K, C) \backslash C$ is locally viable.

Let $D \subset K$ such that $D \backslash C$ is locally viable. Since $C \subset \mathrm{Capt}^{K}_{F}(C) \subset \mathrm{Viab}_F(K, C)$, let us take x in $D \backslash C$. Either x belongs to the viability kernel $\mathrm{Viab}_F(K)$, or every solution starting from x leaves K, and thus $D \backslash C$, in finite time, i.e., reaches C or leaves D ouside C in finite time. Assume that no solution reaches C in finite time and derive a contradiction. In this case, all solutions leave D in finite time, i.e., $\tau^{\sharp}_{D}(x)$ is finite. At least one of them, a solution $x^{\sharp}(\cdot) \in \mathcal{S}_F(x)$ maximizes $\tau_D(x(\cdot))$:

$$\tau^{\sharp}_{D}(x) \;:=\; \sup_{x(\cdot) \in \mathcal{S}_F(x)} \tau_D(x(\cdot)) \;=\; \tau_D(x^{\sharp}(\cdot))$$

Such a solution does exist by Theorem 6.1.24, since D is closed and F is Marchaud. Let us set $x^{\sharp} := x^{\sharp}(\tau^{\sharp}_{D}(x))$, which belongs to $D \backslash C$. Since $D \backslash C$ is locally viable, one could associate with $x^\sharp \in D\backslash C$ a solution $y(\cdot) \in \mathcal{S}_F(x^\sharp)$ and $T > 0$ such that $y(\tau) \in D\backslash C$ for all $\tau \in [0,T]$. Concatenating this solution to $x^\sharp(\cdot)$, we obtain a solution viable in D on an interval $[0, \tau_D^\sharp(x) + T]$, which contradicts the definition of $x^\sharp(\cdot)$.

Furthermore, $x^\sharp(\cdot)$ is viable in K on $[0, \tau_D^\sharp(x)]$ since $D \subset K$. This implies that $D \subset \mathrm{Capt}_F^K(C) \subset \mathrm{Viab}_F(K,C)$. □

Theorem 6.3.9 *Let $C \subset D$. Then the viability kernel* $\mathrm{Viab}_F(K,C)$ *is the* smallest *subset D between C and K isolated in K by F such that $K\backslash D$ is a repeller.*

Proof — Proposition 6.3.6 implies that the viability kernel $\mathrm{Viab}_F(K,C)$ is isolated in K by F and that $K\backslash D$ is a repeller.

Conversely, if $K\backslash D$ is a repeller, then the viability kernel $\mathrm{Viab}_F(K)$ is contained in the viable-capture basin $\mathrm{Capt}_F^K(D)$:

$$\mathrm{Viab}_F(K\backslash D) \;=\; \emptyset \;\Rightarrow\; \mathrm{Viab}_F(K) \;\subset\; \mathrm{Capt}_F^K(D)$$

Otherwise, let us assume that $\mathrm{Viab}_F(K)$ is not contained in $\mathrm{Capt}_F^K(D)$: There exists x belonging to $\mathrm{Viab}_F(K)\backslash\mathrm{Capt}_F^K(D)$ and $x(\cdot)$ a solution to the differential inclusion $x' \in F(x)$ starting at x and viable in K. Since x does not belong to $\mathrm{Capt}_F^K(D)$, it cannot reach D in finite, so that we deduce that $x(\cdot)$ is viable in $K\backslash D$. This contradicts the assumption that $K\backslash D$ is a repeller.

Since we assumed furthermore that D is isolated in K by F, we infer that

$$\mathrm{Viab}_F(K) \;\subset\; \mathrm{Capt}_F^K(D) \;=\; D$$

and thus, that $\mathrm{Viab}_F(K,C) \subset \mathrm{Viab}_F(K,D) = \mathrm{Viab}_F(K) \cup \mathrm{Capt}_F^K(D) = D$.
□

Theorems 6.3.8 and 6.3.9 imply the following characterization of viability kernels with targets:

Theorem 6.3.10 *Let us assume that F is Marchaud and that the subsets C and K are closed and satisfy $C \subset K$. The viability kernel* $\mathrm{Viab}_F(K,C)$

of a subset K with target C under F is the **unique** *closed subset satisfying $C \subset D \subset K$ and*

$$\begin{cases} i) & D\backslash C \text{ is locally viable under } F \\ ii) & D \text{ is isolated in } K \text{ by } F \ (\mathrm{Capt}_F^K(D) \ = \ D) \\ iii) & K\backslash D \text{ is a repeller } (\mathrm{Viab}_F(K\backslash D) \ = \ \emptyset) \end{cases} \tag{6.9}$$

