IX IMGTA
ISCHIA

Le 14 Juillet 2001

Dynamic Core

of Fuzzy Dynamical Cooperative Games

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Presentation

We use in this talk the viability/capturability approach for studying the problem of characterizing the dynamic core of a dynamic cooperative game defined in a characteristic function form.

In order to allow coalitions to evolve, we embed them in the set of fuzzy coalitions.

Hence, we define the dynamic core as a set-valued map associating with each fuzzy coalition and each time the set of allotments such that their payoffs at that time to the fuzzy coalition are larger than or equal to the one assigned by the characteristic function of the game.

Answers:

We shall characterize this core through the (generalized) derivatives of a valuation function associated with the game: We shall

1. **provide its explicit formula,**
2. **characterize its epigraph as a viable-capture basin of the epigraph of the characteristic function of the fuzzy dynamical cooperative game,**
3. **use the tangential properties of such basins for proving that the valuation function is a solution to a Hamilton-Jacobi-Isaacs partial differential equation**
4. **and use this function and its derivatives for characterizing the dynamic core.**

Dynamic Cooperative Game Theory

is a recent line of research opened by Leon Petrosjan, Alain Haurie and others:

Filar & Petrosjan (2000) wrote :

"Bulk of the literature dealing with cooperative games (in characteristic function form) do not address issues related to the evolution of a solution concept over time. However, most conflict situations are not "one shot" games but continue over some time horizon which may be limited *a priori* by the game rules, or terminate when some specified conditions are attained."

Here, we deal also with the evolution of coalitions.

Fuzzy Coalitions

We embed the family of subsets of a (discrete) set N of n players to the space R^n through the map χ associating with any coalition $A \in \mathcal{P}(N)$ its characteristic function $\chi_A \in \{0,1\}^n \subset R^n$.

The family of fuzzy sets is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in R^n.

Player i participates fully in x if $x_i = 1$, does not participate at all if $x_i = 0$ and participates in a fuzzy way if $x_i \in]0,1[$.

Remark: Probabilities

This canonical embedding is more adapted to the nature of the power set $\mathcal{P}(N)$ than the universal embedding of a discrete set M of m elements to R^m by the Dirac measure associating with any $j \in M$ the jth element of the canonical basis of R^m. The convex hull of the image of M by this embedding is the probability simplex of R^m.

L.A. Zadeh, 1965

introduced fuzzy sets. Since then, it has been wildly successful, even in many areas outside mathematics!:

Quotation of the late François Mitterand, president of the French Republic (1981-1995):

"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...

(Today, we swim in the pure poetry of fuzzy subsets)!

In "La lutte finale", Michel Lafon (1994), p.69 by A. Bercoff.

Toll Sets

We may replace the characteristic functions of sets by indicators taking their values in $[0, +\infty]$ and take their convex combinations to provide an alternative allowing us to speak of "toll-sets": They are nonnegative cost functions assigning to each element its cost of belonging, $+\infty$ if it does not belong to the toll set. The set of elements with finite positive cost do form the "fuzzy boundary" of the toll set, the set of elements with zero cost its "core". This has been done to adapt viability theory to "fuzzy viability theory".

Actually, the Cramer transform

$$C_\mu(p) \ := \ \sup_{x \in R^n} \left(\langle p, x \rangle - \log \left(\int_{R^n} e^{\langle x, y \rangle} d\mu(y) \right) \right)$$

maps probability measures to toll sets. In particular, it transforms convolution products of density functions to inf-convolutions of extended functions, Gaussians to squares of norms, etc.

(Fuzzy) static cooperative games

with side-payments are described by a characteristic function U assigning to each fuzzy coalition $x \in [0,1]^n$ a lower bound $U(x)$ to gains or payoffs $y := \langle p, x\rangle$ associated with an allotment $p \in R^n_+$.

The concepts of Shapley value and core coincide with the (generalized) gradient $\partial U(x_N)$ of the "characteristic function" $U : [0,1]^n \mapsto R_+$ at the "grand coalition" $x_N := (1,\ldots,1)$, the characteristic function of $N := \{1,2,\ldots,n\}$.

