Dynamic Core of Fuzzy Dynamical Cooperative Games

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Abstract

We use in this paper the viability/capturability approach for studying the problem of characterizing the dynamic core of a dynamic cooperative game defined in a characteristic function form. In order to allow coalitions to evolve, we embed them in the set of fuzzy coalitions. Hence, we define the dynamic core as a set-valued map associating with each fuzzy coalition and each time the set of allotments such that their payoffs at that time to the fuzzy coalition are larger than or equal to the one assigned by the characteristic function of the game. We shall characterize this core through the (generalized) derivatives of a valuation function associated with the game. We shall provide its explicit formula, characterize its epigraph as a viable-capture basin of the epigraph of the characteristic function of the fuzzy dynamical cooperative game, use the tangential properties of such basins for proving that the valuation function is a solution to a Hamilton-Jacobi-Isaacs partial differential equation and use this function and its derivatives for characterizing the dynamic core.

Introduction

This paper takes up on a recent line of research, *dynamic cooperative game theory*, opened by Leon Petrosjan (see for instance [100, Petrosjan] and [101, Petrosjan & Zenkevitch]), Alain Haurie ([86, Haurie]), Jerzy Filar and others. We quote the first lines of [73, Filar & Petrosjan] : *"Bulk of the literature dealing with cooperative games (in characteristic function form) do not address issues related to the evolution of a solution concept over time. However, most conflict situations are not "one shot" games but continue over some time horizon which may be limited* a priori *by the game rules, or terminate when some specified conditions are attained."* This paper, however, deals also with the evolution of coalitions.

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1. **Fuzzy Coalitions and their Evolution**

For that purpose, since cooperative games deal with the behavior of coalitions of players, the first definition of a coalition which comes to mind being that of a subset of players $A \subset N$ is not adequate for tackling dynamical models of evolution of coalitions since the 2^n coalitions range over a finite set, preventing us from using analytical techniques. One way to overcome this difficulty is to embed the family of subsets of a (discrete) set N of n players to the space $\mathbf{R}^n$ through the map χ associating with any coalition $A \in \mathcal{P}(N)$ its characteristic function[2] $\chi_A \in \{0,1\}^n \subset \mathbf{R}^n$.

By definition, the family of fuzzy sets[3] is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in $\mathbf{R}^n$. Therefore, we can write any fuzzy set in the form

$$x \;=\; \sum_{A\in\mathcal{P}(N)} m_A\chi_A \;\text{ where }\; m_A \geq 0 \;\;\&\;\; \sum_{A\in\mathcal{P}(N)} m_A = 1$$

The memberships are then equal to

$$\forall\, i \in N,\;\; x_i \;=\; \sum_{A\ni i} m_A$$

Consequently, if m_A is regarded as the probability for the set A to be formed, the membership of the player i to the fuzzy set x is the sum of the probabilities of the coalitions to which player i belongs. Player i participates fully in x if $x_i = 1$, does not participate at all if $x_i = 0$ and participates in a fuzzy way if $x_i \in]0,1[$.

Actually, this idea of using fuzzy coalitions has already been used in the framework of cooperative games with and without side-payments described by a characteristic function[4] $\mathbf{u}$ assigning to each fuzzy coalition $x \in [0,1]^n$ a lower bound $\mathbf{u}(x)$ to gains or payoffs $y := \langle p, x\rangle$ associated with an allotment $p \in \mathbf{R}^n_+$ (see [3, 4, Aubin], [2, Aubin, Chapter 12], [7, Aubin, Chapter 13], the books [97, Mares] and [98, Mishizaki & Sokawa], and [48, 49, 50, Basile], [47, Basile, De Simone & Graziano], [1, Allouch & Florenzano], [74, Florenzano]). Fuzzy coalitions have been used in dynamical models of cooperative

[2] This canonical embedding is more adapted to the nature of the power set $\mathcal{P}(N)$ than the universal embedding of a discrete set M of m elements to $\mathbf{R}^m$ by the Dirac measure associating with any $j \in M$ the jth element of the canonical basis of $\mathbf{R}^m$. The convex hull of the image of M by this embedding is the probability simplex of $\mathbf{R}^m$. Hence fuzzy sets offer a "dedicated convexification" procedure of the discrete power set $M := \mathcal{P}(N)$ instead of the universal convexification procedure of frequencies, probabilities, mixed strategies derived from its embedding in $\mathbf{R}^m = \mathbf{R}^{2^n}$.