6.3.4 Properties of Viability Kernels

Taking $C = \emptyset$, Theorem 6.3.1 follows from Lemma 6.3.4 and the following Theorem 6.3.11:

Theorem 6.3.11 *Let us assume that F is Marchaud and that K is closed. The viability kernel* $\mathrm{Viab}_F(K)$ *of a subset K is*

(a) the largest *closed subset of K viable under F*

(b) the **unique** *closed subset $D \subset K$ satisfying*

$$\begin{cases} i) & D \text{ is viable under } F \\ ii) & D \text{ is isolated in } K \text{ by } F \ (\mathrm{Capt}_F^K(D) \ = \ D) \\ iii) & K\backslash D \text{ is a repeller } (\mathrm{Viab}_F(K\backslash D) \ = \ \emptyset) \end{cases} \tag{6.10}$$

Proof — Indeed, if $C = \emptyset$, then $K := K\backslash\emptyset$ is locally viable is viable under F if and only if it is viable under F because the growth of the set-valued map F being linear, no solution explodes in finite time, so that a solution locally viable in K can be extended to a solution globally viable in K. □

6.3.5 Properties of Viable-Capture Basins

We point out the following obvious properties[3] of the viable-capture basins:

Proposition 6.3.12 *Let $C \subset K$ be closed subsets.*

[3]See [179, Quincampoix] for other properties in the framework of invariance envelopes of Lipschitz maps.

(a) *The capture basin* $\mathrm{Capt}_F(C)$ *is the smallest backward invariant containing* C *and* $\mathrm{Capt}_F(C)\backslash C$ *is locally viable. The capture basin of any union of subsets* C_i $(i \in I)$ *is the union of the capture basins of the* C_i.

(b) *The complement* $\mathrm{Capt}_F^K(C)\backslash C$ *of* C *in the viable-capture basin* $\mathrm{Capt}_F^K(C)$ *is locally viable and*

$$\mathrm{Capt}_F^K(C) \; = \; \mathrm{Capt}_F^{\mathrm{Capt}_F^K(C)}(C)$$

(c) *The viable-capture basin* $\mathrm{Capt}_F^K(C)$ *is isolated in* K *by* F*:*

$$\mathrm{Capt}_F^K(\mathrm{Capt}_F^K(C)) \; := \; \mathrm{Capt}_F^K(C)$$

i.e., for all $x \in K\backslash\mathrm{Capt}_F(K)$*, all solutions leave* K *before possibly reaching* $\mathrm{Capt}_F^K(C)$*, or, equivalently, are viable in* $K\backslash\mathrm{Capt}_F^K(C)$ *before leaving* K.

(d) *When* $C \subset K$ *where* K *is backward invariant, then the viable-capture basin and the capture basin of* C *coincide;*

$$\mathrm{Capt}_F^K(C) \; = \; \mathrm{Capt}_F(C) \; \subset \; K$$

Proof — The semi group property of Proposition 6.1.9 implies that the capture basin is the union of the backward reachable subsets:

$$\bigcup_{t\geq 0} \vartheta_{-F}(t,C)$$

and is backward invariant. If K is backward invariant, each backward reachable set $\vartheta_{-F}(t,C)$ is contained in K, so that

$$\mathrm{Capt}_F(C) \; = \; \bigcup_{t\geq 0} \vartheta_{-F}(t,C) \; \subset \; K$$

Since the intersection of backward invariant subsets is backward invariant, the capture basin is contained in the smallest backward invariant subset containing C.

If x belongs to $\mathrm{Capt}_F^K(C)\backslash C$, then there exists a solution to the differential inclusion $x' \in F(x)$ starting from x which reaches C before leaving K, and thus, which is viable in $\mathrm{Capt}_F^K(C)\backslash C$ on some nonempty interval.

Let us set $D := \mathrm{Capt}_F^K(C)$ and assume that there exists $x_0 \in \mathrm{Capt}_F^K(D)\backslash D$, and therefore, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ viable in K until it reaches $D := \mathrm{Capt}_F^K(C)$ in finite time T at $x(T) \in \mathrm{Capt}_F^K(C)$. We concatenate it with a solution starting at time T at $x(T)$ viable in K until it reaches C at finite time S. This means that x_0 belongs to $D := \mathrm{Capt}_F^K(C)$, a contradiction.