For instance, when the characteristic function of the static cooperative game U is concave, positively homogeneous and continuous on the interior of R_+^n, one check that the generalized gradient $\partial U(x_N)$ is not empty and coincides with the subset of allotments $p := (p_1, \ldots, p_n) \in R_+^n$ accepted by all fuzzy coalitions in the sense that

$$\forall \, x \in [0,1]^n, \;\; \langle p, x \rangle \; = \; \sum_{i=1}^{n} p_i x_i \; \geq \; U(x) \tag{1}$$

and that, for the grand coalition $x_N := (1, \ldots, 1)$,

$$\langle p, x_N \rangle \; = \; \sum_{i=1}^{n} p_i \; = \; U(x_N)$$

state that for any evolution $t \mapsto x(t)$ of fuzzy coalitions, the payoff $y(t) := \langle p(t), x(t) \rangle$ should satisfy one of the (at least) three following rules:

1. **at a prescribed final time T of the end of the game:**

$$y(T) \;:=\; \sum_{i=1}^{n} p_i(T) x_i(T) \;\geq\; U(x(T))$$

2. **during the whole time span of the game:**

$$\forall\, t \in [0, T],\;\; y(t) \;:=\; \sum_{i=1}^{n} p_i(t) x_i(t) \;\geq\; U(x(t))$$

3. **at the first winning time $t^* \in [0, T]$ when**

$$y(t^*) \;:=\; \sum_{i=1}^{n} p_i(t^*) x_i(t^*) \;\geq\; U(x(t^*))$$

   at which time the game stops.

The Question Raised

The above conditions require to find — for each of the above three rules of the game — an evolution of an allotment $p(t) \in R^n$ such that, for all evolutions of fuzzy coalitions $x(t) \in [0,1]^n$ starting at x, the corresponding rule of the game

$$\begin{cases} i) \quad \sum_{i=1}^{n} p_i(T)x_i(T) \ \geq \ U(x(T)) \\ ii) \quad \forall\, t \in [0,T], \ \ \sum_{i=1}^{n} p_i(t)x_i(t) \ \geq \ U(x(t)) \\ iii) \quad \exists\, t^\star \in [0,T] \ \ \textbf{such that} \\ \qquad \sum_{i=1}^{n} p_i(t^\star)x_i(t^\star) \ \geq \ U(x(t^\star)) \end{cases} \tag{2}$$

must be satisfied

Formal Definition of the Dynamical Core

Therefore, for each one of the above three rules of the game (2), a concept of dynamical core should provide a set-valued map $\Gamma : R_+ \times [0,1]^n \rightsquigarrow R^n$ associating with each time t and any fuzzy coalition x a set $\Gamma(t,x)$ of allotments $p \in R_+^n$ such that, taking $p(t) \in \Gamma(T-t, x(t))$, and in particular, $p(0) \in \Gamma(T, x(0))$, the chosen above condition is satisfied.

This is the purpose of this talk.

The Dynamics

1. **the evolution of coalitions** $x(t) \in R^n$ **is governed by differential inclusions**

$$x'(t) \;:=\; f(x(t), v(t)) \text{ where } v(t) \;\in\; Q(x(t))$$

   where $v(t)$ **are perturbations,**

2. **static constraints**

$$\forall\, x \in [0,1]^n, \;\; p \;\in\; P(x) \;\subset\; R^n_+$$

   and dynamic constraints on the velocities of the allotments $p(t) \in R^n_+$ **of the form**

$$\langle p'(t), x(t)\rangle \;=\; -m(x(t), p(t), v(t))\langle p(t), x(t)\rangle$$

3. **from which we deduce the velocity** $y'(t) = \langle p(t), f(x(t), v(t))\rangle - m(x(t), p(t))y(t)$ **of the payoff** $y(t) := \langle p(t), x(t)\rangle$ **of the fuzzy coalition** $x(t)$**.**

A feedback $\widetilde{p}$ **is a selection of the set-valued map** P **in the sense that for any** $x \in [0,1]^n$**,** $\widetilde{p}(x) \in P(x)$**. We thus associate with any feedback** $\widetilde{p}$ **the set** $\mathcal{C}_{\widetilde{p}}(x)$ **of** $(x(\cdot), y(\cdot), v(\cdot))$ **solutions to**

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), v(t)) \\ ii) & y'(t) & = & \langle \widetilde{p}(x(t)), f(x(t), v(t)) \rangle \\ & \multicolumn{3}{l}{-y(t) m(x(t), \widetilde{p}(x(t)), v(t))} \\ & \multicolumn{3}{l}{\text{where } \; v(t) \; \in \; Q(x(t))} \end{array} \right. \tag{3}$$

Towards the Characterization of the Dynamical Core

We shall characterize the dynamical core of the fuzzy dynamical cooperative game in terms of the derivatives of a valuation function that we now define.