[3] This concept of fuzzy set was introduced in 1965 by L. A. Zadeh. Since then, it has been wildly successful, even in many areas outside mathematics!. Lately, we found in *"La lutte finale"*, Michel Lafon (1994), p.69 by A. Bercoff the following quotation of the late François Mitterand, president of the French Republic (1981-1995): *"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...* (Today, we swim in the pure poetry of fuzzy subsets)!

[4] not to be confused with characteristic functions of sets !

games in [22, Aubin & Cellina, Chapter 4], economic theory in [13, Aubin, Chapter 5] and in [20, Aubin].

For instance, it has been shown that in the framework of static cooperative games with side payments involving fuzzy coalitions, *the concepts of Shapley value and core coincide with the (generalized) gradient*[5] $\partial\mathbf{u}(x_N)$ *of the "characteristic function"* $\mathbf{u} : [0,1]^n \mapsto \mathbf{R}_+$ *at the "grand coalition"* $x_N := (1,\ldots,1)$, *the characteristic function of* $N := \{1,2,\ldots,n\}$.

For instance, when the characteristic function of the static cooperative game $\mathbf{u}$ is concave, positively homogeneous and continuous on the interior of $\mathbf{R}^n_+$, one check[6] that the generalized gradient $\partial\mathbf{u}(x_N)$ is not empty and coincides with the subset of allotments $p := (p_1,\ldots,p_n) \in \mathbf{R}^n_+$ accepted by all fuzzy coalitions in the sense that

$$\forall\, x \in [0,1]^n,\ \ \langle p, x\rangle \ = \ \sum_{i=1}^n p_i x_i \ \geq \ \mathbf{u}(x) \tag{1}$$

and that, for the grand coalition $x_N := (1,\ldots,1)$,

$$\langle p, x_N\rangle \ = \ \sum_{i=1}^n p_i \ = \ \mathbf{u}(x_N)$$

2. Fuzzy Dynamic Cooperative Games

In a dynamical context, (fuzzy) coalitions evolve, so that static conditions (1) should be replaced by conditions[7] stating that for any evolution $t \mapsto x(t)$ of fuzzy coalitions, the payoff $y(t) := \langle p(t), x(t)\rangle$ should be larger than or equal to $\mathbf{u}(x(t))$ according (at least) to one of the three following rules:

(a) at a prescribed final time T of the end of the game:

$$y(T) \ := \ \sum_{i=1}^n p_i(T) x_i(T) \ \geq \ \mathbf{u}(x(T))$$

(b) during the whole time span of the game:

$$\forall\, t \in [0,T],\ \ y(t) \ := \ \sum_{i=1}^n p_i(t) x_i(t) \ \geq \ \mathbf{u}(x(t))$$

[5] The differences between these concepts for usual games is explained by the different ways one "fuzzyfy" a characteristic function defined on the set of usual coalitions. See [3, 4, Aubin], [2, Aubin, Chapter 12] and [7, Aubin, Chapter 13].

[6] See [3, 4, Aubin], [2, Aubin, Chapter 12] and [7, Aubin, Chapter 13].

[7] Naturally, the privileged role played by the grand coalition in the static case must be abandoned, since the coalitions evolve, so that the grand coalition eventually loses its status.

(c) at the first winning time $t^* \in [0,T]$ when

$$y(t^*) \ := \ \sum_{i=1}^{n} p_i(t^*)x_i(t^*) \ \geq \ \mathbf{u}(x(t^*))$$

at which time the game stops.

Summarizing, the above conditions require to find — for each of the above three rules of the game — an evolution of an allotment $p(t) \in \mathbf{R}^n$ such that, for all evolutions of fuzzy coalitions $x(t) \in [0,1]^n$ starting at x, the corresponding rule of the game

$$\left\{ \begin{array}{ll} i) & \sum_{i=1}^{n} p_i(T)x_i(T) \ \geq \ \mathbf{u}(x(T)) \\ ii) & \forall\, t \in [0,T], \ \sum_{i=1}^{n} p_i(t)x_i(t) \ \geq \ \mathbf{u}(x(t)) \\ iii) & \exists\, t^* \in [0,T] \ \text{ such that } \sum_{i=1}^{n} p_i(t^*)x_i(t^*) \ \geq \ \mathbf{u}(x(t^*)) \end{array} \right. \tag{2}$$