When $C \subset K$ where K is assumed to be backward invariant, then the capture basin $\mathrm{Capt}_F(C)$ is contained in K. □

Remark — Denote by $\mathcal{P}(X)$ and $\mathcal{F}(X)$ the families of subsets and closed subsets of X respectively. The map $C \in \mathcal{P}(X) \mapsto \mathrm{Capt}_F(C) \in \mathcal{P}(X)$ is increasing, extensive and idempotent. Such maps are called closings. Proposition 6.3.12 implies that the map $C \in \mathcal{P}(C) \mapsto \mathrm{Capt}_F(C) \in \mathcal{F}(X)$ is also the Galois transform of the map $C \mapsto \mathrm{Inv}_{-F}(C)$:

$$\mathrm{Capt}_F(C) \ := \ \bigcap_{\mathrm{Inv}_{-F}(M)\supset C} M \ := \ \text{Galois transform of } C \mapsto \mathrm{Inv}_{-F}(C)$$

If a power map $B : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ mapping closed subsets to closed subsets is either increasing or decreasing for the inclusion preorder, the (algebraic) Tarski Fixed-Point Theorem implies the existence of a fixed set $\widehat{C}$ such that

$$\widehat{C} \ = \ \mathrm{Capt}_F(B(\widehat{C}))$$

and a fixed set $\widetilde{C}$ such that

$$\widetilde{C} \ = \ B(\mathrm{Capt}_F(\widetilde{C}))$$

See Chapter 7 of [13, Aubin] for more details. □

Theorem 6.3.2 follows from Lemma 6.3.4 and the following Theorem 6.3.13:

Theorem 6.3.13 *Let us assume that F is Marchaud and that a closed subset $C \subset K$ satisfies property*

$$\mathrm{Viab}_F(K\backslash C) \ = \ \emptyset \tag{6.11}$$

Then the viable-capture basin $\mathrm{Capt}_F^K(C)$ *is the* **unique** *closed subset D satisfying $C \subset D \subset K$ and*

$$\begin{cases} i) & D\backslash C \text{ is locally viable under } F \\ ii) & D \text{ is isolated in } K \text{ by } F \ (\mathrm{Capt}_F^K(D) \ = \ D) \end{cases} \tag{6.12}$$

6.4 The Barrier Property

The boundary of the viability kernel satisfies the barrier property:

Definition 6.4.1 *If $D \subset K$, the* boundary $\partial_K(D)$ of D relative to K *is the subset*

$$\partial_K(D) \ := \ \overline{D} \cap \overline{(K \backslash D)}$$

and the subset $\partial D := \partial_X(D)$ is called the boundary of D. *We shall say that a subset $D \subset K$ enjoys the* barrier property relative to K under F *if for every $x \in \partial_K(D)$, all solutions starting from x viable in D are actually viable in the boundary $\partial_K(D)$ of D relative to K until they reach the boundary of K. It enjoys the (global)* barrier property *if all solutions starting from the boundary of D viable in D until they leave K are actually viable in the boundary of D until they leave K.*

We see at once that

$$\partial_K(D) \cap \mathrm{Int}(K) \ = \ \partial D \cap \mathrm{Int}(K)$$

and that

$$\text{if } D \ \subset \ \mathrm{Int}(K), \ \text{ then } \partial_K(D) \ = \ \partial D$$

Remark on the barrier property: — The "barrier property" of the viability kernel of a closed subset has been discovered by Marc Quincampoix in [178, Quincampoix] and generalized by Pierre Cardaliaguet in [79, Cardaliaguet]. It plays an important role in control theory and the theory of differential games, because every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or equivalently, no solution starting from outside the viability kernel can cross its boundary: such solutions can only remain on the boundary of the viability kernel, or leave it.

This is a semi-permeability property of the viability kernel, which is very important in terms of interpretation. Viability is indeed a very fragile property, which cannot be reestablished from the outside: In other words, love it or leave it ... □

Theorem 6.4.2 *If F is Marchaud and Lipschitz, then the viability kernel* $\text{Viab}_F(K, C)$ *of a closed subset K with a closed target $C \subset K$ under F enjoys the barrier property relative to K.*