For each rule of the game (2), the set $\mathcal{V}^\sharp$ **of initial conditions** (T, x, y) **such that there exists a feedback** $x \mapsto \widetilde{p}(x) \in P(x)$ **such that, for all perturbations** $t \in [0, T] \mapsto v(t) \in Q(x(t))$**, for all solutions to system (3) of differential equations satisfying** $x(0) = x$**,** $y(0) = y$**, the corresponding condition (2) is satisfied, is called the guaranteed valuation set.**

The Valuation Function

Knowing it, we deduce the valuation function

$$V^\sharp(T,x) \ := \ \inf\{y \mid (T,x,y) \in \mathcal{V}\}$$

providing the cheapest initial payoff allowing to satisfy the viability/capturability conditions (2). It satisfies the initial condition:

$$V^\sharp(0,x) \ := \ U(x)$$

We associate with the characteristic function $U : [0,1]^n \mapsto R \cup \{+\infty\}$ **of the dynamical cooperative game the functional**

$$\begin{cases} J_U(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \\ := \ e^{\int_0^t m(x(s),\widetilde{p}(x(s)),v(s))ds} U(x(t)) \\ \\ - \displaystyle\int_0^t e^{\int_0^\tau m(x(s),\widetilde{p}(x(s)),v(s))ds} \langle \widetilde{p}(x(\tau)), f(x(\tau),v(\tau)) \rangle d\tau \end{cases}$$

1. **prescribed end rule: We obtain**

$$V^{\sharp}_{(\mathbf{0},U_\infty)}(T,x) \;:=$$
$$\inf_{\widetilde{p}(x)\in P(x)} \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} J_U(T;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{4}$$

2. **time span rule: We obtain**

$$V^{\sharp}_{(U,U_\infty)}(T,x) \;:=$$
$$\inf_{\widetilde{p}(x)\in P(x)} \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} \sup_{t\in[0,T]} J_U(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{5}$$

3. **first winning time rule: We obtain**

$$V^{\sharp}_{(\mathbf{0},U)}(T,x) \;:=$$
$$\inf_{\widetilde{p}(x)\in P(x)} \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} \inf_{t\in[0,T]} J_U(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{6}$$

Although these functions are only lower semicontinuous, one can define epi-derivatives of lower semicontinuous functions (or generalized gradients) in adequate ways and compute the core Γ: for instance, when the valuation function is differentiable, we shall prove that Γ associates with any $(t,x) \in R_+ \times R^n$ the subset $\Gamma(t,x)$ of allotments $p \in P(x)$ satisfying

$$\sup_{v \in Q(x)} \left(\sum_{i=1}^{n} \left(\frac{\partial V^\sharp(t,x)}{\partial x_i} - p_i \right) f_i(x,v) + m(x,p,v) V^\sharp(t,x) \right)$$
$$\leq \ \frac{\partial V^\sharp(t,x)}{\partial t}$$

The valuation function is a (generalized) solution to

Hamilton-Jacobi-Isaacs partial differential equation

$$-\frac{\partial V(t,x)}{\partial t} + \inf_{p \in P(x)} \sup_{v \in Q(x)} \left(\sum_{i=1}^{n} \left(\frac{\partial V(t,x)}{\partial x_i} - p_i \right) f_i(x,v) \right.$$
$$\left. + m(x,p,v) V(t,x) \right) = 0$$

satisfying the initial condition

$$V(0,x) = U(x)$$

on each of the subsets

1. **prescribed end rule:**

$$\Omega_{(\mathbf{0},U_\infty)}(\mathbf{V}) := \{(t,x) \mid t > 0 \;\&\; V(t,x) \geq 0\}$$

2. **time span rule**

$$\Omega_{(U,U_\infty)}(\mathbf{V}) := \{(t,x) \mid t > 0 \;\&\; V(t,x) \geq U(x)\}$$

3. first winning time rule

$$\Omega_{(\mathbf{0},U)}(\mathbf{V}) \ := \ \{(t,x) \mid t > 0 \ \& \ U(x) \ > \ V(t,x) \ \geq \ 0\}$$

The Static Case as a Limiting Case

When $m(x,p,v) = 0$**,** $f(x,v) = v$ **and** $Q(x) = rB$**, then** $\Gamma(t,x)$ **is the set of** $p \in P(x)$ **satisfying**

$$r\left\|\frac{\partial V^\sharp(t,x)}{\partial x} - p\right\| \;=\; \frac{\partial V^\sharp(t,x)}{\partial t}$$