must be satisfied

Therefore, for each one of the above three rules of the game (2), a concept of dynamical core should provide a set-valued map $\Gamma : \mathbf{R}_+ \times [0,1]^n \rightsquigarrow \mathbf{R}^n$ associating with each time t and any fuzzy coalition x a set $\Gamma(t,x)$ of allotments $p \in \mathbf{R}^n_+$ such that, taking $p(t) \in \Gamma(T-t, x(t))$, and in particular, $p(0) \in \Gamma(T, x(0))$, the chosen above condition is satisfied. This is the purpose of this paper.

Naturally, this makes sense only once the dynamics of the coalitions and of the allotments are given. We shall assume that

(a) the evolution of coalitions $x(t) \in \mathbf{R}^n$ is governed by differential inclusions

$$x'(t) \ := \ f(x(t), v(t)) \text{ where } v(t) \ \in \ Q(x(t))$$

where $v(t)$ are perturbations,

(b) static constraints

$$\forall\, x \in [0,1]^n, \ \ p \ \in \ P(x) \ \subset \ \mathbf{R}^n_+$$

and dynamic constraints on the velocities of the allotments $p(t) \in \mathbf{R}^n_+$ of the form

$$\langle p'(t), x(t) \rangle \ = \ -\mathbf{m}(x(t), p(t), v(t)) \langle p(t), x(t) \rangle$$

(c) from which we deduce the velocity $y'(t) = \langle p(t), f(x(t), v(t)) \rangle - \mathbf{m}(x(t), p(t))y(t)$ of the payoff $y(t) := \langle p(t), x(t) \rangle$ of the fuzzy coalition $x(t)$.

The evolution of the fuzzy coalitions is thus parametrized by allotments and perturbations, i.e., is governed by a dynamic game

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), v(t)) \\ ii) & y'(t) & = & \langle p(t), f(x(t), v(t)) \rangle - \mathbf{m}(x(t), p(t)) y(t) \\ iii) & \text{where } p(t) & \in & P(x(t)) \ \& \ v(t) \in Q(x(t)) \end{array} \right. \quad (3)$$

A feedback $\widetilde{p}$ is a selection of the set-valued map P in the sense that for any $x \in [0,1]^n$, $\widetilde{p}(x) \in P(x)$. We thus associate with any feedback $\widetilde{p}$ the set $\mathcal{C}_{\widetilde{p}}(x)$ of triples $(x(\cdot), y(\cdot), v(\cdot))$ solutions to

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), v(t)) \\ ii) & y'(t) & = & \langle \widetilde{p}(x(t)), f(x(t), v(t)) \rangle - y(t) \mathbf{m}(x(t), \widetilde{p}(x(t)), v(t)) \\ & \text{where } v(t) & \in & Q(x(t)) \end{array} \right. \quad (4)$$

3. **Characterization of the Dynamical Core**

We shall characterize the dynamical core of the fuzzy dynamical cooperative game in terms of the derivatives of a valuation function that we now define.

For each rule of the game (2), the set $\mathcal{V}^\sharp$ of initial conditions (T, x, y) such that there exists a feedback $x \mapsto \widetilde{p}(x) \in P(x)$ such that, for all perturbations $t \in [0, T] \mapsto v(t) \in Q(x(t))$, for all solutions to system (4) of differential equations satisfying $x(0) = x$, $y(0) = y$, the corresponding condition (2) is satisfied, is called the guaranteed valuation set[8].

Knowing it, we deduce the valuation function

$$V^\sharp(T, x) \ := \ \inf\{y \mid (T, x, y) \in \mathcal{V}\}$$

providing the cheapest initial payoff allowing to satisfy the viability/capturability conditions (2). It satisfies the initial condition:

$$V^\sharp(0, x) \ := \ \mathbf{u}(x)$$

In each of the three cases, we shall compute explicitly the valuation functions as inf-sup of underlying criteria we shall uncover. For that purpose, we associate with the characteristic function $\mathbf{u} : [0,1]^n \mapsto \mathbf{R} \cup \{+\infty\}$ of the dynamical cooperative game the functional

$$\left\{ \begin{array}{l} J_{\mathbf{u}}(t; (x(\cdot), v(\cdot)); \widetilde{p})(x) \ := \ e^{\int_0^t \mathbf{m}(x(s), \widetilde{p}(x(s)), v(s)) ds} \mathbf{u}(x(t)) \\ \displaystyle - \int_0^t e^{\int_0^\tau \mathbf{m}(x(s), \widetilde{p}(x(s)), v(s)) ds} \langle \widetilde{p}(x(\tau)), f(x(\tau), v(\tau)) \rangle d\tau \end{array} \right.$$