Proof — Let x belong to $\partial_K(\text{Viab}_F(K, C)$ and $x(\cdot) \in \mathcal{S}_F(x)$ be a solution viable in K forever ($\varpi^\sharp_{(K,C)}(x(\cdot)) = +\infty$) or until it reaches C at finite time $\varpi^\sharp_{(K,C)}(x(\cdot)) < +\infty$. Let $x_n \in K\backslash\text{Viab}_F(K, C)$ converge to x. By the Filippov Theorem 6.1.11, there exists a solution $x_n(\cdot) \in \mathcal{S}_F(x_n)$ such that

$$\forall\, t \geq 0, \;\; \|x(t) - x_n(t)\| \;\leq\; e^{\lambda t}\|x - x_n\|$$

where λ is the Lipschitz constant of F. Hence the solutions $x_n(\cdot)$ converge to the solution $x(\cdot)$. Since $\text{Viab}_F(K, C)$ is isolated, we know that for every n,

$$\forall\, t \leq \tau_K(x_n(\cdot)), \;\; x_n(t) \in K\backslash\text{Viab}_F(K, C)$$

Since $\varpi_{\partial K}(x_n(\cdot)) \;\leq\; \tau_K(x_n(\cdot))$ and since the functional $x(\cdot) \mapsto \varpi_{\partial K}(x(\cdot))$ is lower semicontinuous, we infer that for every $t < \varpi_{\partial K}(x(\cdot))$ there exists $N > 0$ such that for any $n \geq N$,

$$t < \varpi_{\partial K}(x_n(\cdot)) \;\leq\; \tau_K(x_n(\cdot))$$

and thus, that $x_n(t)$ belongs to $K\backslash\text{Viab}_F(K, C)$. Taking the limit, we infer that $x(t)$ belongs to $\overline{K\backslash\text{Viab}_F(K, C)}$. Hence $x(t)$ belongs to the boundary $\partial_K(\text{Viab}_F(K, C))$ relative to K whenever $t < \varpi_{\partial K}(x(\cdot))$. $\square$

Taking $C = \emptyset$, we obtain the original Quincampoix Theorem:

Theorem 6.4.3 (Quincampoix) *If F is Marchaud and Lipschitz, then the viability kernel* $\text{Viab}_F(K)$ *enjoys the barrier property relative to K.*

In particular, if $\text{Viab}_F(K) \subset \text{Int}(K)$*, then the boundary of the viability kernel enjoys the barrier property.*

6.5 Viability Kernels of Backward Invariant Sets

We obtain further properties when K is backward invariant under F. To begin with, Proposition 6.3.12 implies that the capture basin $\text{Capt}_F(C) =$

$\bigcup_{T\geq 0}\vartheta_{-F}(T,C)$ — being the smallest backward invariant containing C — is contained in K and equal to $\text{Capt}_F^K(C)$, so that

$$\text{Viab}_F(K,C) \;=\; \text{Viab}_F(K)\cup\text{Capt}_F(C)$$

Let us mention another useful property:

Theorem 6.5.1 *Assume that F is Marchaud and that $C\subset K$ and K are closed and that K is (globally) backward invariant under F. Then the viability kernel* $\text{Viab}_F(K,C)$ *of K with target C under F is the* **unique** *closed subset D satisfying $C\subset D\subset K$ and*

$$\begin{cases} i) & D\backslash C \text{ is locally viable under } F \\ ii) & D \text{ is backward invariant under } F \text{ (or, equivalently, } X\backslash D \text{ is invariant under } F) \\ iii) & K\backslash D \text{ is a repeller under } F \end{cases}$$

Furthermore, if F is also Lipschitz, then the viability kernel $\text{Viab}_F(K,C)$ *of K with target C under F enjoys the* barrier property*.*

Proof — To say that K is backward invariant amounts to saying that the complement of K is invariant thanks to Theorem 6.1.20. Therefore, $\text{Viab}_F(K,C)$ being isolated and $K\backslash\text{Viab}_F(K,C)$ being a repeller, all solutions starting from $K\backslash\text{Viab}_F(K,C)$ leave K in finite time before possibly hitting C. Actually, they never reach C because the complement $X\backslash K$ is invariant. Hence we have checked that the complement $X\backslash\text{Viab}_F(K,C)$ of the viability kernel of K with target C is invariant. Theorem 6.1.20 implies that the viability kernel $\text{Viab}_F(K,C)$ of K with target C is backward invariant.