Letting the radius $r \to \infty$**, we obtain** $p = \frac{\partial V^\sharp(t,x)}{\partial x}$ **and** $\frac{\partial V^\sharp(t,x)}{\partial t} = 0$**.**

Since $V^\sharp(0,x) = U(x)$**, we infer that in this case** $\Gamma(t,x) = \frac{\partial U(x)}{\partial x}$**, i.e., the Shapley value of the fuzzy static cooperative game when the characteristic function** U **is differentiable and positively homogenous, and the core of the fuzzy static cooperative game when the characteristic function** U **is concave, continuous and positively homogenous.**

Actually, the solution of the above partial differential equation is taken in the "contingent sense", where the directional derivatives are the contingent epiderivatives $D_{\uparrow}V(t,x)$ of V at (t,x). They are defined by

$$D_{\uparrow}V(t,x)(\lambda,v) \ := \ \liminf_{h\to 0+,\ u\to v} \frac{V(t+h\lambda, x+hu)}{h}$$

The dynamical core Γ of the corresponding fuzzy dynamical cooperative game is equal to

$$\begin{cases} \Gamma(t,x) \ := \ \{p \in P(x) \ \textbf{such that} \\ \sup_{v\in Q(x)}(D_{\uparrow}V^{\sharp}(t,x)(-1,f(x,v)) - \langle p, f(x,v)\rangle \\ +m(x,p,v)V^{\sharp}(t,x)) \ \leq \ 0\} \end{cases}$$

where $V^{\sharp}$ is the corresponding value function.

We prove that for each feedback $\widetilde{p}(t,x) \in \Gamma(t,x)$, selection of the dynamical core Γ, all evolutions $(x(\cdot), v(\cdot))$ of the system

$$\left\{ \begin{array}{lll} i) & x'(t) & = & f(x(t), v(t)) \\ ii) & y'(t) & = & \langle \widetilde{p}(T-t, x(t)), x(t) \rangle \\ & & & -m(x(t), \widetilde{p}(T-t, x(t))y(t) \\ iii) & v(t) & \in & Q(x(t)) \end{array} \right. \tag{7}$$

satisfy the rule of the game.

Extended Functions and their Epigraphs

A function $U : X \mapsto R \cup \{+\infty\}$ is called an extended (real-valued) function. Its domain is the set of points at which U is finite:

$$\text{Dom}(U) \ := \ \{x \in X \ \mid \ U(x) < +\infty\}$$

that embodies underlying state constraints: in particular, we shall assume that $U(x) := +\infty$ whenever $x \notin [0,1]^n$ to take into account that x is a fuzzy coalition.

The southern border of a subset $\mathcal{V} \subset X \times R_+$ the function $V_{\mathcal{V}} : X \mapsto R_+ \cup \{+\infty\}$ defined by

$$V_{\mathcal{V}}(x) \ := \ \inf_{(x,w) \in \mathcal{V}} w \ \in \ \overline{R}$$

Extended Characteristic Functions

In order to treat the three rules of the game

$$\begin{cases} i) \quad \displaystyle\sum_{i=1}^{n} p_i(T)x_i(T) \ \geq \ U(x(T)) \\ ii) \quad \forall\, t \in [0,T], \ \displaystyle\sum_{i=1}^{n} p_i(t)x_i(t) \ \geq \ U(x(t)) \\ iii) \quad \exists\, t^* \in [0,T] \ \textbf{such that} \\ \qquad \displaystyle\sum_{i=1}^{n} p_i(t^*)x_i(t^*) \ \geq \ U(x(t^*)) \end{cases} \tag{8}$$

as particular cases of a more general framework, we introduce two nonnegative extended functions b and c (characteristic functions of the cooperative games) satisfying

$$\forall\, (t,x) \in R_+ \times R_+^n \times R^n, \ 0 \leq \mathbf{b}(t,x) \ \leq \ \mathbf{c}(t,x) \leq +\infty$$

Extended Rules of the Game

By associating with the initial characteristic function U of the game adequate pairs $(\mathbf{b}, \mathbf{c})$ of extended functions, we shall replace the three above requirements by the requirements

$$\begin{cases} i) & \forall\, t \in [0, t^\star], \quad y(t) \geq \mathbf{b}(T - t, x(t)) \\ & (\textbf{dynamical constraints}) \\ ii) & y(t^\star) \geq \mathbf{c}(T - t^\star, x(t^\star)) \\ & (\textbf{objective}) \end{cases} \tag{9}$$