[8] One can also define the conditional valuation set $\mathcal{V}^\flat$ of initial conditions (T, x, y) such that for all perturbations v, there exists an evolution of the allotment $p(\cdot)$ such that viability/capturability conditions (2) are satisfied. We omit this study for the sake of brevity, since it is parallel to the one of guaranteed valuation sets.

We shall associate with it and with each of the three rules of the game the three corresponding valuation functions[9]:

(a) **prescribed end rule**: We obtain

$$V^\sharp_{(\mathbf{0},\mathbf{u}_\infty)}(T,x) \ := \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} J_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{5}$$

(b) **time span rule**: We obtain

$$V^\sharp_{(\mathbf{u},\mathbf{u}_\infty)}(T,x) \ := \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} \ \sup_{t\in[0,T]} J_{\mathbf{u}}(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{6}$$

(c) **first winning time rule**: We obtain

$$V^\sharp_{(\mathbf{0},\mathbf{u})}(T,x) \ := \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} \ \inf_{t\in[0,T]} J_{\mathbf{u}}(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \tag{7}$$

Although these functions are only lower semicontinuous, one can define epiderivatives of lower semicontinuous functions (or generalized gradients) in adequate ways and compute the core Γ: for instance, when the valuation function is differentiable, we shall prove that Γ associates with any $(t,x)\in\mathbf{R}_+\times\mathbf{R}^n$ the subset $\Gamma(t,x)$ of allotments $p\in P(x)$ satisfying

$$\sup_{v\in Q(x)}\left(\sum_{i=1}^n\left(\frac{\partial V^\sharp(t,x)}{\partial x_i}-p_i\right)f_i(x,v)+\mathbf{m}(x,p,v)V^\sharp(t,x)\right) \ \leq \ \frac{\partial V^\sharp(t,x)}{\partial t}$$

The valuation function $V^\sharp$ is actually a solution to the nonlinear Hamilton-Jacobi-Isaacs partial differential equation

$$-\frac{\partial \mathbf{v}(t,x)}{\partial t}+\inf_{p\in P(x)}\ \sup_{v\in Q(x)}\left(\sum_{i=1}^n\left(\frac{\partial \mathbf{v}(t,x)}{\partial x_i}-p_i\right)f_i(x,v)+\mathbf{m}(x,p,v)\mathbf{v}(t,x)\right) \ = \ 0$$

satisfying the initial condition

$$\mathbf{v}(0,x) \ = \ \mathbf{u}(x)$$

on each of the subsets

(a) **prescribed end rule**:

$$\Omega_{(\mathbf{0},\mathbf{u}_\infty)}(\mathbf{v}) \ := \ \{(t,x)\mid t>0 \ \& \ \mathbf{v}(t,x) \ \geq \ 0\}$$

[9] The notations $(\mathbf{0},\mathbf{u}_\infty)$, $(\mathbf{u},\mathbf{u}_\infty)$, $(\mathbf{0},\mathbf{u})$ will be explained later.

(b) **time span rule**

$$\Omega_{(\mathbf{u},\mathbf{u}\infty)}(\mathbf{v}) \;:=\; \{(t,x) \mid t>0 \;\&\; \mathbf{v}(t,x) \;\geq\; \mathbf{u}(x)\}$$

(c) **first winning time rule**

$$\Omega_{(\mathbf{0},\mathbf{u})}(\mathbf{v}) \;:=\; \{(t,x) \mid t>0 \;\&\; \mathbf{u}(x) \;>\; \mathbf{v}(t,x) \;\geq\; 0\}$$