When F is assumed to be Lipschitz, we deduce from the invariance of $X\backslash\text{Viab}_F(K,C)$ and from Proposition 6.1.16 that the closure $\overline{X\backslash\text{Viab}_F(K,C)}$ is invariant. Therefore, any solution starting from

$$x \;\in\; \partial(\text{Viab}_F(K,C)) = \text{Viab}_F(K,C)\cap\overline{X\backslash\text{Viab}_F(K,C)}$$

viable in $\text{Viab}_F(K,C)$ on the interval $[0,\tau_K(x(\cdot))[$ is also viable in $\overline{X\backslash\text{Viab}_F(K,C)}$, and thus, on the boundary $\partial(\text{Viab}_F(K),C)$ of the viability kernel. □

When $C=\emptyset$, we deduce that the viability kernel of a closed backward invariant subset is both viable and backward invariant:

Theorem 6.5.2 *Assume that F is Marchaud and that a closed subset $C \subset X$ is closed and (globally) backward invariant under F. Then the viability kernel* $\text{Viab}_F(K)$ *of K under F is the* **unique** *closed subset $D \subset K$ satisfying*

$$\begin{cases} i) & D \text{ is viable under } F \\ ii) & D \text{ is backward invariant under } F \text{ (or, equivalently, } X \backslash D \text{ is invariant under } F) \\ iii) & K \backslash D \text{ is a repeller under } F \end{cases}$$

Furthermore, if F is also Lipschitz, then the viability kernel $\text{Viab}_F(K, C)$ *of K with target C under F enjoys the globa* barrier property.

When $K \backslash C$ is a repeller, we obtain the following consequence:

Theorem 6.5.3 *Let us assume that F is Marchaud, that the closed subsets $C \subset K$ and K are closed, that* $\text{Viab}_F(K \backslash C) = \emptyset$ *and that K is backward invariant under F.*

Then the capture basin $\text{Capt}_F(C)$ *is equal to* $\bigcup_{T \geq 0} \vartheta_{-F}(T, C)$ *and is the* **unique** *closed subset D which satisfying $C \subset D \subset K$ and*

$$\begin{cases} i) & D \backslash C \text{ is locally viable under } F \\ ii) & D \text{ is backward invariant under } F \text{ (or, equivalently, } X \backslash D \text{ is invariant under } F) \end{cases}$$

We now mention this useful result:

Proposition 6.5.4 (Cardaliaguet) *Assume that F is Marchaud and K is a closed subset. The limit-sets of the backward solutions are contained in the viability kernel of K.*

Proof — Let us consider any backward solution $y(\cdot) \in \mathcal{S}_{-F}(y_0)$ and let us choose any cluster point $y_\star := \lim_{n \to +\infty} y(t_n)$ of $y(\cdot)$. Therefore, $x_n(t) = y(t_n - t)$ is a forward solution starting at $x_n(0) = y(t_n)$ and satisfying $x_n(t_n) = y_0$. By Theorem 6.1.6, a subsequence (again denoted by) $x_n(\cdot)$ converges to a solution $x(\cdot) \in \mathcal{S}_F(y_\star)$. Since for every n such that $t_n \geq t$, $x_n(t) = y(t_n - t)$ belongs to K assumed to be backward invariant, we deduce that for every $t \geq 0$, $x(t)$ belongs to K. Hence $y_\star$ belongs to the viability kernel of K. □

In particular, we infer that a compact backward invariant subset has a nonempty viability kernel:

Corollary 6.5.5 *Assume that F is Marchaud and K is a compact subset backward invariant under F. Then the viability kernel* $\text{Viab}_F(K)$ *is nonempty and both viable and backward invariant.*

Let us point out this other consequence useful for "localizing" the viability kernel:

Corollary 6.5.6 *Assume that F is Marchaud, that K is a closed locally backward viable subset and that $C \subset K$ satisfies*

(a) C is closed and backward invariant under F,

(b) $K\backslash C$ is a repeller under F.

Then the viability kernels $\text{Viab}_F(K)$ *and* $\text{Viab}_F(C)$ *of K and C coincide.*

Proof — The viability kernel $\text{Viab}_F(C)$ is viable and backward invariant by Proposition 6.5.2. Its complement $K\backslash\text{Viab}_F(C)$ is a repeller, because both $C\backslash\text{Viab}_F(C)$ and $K\backslash C$ are repellers. Since K is assumed backward locally viable, Theorem 6.3.1 implies that $\text{Viab}_F(C) = \text{Viab}_F(K)$ is the viability kernel of K. □

When K is not backward invariant, we can consider its backward invariance kernel and its viability kernel:

Proposition 6.5.7 *Let K be a subset. Then* $\text{Viab}_F(\text{Inv}_{-F}(K))$ *is the largest subset viable and backward invariant contained in K.*

Proof — First, we know that $M := \text{Viab}_F(\text{Inv}_{-F}(K))$ is backward invariant by Theorem 6.5.2.