Prescribed Time Rules

Problems with prescribed final time are obtained with objective functions satisfying the condition

$$\forall\, t > 0, \;\; \mathbf{c}(t,x) \;:=\; +\infty$$

In this case, $t^\star = T$ **and condition**

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,t^\star], \;\; y(t) \geq \mathbf{b}(T-t, x(t)) \\ & (\textbf{dynamical constraints}) \\ ii) & y(t^\star) \geq \mathbf{c}(T-t^\star, x(t^\star)) \\ & (\textbf{objective}) \end{array} \right. \tag{10}$$

boils down to

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,T], \;\; y(t) \geq b(T-t, x(t)) \\ ii) & y(T) \geq c(0, x(T)) \end{array} \right.$$

Indeed, since $y(t^\star)$ **is finite and since** $\mathbf{c}(T-t^\star, x(t^\star))$ **is infinite whenever** $T-t^\star > 0$**, we infer from inequality ii) that** $T-t^\star$ **must be equal to** 0**.**

Examples

Setting

$$\mathbf{0}(t,x) \ = \begin{cases} 0 & \text{if} \quad t \geq 0, \\ +\infty & \text{if} \quad t < 0 \end{cases} \quad U_\infty(t,x) \ := \begin{cases} U(x) & \text{if} \quad t = 0 \\ +\infty & \text{if} \quad t > 0 \end{cases}$$

We can recover the three rules of the game

1. **We take b**$(t,x) := 0$ **and c**$(t,x) = U_\infty(x)$**, we obtain the prescribed final time rule (2)i).**
2. **We take b**$(t,x) := U(x)$ **and b**$(t,x) := U_\infty(x)$**, we obtain the span time rule (2)ii).**
3. **We take b**$(t,x) := 0$ **and c**$(t,x) = U(x)$**, we obtain the first winning time rule (2)iii).**

Valuation Set

The guaranteed valuation subset $\mathcal{V}^\sharp \subset R_+ \times R^n \times R_+$ of triples (T, x, y) made of the final time T, the initial fuzzy coalition x and the initial payoff y such that there exists a feedback $x \mapsto \widetilde{p}(x) \in P(x)$ such that, for all perturbations $t \in [0, T] \mapsto v(t) \in Q(x(t))$, for all solutions to system (3) of differential equations satisfying $x(0) = x$, $y(0) = y$, there exists a time $t^* \in [0, T]$ such that conditions (9):

$$\begin{cases} i) \quad \forall\, t \in [0, t^*], \;\; y(t) \geq \mathbf{b}(T - t, x(t)) \\ ii) \;\; y(t^*) \geq \mathbf{c}(T - t^*, x(t^*)) \end{cases}$$

are satisfied.

associates with any final time T and initial coalition x the smallest payoff $V^\sharp(T,x)$:

$$V^\sharp_{(\mathbf{b},\mathbf{c})}(T,x) \ := \inf_{(T,x,y)\in\mathcal{V}^\sharp} y \tag{11}$$

The function $(T,x)\mapsto V^\sharp_{(\mathbf{b},\mathbf{c})}(T,x)$ is called the guaranteed valuation function of the allotment, i.e., the minimal initial payoff y satisfying the two constraints

$$\begin{cases} i) & \forall\, t\in[0,t^*], \ \ y(t)\geq \mathbf{b}(T-t,x(t)) \\ ii) & y(t^*)\geq \mathbf{c}(T-t^*,x(t^*)) \end{cases}$$

$$\begin{cases} J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \\ := \; e^{\int_0^t m(x(s),\widetilde{p}(x(s)),v(s))ds}\mathbf{c}(T-t,x(t)) \\ \\ -\displaystyle\int_0^t e^{\int_0^\tau m(x(s),\widetilde{p}(x(s)),v(s))ds}\langle\widetilde{p}(x(\tau)),f(x(\tau),v(\tau))\rangle d\tau \end{cases}$$

(where $t\in[0,T]$), $K_{\mathbf{b}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) :=$

$$\sup_{s\in[0,t]} J_{\mathbf{b}}(s;(x(\cdot),v(\cdot));\widetilde{p})(T,x)$$

and

$$\begin{cases} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; := \\ \max\left(J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x),\right. \\ \left.K_{\mathbf{b}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x)\right) \end{cases}$$