Remark: The Static Case as a Limiting Case — Let us consider the case when $\mathbf{m}(x,p,v)=0$ (self-financing of fuzzy coalitions) and when the evolution of coalitions is governed by $f(x,v)=v$ and $Q(x)=rB$. Then the dynamical core is the subset $\Gamma(t,x)$ of allotments $p\in P(x)$ satisfying on $\Omega(V^\sharp)$ the equation[10]

$$r\left\|\frac{\partial V^\sharp(t,x)}{\partial x}-p\right\| \;=\; \frac{\partial V^\sharp(t,x)}{\partial t}$$

Now, assuming that the data and the solution are smooth we deduce formally that letting the radius $r\to\infty$, we obtain as a limiting case that $p=\frac{\partial V^\sharp(t,x)}{\partial x}$ and that $\frac{\partial V^\sharp(t,x)}{\partial t}=0$. Since $V^\sharp(0,x)=\mathbf{u}(x)$, we infer that in this case $\Gamma(t,x)=\frac{\partial \mathbf{u}(x)}{\partial x}$, i.e., the Shapley value of the fuzzy static cooperative game when the characteristic function $\mathbf{u}$ is differentiable and positively homogenous, and the core of the fuzzy static cooperative game when the characteristic function $\mathbf{u}$ is concave, continuous and positively homogenous. □

Actually, the solution of the above partial differential equation is taken in the "contingent sense", where the directional derivatives are the contingent epiderivatives $D_\uparrow\mathbf{v}(t,x)$ of $\mathbf{v}$ at (t,x). They are defined by

$$D_\uparrow\mathbf{v}(t,x)(\lambda,v) \;:=\; \liminf_{h\to 0+,\;u\to v}\frac{\mathbf{v}(t+h\lambda,x+hu)}{h}$$

(see for instance [27, Aubin & Frankowska] and [108, Rockafellar & Wets]). In this case, for each rule of the game, the dynamical core Γ of the corresponding fuzzy dynamical cooperative game is equal to

$$\begin{cases} \Gamma(t,x) \;:=\; \{p\in P(x) \;\text{ such that} \\ \sup_{v\in Q(x)}(D_\uparrow V^\sharp(t,x)(-1,f(x,v)) - \langle p, f(x,v)\rangle + \mathbf{m}(x,p,v)V^\sharp(t,x)) \;\leq\; 0\}\end{cases}$$

[10] when $p=0$, we find the eikonal equation.

where $V^\sharp$ is the corresponding value function. We shall prove that for each feedback $\widetilde{p}(t,x) \in \Gamma(t,x)$, selection of the dynamical core Γ, all evolutions $(x(\cdot), v(\cdot))$ of the system

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), v(t)) \\ ii) & y'(t) & = & \langle \widetilde{p}(T-t, x(t)), x(t)\rangle - \mathbf{m}(x(t), \widetilde{p}(T-t, x(t)) y(t) \\ iii) & v(t) & \in & Q(x(t)) \end{array} \right. \tag{8}$$

satisfy the corresponding condition (2).

4. **Outline**

   The paper is organized as follows

   (a) We shall present fuzzy dynamical cooperative games (allotments and payoff, dynamics, dynamical constraints on allotments, dynamics of the payoff, characteristic functions), raise the questions and provide some answers : underlying criteria, Hamilton-Jacobi-Isaacs variational inequalities and the derivation of the dynamical core,

   (b) outline the viability/capturability strategy,

   (c) study and characterize guaranteed viable-capture basins of targets under dynamical games and use these results for studying intertemporal dynamical games problems,

   (d) characterize guaranteed viable-capture basins in terms of tangential conditions, deduce that the valuation function is the solution to Hamilton-Jacobi-Isaacs variational inequalities and derive the regulation map and the adjustment law.

1 Fuzzy Dynamical Cooperative Game

Let us consider the set N of n players $i = 1, \ldots, n$ and the set $[0,1]^n$ of fuzzy coalitions[11]. The components of the state variable $x := (x_1, \ldots, x_n) \in [0,1]^n$ are the rates of participation in the fuzzy coalition x of player $i = 1, \ldots, n$.

[11] The choice of "cooperative" fuzzy coalitions $x \in [0,1]^n$ is arbitrary.