Consider now any subset $C \subset K$ viable and backward invariant under F. Since C is backward invariant under F, then $C \subset \text{Inv}_{-F}(K)$. Since C is a viable subset of $\text{Inv}_{-F}(K)$, it is contained in its viability kernel $\text{Viab}_F(\text{Inv}_{-F}(K)$. □

6.6 Frankowska's and Viscosity Property of Viability Kernels

Using the Viability Theorem 6.1.6 and the Invariance Theorem 6.1.15, we deduce that the viability kernels and the viable-capture basins enjoy tangential conditions.

To begin with, property (6.7) allows us to deduce the following tangential characterization of isolated subset by a set-valued map:

Theorem 6.6.1 *Let us assume that F is Lipschitz. Then a closed subset $D \subset K$ is isolated in K by F in the sense that $D = \text{Capt}_F^K(D)$ if and only if*

$$\begin{cases} i) & \forall\, x \in D \cap \text{Int}(K), \ \ -F(x) \ \subset \ T_D(x) \\ ii) & \forall\, x \in D \cap \partial K, \ \ -F(x) \ \subset \ T_D(x) \cup T_{X\backslash K}(x) \end{cases} \tag{6.13}$$

or, equivalently in normal form, if and only if

$$\begin{cases} i) & \forall\, x \in D \cap \text{Int}(K), \ \ \forall\, p \in N_D(x), \ \ \sigma(F(x), -p) \ \leq \ 0 \\ ii) & \forall\, x \in D \cap \partial K, \ \ \forall\, p \in N_D(x) \cap N_{\overline{X\backslash K}}(x), \ \ \sigma(F(x), -p) \ \leq \ 0 \end{cases} \tag{6.14}$$

We thus introduce the following Frankowska property:

Definition 6.6.2 *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K. We shall say that they satisfy the* Frankowska property *if*

$$\begin{cases} i) & \forall\, x \in D\backslash C, \ \ F(x) \cap T_D(x) \ \neq \ \emptyset \\ ii) & \forall\, x \in D \cap \text{Int}(K), \ \ -F(x) \ \subset \ T_D(x) \\ iii) & \forall\, x \in D \cap \partial K, \ \ -F(x) \ \subset \ T_D(x) \cup T_{X\backslash K}(x) \end{cases} \tag{6.15}$$

or, equivalently, by duality, satisfying the "normal conditions"

$$\begin{cases} i) & \forall\, x \in D\backslash C, \forall\, p \in N_D(x), \ \ \sigma(F(x), -p) \ \geq \ 0 \\ i) & \forall\, x \in D \cap \text{Int}(K), \ \ \forall\, p \in N_D(x), \ \ \sigma(F(x), -p) \ \leq \ 0 \\ ii) & \forall\, x \in D \cap \partial K, \ \ \forall\, p \in N_D(x) \cap N_{\overline{X\backslash K}}(x), \ \ \sigma(F(x), -p) \ \leq \ 0 \end{cases} \tag{6.16}$$

When K is assumed further to be backward locally invariant, the above conditions (6.15) and (6.16) boil down to

$$\begin{cases} i) & \forall\, x \in D\backslash C, \ \ F(x) \cap T_D(x) \ \neq \ \emptyset \\ ii) & \forall\, x \in D, \ \ -F(x) \subset T_D(x) \end{cases} \tag{6.17}$$

and

$$\begin{cases} i) & \forall\, x \in D\backslash C,\ \forall\, p \in N_D(x), \quad \sigma(F(x), -p) \;=\; 0 \\ ii) & \forall\, x \in C,\ \forall\, p \in N_D(x), \quad \sigma(F(x), -p) \;\leq\; 0 \end{cases} \tag{6.18}$$

respectively.

The equivalence between the tangential and normal formulations follows from Theorems 6.1.20 and6.1.18.

Theorem 6.6.3 *Let us assume that F is Marchaud and that $C \subset K$ and K are closed. The viability kernel $\mathrm{Viab}_F(K, C)$ of the subset K with target C under F is*

(a) the **largest** *closed subset D of K satisfying*

$$\forall\, x \in D\backslash C, \quad F(x) \cap T_D(x) \;\neq\; \emptyset$$

(b) when F is assumed to be also Lipschitz, the viability kernel $\mathrm{Viab}_F(K, C)$ is the **unique** *closed subset $D \subset K$ satisfying*

i. the Frankowska property (6.15) (or its dual formulation (6.16)),

ii. $K\backslash D$ is a repeller.