Formula for the Valuation Function

The guaranteed valuation function $(T,x,p) \mapsto V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x)$ **is equal to** $V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x) =$

$$\inf_{\widetilde{p}(x)\in P(x)} \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)}$$

$$\inf_{t\in[0,T]} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x)$$

satisfies the initial condition

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(0,x) \;=\; \mathbf{c}(0,x)$$

and inequalities: $\forall\ (T,x) \in R_+ \times R^n \times R^n$,

$$0 \;\leq\; \mathbf{b}(T,x) \;\leq\; V^{\sharp}(T,x) \;\leq\; \mathbf{c}(T,x)$$

We translate the viability/capturability conditions

$$\begin{cases} i) & \forall\, t \in [0, t^*], \quad y(t) \geq \mathbf{b}(T - t, x(t)) \\ ii) & y(t^*) \geq \mathbf{c}(T - t^*, x(t^*)) \end{cases}$$

in the following geometric form:

$$\begin{cases} i) & \forall\, t \in [0, t^*], \quad (T - t, x(t), y(t)) \in \mathcal{E}p(\mathbf{b}) \\ & \text{(viability constraint)} \\ ii) & (T - t^*, x(t^*), y(t^*)) \in \mathcal{E}p(\mathbf{c}) \\ & \text{(capturability of a target)} \end{cases} \tag{12}$$

The Auxiliary Dynamical Game

The evolution of $(T-t, x(t), y(t))$ **is governed by the dynamical game**

$$\left\{\begin{array}{ll} i) & \tau'(t) \ = \ -1 \\ ii) & \forall\, i = 0,\ldots,n, \ \ x_i'(t) \ = \ f_i(x(t), v(t)) \\ iii) & y'(t) \ = \ \langle p(t), f(x(t), v(t))\rangle \\ & -y(t)m(x(t), p(t), v(t)) \\ & \text{where } p(t) \ \in \ P(x(t)) \ \& \ v(t) \ \in \ Q(x(t)) \end{array}\right. \quad (13)$$

starting at (T, x, y)**.**

We summarize it in the form

$$\begin{cases} i) & z'(t) \ \in \ g(z(t), u(t), v(t)) \\ ii) & u(t) \ \in \ P(z(t)) \ \& \ v(t) \in Q(z(t)) \end{cases}$$

where $z := (\tau, x, y) \in R \times R^n \times R$**,** $u := p$**,** $g : R \times R^n \times R \rightsquigarrow R \times R^n \times R^n \times R$ **is defined by** $g(z, v)$

$$= (-1, f(x, v), u, -m(x, u, v)y + \langle u, f(x, v)\rangle)$$

where u **ranges over** $P(z) := P(x)$ **and** v **over** $Q(z) := Q(x)$**.**

We say that a selection $z \mapsto \widetilde{p}(z) \in P(z)$ **is a feedback, regarded as a strategy. One associates the evolutions governed by**

$$z'(t) \ = \ g(z(t), \widetilde{p}(z(t)), v(t))$$

starting at time 0 **at** z**.**

Guaranteed Capture Basins

Let K and $C \subset K$ be two subsets of Z.

The guaranteed viable-capture basin of the target C viable in K is the set of elements $z \in K$ such that there exists a continuous feedback $\widetilde{p}(z) \in P(z)$ such that for every $v(\cdot) \in Q(z(\cdot))$, for every solutions $z(\cdot)$ to $z' = g(z, \widetilde{p}(z), v)$, there exists $t^* \in R_+$ such that the viability/capturability conditions

$$\begin{cases} i) & \forall\, t \in [0, t^*], \quad z(t) \in K \\ ii) & \qquad\qquad\quad\;\; z(t^*) \in C \end{cases}$$

are satisfied.

The guaranteed valuation subset $\mathcal{V}^\sharp$ defined above is the "southern border" of the guaranteed viable-capture basin under the dynamical game of the epigraph of the function c viable in the epigraph of the function b.

The Strategy

Since we have related the guaranteed valuation problem to the much simpler — although more abstract — study of guaranteed viable-capture basin of a target and other guaranteed viability/capturability issues for dynamical games,

1. **we first "solve" these "viability/capturability problems" for dynamical games at this general level, and in particular, study the tangential conditions enjoyed by the guaranteed viable-capture basins**
2. **and translate translating tangential conditions to give a meaning to the concept of a generalized solution (Frankowska's episolutions or, by duality, viscosity solutions) to Hamilton-Jacobi-Isaacs variational inequalities.**