We could, for instance, introduce negative memberships when players enter a coalition with aggressive intents. This is mandatory if one wants to be realistic ! A positive membership is interpreted as a cooperative participation of the player i in the coalition, while a negative membership is interpreted as a non-cooperative participation of the ith player in the generalized coalition. In what follows, one can replace the cube $[0,1]^n$ by any product $\prod_{i=1}^{n}[\lambda_i, \mu_i]$ for describing the cooperative or noncooperative behavior of the consumers.

We can still enrich the description of the players by representing each player i by what psychologists call her 'behavior profile' as in [33, Aubin, Louis-Guerin & Zavalloni]. We consider q 'behavioral qualities' $k = 1, \ldots, q$, each with a unit of measurement. We also suppose that a behavioral quantity can be measured (evaluated) in terms of a real number (positive or negative) of units. A behavior profile is a vector $a = (a_1, \ldots, a_q) \in \mathbf{R}^q$ which specifies the quantities a_k of the q qualities k attributed to the player. Thus, instead of representing

An allotment is an element

$$p := (p_1, \dots, p_n) \in \mathbf{R}_+^n$$

describing the payoff of player $i = 1, \dots, n$ in the game. The associated payoff of the coalition (or the coalition-payoff) y is defined by

$$y \ := \ \langle p, x \rangle \ = \ \sum_{i=1}^{n} p_i x_i$$

We now introduce constraints on the allotments gathered by a fuzzy coalition x. They range over a subset $P(x) = P(x_1, \dots, x_n) \subset \mathbf{R}_+^n$ that can be a constant set $P \subset \mathbf{R}^n$ or can depend on the fuzzy coalitions.

We assume that the velocity of the fuzzy coalitions of players are uncertain, in a contingent (i.e., nonstochastic) or "tychastic" way: they depend upon a parameter — usually known under the name or a perturbation, or disturbance, or *tyche* — $v(t) \in Q(x(t)) \subset \mathcal{V}$ and evolve according to the perturbed — or tychastic — system of differential equations

$$\forall \, i = 1, \dots, n, \ \ x_i'(t) \ = \ f_i(x(t), v(t))$$

or, in a more compact form,

$$x'(t) \ = \ f(x(t), v(t)), \ \ v(t) \ \in Q(x(t))$$

In agreement with the tradition, the standard example of dynamics should be define by

$$f(x, v) \ := \ v \ \& \ Q(x) \ := \ rB, \ \text{ the ball of radius } r$$

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each player by a letter of the alphabet, she is described as an element of the vector space $\mathbf{R}^q$. We then suppose that each player may implement all, none, or only some of her behavioral qualities when she participates in a social coalition. Consider n players represented by their behavior profiles in $\mathbf{R}^q$. Any matrix $x = (x_i^k)$ describing the levels of participation $x_i^k \in [-1, +1]$ of the behavioral qualities k for the n players i is called a **social coalition**. Extension of the following results to social coalitions is straightforward.

Technically, the choice of the scaling $[0, 1]$ inherited from the tradition built on integration and measure theory is not adequate for describing convex sets. When dealing with convex sets, we have to replace the characteristic functions by indicators taking their values in $[0, +\infty]$ and take their convex combinations to provide an alternative allowing us to speak of "fuzzy" convex sets. Therefore, "toll-sets" are nonnegative cost functions assigning to each element its cost of belonging, $+\infty$ if it does not belong to the toll set. The set of elements with finite positive cost do form the "fuzzy boundary" of the toll set, the set of elements with zero cost its "core". This has been done to adapt viability theory to "fuzzy viability theory". See Chapter 10 of [12, Aubin] and [26, Aubin & Dordan] for more details.

Actually, the Cramer transform

$$C_\mu(p) \ := \ \sup_{x \in \mathbf{R}^n} \left(\langle p, x \rangle - \log \left(\int_{\mathbf{R}^n} e^{\langle x, y \rangle} d\mu(y) \right) \right)$$

maps probability measures to toll sets. In particular, it transforms convolution products of density functions to inf-convolutions of extended functions, Gaussians to squares of norms, etc. See [26, Aubin & Dordan] for more informations on this topic.

allowing the coalitions to evolve in all directions without any constraint. It seems to us more reasonable to take into account dedicated dynamics governing the evolution of coalitions.