As a consequence, we obtain the following tangential characterization of viable-capture basins:

Theorem 6.6.4 *Let us assume that F is Marchaud, that K is closed and that a closed subset C satisfies $\mathrm{Viab}_F(K\backslash C) = \emptyset$. Then the viable-capture basin $\mathrm{Capt}_F^K(C)$ is*

(a) the **largest** *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall\, x \in D\backslash C, \quad F(x) \cap T_D(x) \;\neq\; \emptyset \tag{6.19}$$

(b) if F is Lipschitz, the **unique** *closed subset D satisfying the Frankowska property (6.15) (or its dual formulation (6.16)).*

We now define the following viscosity property:

Definition 6.6.5 *Let us consider a set-valued map* $F : X \rightsquigarrow X$ *and two subsets* $C \subset K$ *and* K*. We shall say that they satisfy the* viscosity property *if*

$$\left\{ \begin{array}{ll} i) & \forall\, x \in D \backslash C, \;\; F(x) \cap T_D(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in \overline{X \backslash D}, \;\; F(x) \subset T_{\overline{X \backslash D}}(x) \end{array} \right. \tag{6.20}$$

and, in normal form,

$$\left\{ \begin{array}{ll} i) & \forall\, x \in D \backslash C, \; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \;\geq\; 0 \\ ii) & \forall\, x \in \overline{X \backslash D}, \; \forall\, p \in N_{\overline{X \backslash D}}(x), \;\; \sigma(F(x), p) \;\leq\; 0 \end{array} \right. \tag{6.21}$$

respectively.

When $C = \emptyset$, we recognize the definition of a discriminating kernel of K of the hamiltionian $H(x, p) := \sigma(F(x), -p)$ given in [79, Cardaliaguet].

Theorem 6.6.6 *Let us assume that* F *is Marchaud and Lipschitz and that* $C \subset K$ *and* K *are closed and that* K *is backward invariant. The viability kernel* $\mathrm{Viab}_F(K, C)$ *of the subset* K *with the target* C *under* F *is the* **unique** *closed subset* $D \subset K$ *satisfying the*

(a) the viscosity property (6.20) (or its dual formulation (6.21)),

(b) $K \backslash D$ *is a repeller.*

6.7 Stability Properties

Let us point out the following stability properties of the viability kernels and the viable-capture basins:

Theorem 6.7.1 *Let us consider a sequence of closed subsets* $K_n \subset K$ *and* $C_n \subset K_n$*. If the set-valued map* F *is Marchaud, then*

$$\mathrm{Limsup}_{n \to +\infty} \mathrm{Viab}_F(K_n, C_n) \;\subset\; \mathrm{Viab}_F(\mathrm{Limsup}_{n \to +\infty} K_n, \mathrm{Limsup}_{n \to +\infty} C_n) \tag{6.22}$$

If we assume furthermore that for all $n \geq 0$, $\mathrm{Viab}_F(K) \subset C_n$*, then*

$$\mathrm{Limsup}_{n \to +\infty} \mathrm{Capt}_F^{K_n}(C_n) \;\subset\; \mathrm{Capt}_F^{\mathrm{Limsup}_{n \to +\infty} K_n}(\mathrm{Limsup}_{n \to +\infty} C_n) \tag{6.23}$$

Proof — Let us set

$$C^\sharp \ := \ \mathrm{Limsup}_{n\to+\infty} C_n \ \& \ K^\sharp \ := \ \mathrm{Limsup}_{n\to+\infty} K_n$$

For proving that

$$\mathrm{Limsup}_{n\to+\infty}\mathrm{Viab}_F(K_n,C_n) \ \subset \ \mathrm{Viab}_F(\mathrm{Limsup}_{n\to+\infty}C_n, \mathrm{Limsup}_{n\to+\infty}K_n)$$

let us consider the limit $x := \lim_{n\to+\infty} x_n \in K^\sharp$ of elements x_n of

$$\mathrm{Viab}_F(K_n,C_n) \ := \ \mathrm{Viab}_F(K_n) \cup \mathrm{Capt}_F^{K_n}(C_n)$$

An infinite subsequence (again denoted by) x_n must belong to $\mathrm{Viab}_F(K_n)$ or to $\mathrm{Capt}_F^{K_n}(C_n)$. Let $x_n(\cdot) \in \mathcal{S}_F(x_n)$ be a solution to the differential inclusion $x' \in F(x)$ starting from x_n and

- viable in K_n forever
- viable in K_n until it reaches C_n at finite time t_n.

Since F is Marchaud, Theorem 6.1.6 implies that a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}_F(x)$ uniformly on compact intervals.

In the first case, we infer that x belongs to the viability kernel of the upper limit $K^\sharp$ since for every $t \geq 0$, $x_n(t) \in K_n$.