Remark: Tychastic Differential Equations and Control — The set-valued map $Q : X \rightsquigarrow \mathcal{V}$ translates mathematically the concept of uncertainty in a contingent (i.e., non-stochastic) or tychastic way to adopt Charles Peirce's terminology[12]: Contingent uncertainty depends upon a parameter — usually known under the name or a perturbation, or disturbance, that could also be called a *tyche*, ranging over a given subset (that could be a fuzzy subset, as it is advocated in [26, Aubin & Dordan]). The size of this subset captures mathematically the concept of "contingency" — instead of "volatility", that by now has a specific meaning in mathematical finance. The larger the subsets $Q(x)$, the more contingent or "tychastic" the system.

Indeed, we are investigating properties (such as the viability/capturability properties) that hold true for every *tyche* — instead of "random", terminology already confiscated by probability theory — or every perturbation or disturbance, and, "robust", in the sense of robust control in control theory.

Controlling a system for solving a problem (such as viability, capturability, intertemporal optimality) whatever the perturbation is the branch of dynamical games known among control specialists as "robust control", that we propose to call "tychastic control" in contrast to "stochastic control".

In tychastic control problems, we have two kinds of uncertainties, one described by the set-valued map Q, describing the unknown contingent uncertainty, and the one described by the set-valued map P, providing a set of available regulation parameters (regulees), here, the allotments, describing what biologists call "pleiotropism". The larger the set-valued map P, the more able is the system to find a regulation parameter or a control to satisfy a given property whatever the perturbation in Q. In some sense, the set-valued map P is an antidote to cure the negative effects of unknown perturbations.

"Tychastic equations" seem to us reasonable candidates for encapsulating the idea underlying robust control and "games against Nature", on which there is an abundant literature. They provide an alternative way of representing uncertainty to usual "stochastic differential equations":

$$\forall\ i = 1, \ldots, n, \ \ dx_i(t) \ = \ f_i(x_i(t))dt + \sigma_i(x_i(t))dW(t)$$

in a stochastic environment.

However, these two choices can be reconciled in the framework of stochastic differential inclusions (see [23, 24, Aubin & Da Prato], [25, Aubin, Da Prato &Frankowska], [69, Da Prato & Frankowska], [38, Bardi & Goatin], [56, Bjork], [60, Buckdahn, Cardaliaguet & Quincam-

[12] See [99, Peirce] among other references of this prolific and profound philosopher. He associates with the Greek concept of necessity, *ananke*, the concept of *anancastic evolution*, anticipating the "chance and necessity" framework that has motivated viability theory in the first place.

poix], [57, 58, Buckdahn, Quincampoix & Rascanu], [59, Buckdahn, Peng, Quincampoix & Rainer], [82, Gautier & Thibault], [89, Jachimiak], [91, Kisielewicz], etc.).

Invariance of a set under a tychastic differential equation requires that for all tyche $v(\cdot)$, the associated solution is viable in the set whereas under a stochastic differential equation, it requires that the stochastic process is viable for almost all ω. Thanks to the equivalence formulas between Itô and Stratonovitch stochastic integrals and to the Strook & Varadhan "Support Theorem"(see for instance [71, Doss], [72, Doss & Priouret], [115, 116, Zabczyk]), and under convenient assumptions, stochastic viability problems are equivalent to invariance problems for tychastic systems and thus, viability problems for stochastic control systems are equivalent to guaranteed viability problems for dynamical games. □

In the context of dynamical cooperative games, a fuzzy coalition $x(t)$ at time t is allowed to change the allotment $p(t)$ at time t in such a way that

$$\sum_{i=1}^{n} p_i'(t)x_i(t) \;=\; -\mathbf{m}(x(t),p(t),v(t))\langle p(t),x(t)\rangle$$

imposing instantaneous exchange of allotments of payments among players is allowed only in the extent that the variation $\langle p'(t),x(t)\rangle$ of the payoff of the fuzzy coalition is equal to the payoff discounted by a given factor depending upon the fuzzy coalition, the allotment and, possibly, the perturbation.

Therefore, under this behavioral rule of the decision maker, the velocity of the evolution of the payoff is equal to $y'(t) = \langle p(t), f(x(t),v(t))\rangle - y(t)\mathbf{m}(x(t),p(t),v(t))$.

An important instance is the case when $\mathbf{m}(x,p,v) = 0$, i.e., when allotments are self-financed by the fuzzy coalition x: This means that along the evolutions of the coalitions and the allotments, the velocity of the payoff satisfies $\langle p'(t),x(t)\rangle = 0$.