In the second case, consider the case when this limit x does not belong to the viability kernel $\mathrm{Viab}_F(K^\sharp)$ of the upper limit $K^\sharp$. This means that $\tau^\sharp_{K^\sharp}(x)$ is finite. Since $x_n(t_n)$ belongs to C_n and for every $s \in [0,t_n]$, $x_n(s)$ belongs to K_n, we deduce that $t_n \leq \tau^\sharp_{K_n}(x_n)$. On the other hand, since F is Marchaud, Theorem 6.1.26 implies that

$$\limsup_{n\to+\infty, x_n \to_{K_n} x} \tau^\sharp_{K_n}(x_n) \ \leq \ \tau^\sharp_{K^\sharp}(x)$$

and thus, for n large enough

$$t_n \ \leq \ \tau^\sharp_{K_n}(x_n) \ \leq \ \tau^\sharp_{K^\sharp}(x) + 1 \ < \ +\infty$$

Therefore, a subsequence (again denoted by) t_n converges to some $t \leq T$. Hence $x_n(t_n) \in C_n$ converges to $x(t)$, which thus belongs to the upper limit $C^\sharp$.

Furthermore, for every $s \in [0,t[$, there exists N such that $t_n \in [s,t[$ whenever $n \geq N$. Since $x_n(s)$ belongs to K_n, we infer that $x(s) \in K^\sharp$ for every $s \in [0,t[$. This implies that

$$t \;\geq\; \varpi_{(K^\sharp,C^\sharp)}(x(\cdot)) \;\geq\; \varpi^\flat_{(K^\sharp,C^\sharp)}(x)$$

Hence x belongs to the capture basin of $C^\sharp$ viable in $K^\sharp$. $\square$

Taking $C_n = \emptyset$, we obtain the following consequence:

Theorem 6.7.2 *Let F be a Marchaud set-valued map and K_n be a sequence of closed subsets. Then*

$$\mathrm{Limsup}_{n\to+\infty}\mathrm{Viab}_F(K_n) \;\subset\; \mathrm{Viab}_F(\mathrm{Limsup}_{n\to+\infty}K_n)$$

For the lower Painlev-Kuratowski limits, we obtain the following stability property of capture basins:

Theorem 6.7.3 *If the set-valued map F is Lipschitz, then for any sequence of closed subsets C_n,*

$$\mathrm{Capt}_F(\mathrm{Liminf}_{n\to+\infty}C_n) \;\subset\; \mathrm{Liminf}_{n\to+\infty}\mathrm{Capt}_F(C_n) \tag{6.24}$$

Proof — For proving that

$$\mathrm{Capt}_F(\mathrm{Liminf}_{n\to+\infty}C_n) \;\subset\; \mathrm{Liminf}_{n\to+\infty}\mathrm{Capt}_F(C_n)$$

let $C^\flat$ denote the lower limit of the subsets C_n. Let us take $x \in \mathrm{Capt}_F(C^\flat)$ and a solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K until it reaches the target $C^\flat$ at time $T < +\infty$ at $c := x(T) \in C^\flat$. Hence the function $t \mapsto y(t) := x(T-t)$ is a solution $y(\cdot) \in \mathcal{S}_{-F}(c)$. Let us consider a sequence of elements $c_n \in C_n$ converging to c.

The Filippov Theorem states that there exist solutions $y_n(\cdot) \in \mathcal{S}_{-F}(c_n)$ such that

$$\|y(t) - y_n(t)\| \;\leq\; e^{\lambda t}\|c - c_n\|$$

where λ is the Lipschitz constant of F. Therefore $x_n := y_n(T)$ converges to x. It is enough to observe that x_n belongs to $\mathrm{Capt}_F(C_n)$ to conclude. $\square$

As a consequence, we obtain the following

Theorem 6.7.4 *Let us consider a sequence of closed subsets* C_n *satisfying* $\mathrm{Viab}_F(K) \subset C_n \subset K$ *and*

$$\mathrm{Lim}_{n\to+\infty} C_n \ := \ \mathrm{Limsup}_{n\to+\infty} C_n \ = \ \mathrm{Liminf}_{n\to+\infty}$$

If the set-valued map F *is Marchaud and if* K *is closed and backward invariant under* F*, then*

$$\mathrm{Lim}_{n\to+\infty} \mathrm{Capt}_F^K(C_n) \ = \ \mathrm{Capt}_F^K(\mathrm{Lim}_{n\to+\infty} C_n) \tag{6.25}$$

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