Hence, the evolution of the coalitions $x(t)$ of the shares, of the allotment and of the payoff $y(t)$ is governed by the two-person dynamical game (3), with which we associate with any feedback $\widetilde{p}$ the set $\mathcal{C}_{\widetilde{p}}(x)$ of triples $(x(\cdot),y(\cdot),v(\cdot))$ solutions to the tychastic equations (4):

$$\begin{cases} i) \quad x'(t) \;=\; f(x(t),v(t)) \\ ii) \quad y'(t) \;=\; \langle \widetilde{p}(x(t)), f(x(t),v(t))\rangle - y(t)\mathbf{m}(x(t),\widetilde{p}(x(t)),v(t)) \\ \qquad \text{where } \; v(t) \in Q(x(t)) \end{cases}$$

By construction, since $y' = \langle p',x\rangle + \langle p,x'\rangle$ whenever $y = \langle p,x\rangle$ in all cases, the constraints

$$y(t) \;=\; \langle p(t),x(t)\rangle$$

are automatically satisfied whenever the initial conditions satisfy $y = \langle p,x\rangle$.

Remark: Path-Dependent Problems — Actually, our study (up to Hamilton-Jacobi-Isaacs partial differential equations) does not depend upon the fact that the evolution of the coalitions is governed by differential equations. It holds true for discretized systems, for path-dependent dynamics, as well as for other kinds of dynamics. We shall use only the properties of the set-valued map $(x, \widetilde{p}) \rightsquigarrow \mathcal{C}_{\widetilde{p}}(x)$, that are shared by solution maps of other dynamical systems.

This study can also adapted to impulsive control systems, allowing to take into account discontinuities in the evolution of fuzzy coalitions, using results of [14, 15, 16, Aubin], [29, Aubin & Haddad], [36, Aubin, Lygeros, Quincampoix, Sastry & Seube], [51, Bensoussan & Menaldi], [67, 68, Cruck], [94, 95, Matveev & Savkin], [111, Saint-Pierre] among many other references. □

1.1 Characteristic Functions of the Cooperative Game

Let us recall that a function $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which $\mathbf{u}$ is finite:

$$\mathrm{Dom}(\mathbf{u}) := \{x \in X \mid \mathbf{u}(x) < +\infty\}$$

that embodies underlying state constraints: in particular, we shall assume that $\mathbf{u}(x) := +\infty$ whenever $x \notin [0,1]^n$ to take into account that x is a fuzzy coalition[13].

Actually, in order to treat the three rules of the game (2) as particular cases of a more general framework, we introduce two nonnegative extended functions $\mathbf{b}$ and $\mathbf{c}$ (characteristic functions of the cooperative games) satisfying

$$\forall\, (t,x) \in \mathbf{R}_+ \times \mathbf{R}^n_+ \times \mathbf{R}^n, \;\; 0 \;\le\; \mathbf{b}(t,x) \;\le\; \mathbf{c}(t,x) \;\le\; +\infty$$

By associating with the initial characteristic function $\mathbf{u}$ of the game adequate pairs $(\mathbf{b}, \mathbf{c})$ of extended functions, we shall replace the requirements (2) by the requirement

$$\left\{ \begin{array}{ll} i) & \forall\, t \in [0,t^\star], \;\; y(t) \ge \mathbf{b}(T-t, x(t)) \;\; \text{(dynamical constraints)} \\ ii) & y(t^\star) \ge \mathbf{c}(T-t^\star, x(t^\star)) \;\; \text{(objective)} \end{array} \right. \tag{9}$$

We extend the functions $\mathbf{b}$ and $\mathbf{c}$ as functions from $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ to $\mathbf{R}_+ \cup \{+\infty\}$ by setting

$$\forall\, t < 0, \;\; \mathbf{b}(t,x) \;=\; \mathbf{c}(t,x) \;=\; +\infty$$

so that nonnegativity constraints on time are automatically taken into account.

[13] If we assume that $\mathbf{u}$ is positively homogenous, it is enough to assume that $\mathbf{u}(x) := +\infty$ whenever $x \notin \mathbf{R}^n_+$.

For instance, *problems with prescribed final time are obtained with objective functions satisfying the condition*

$$\forall\, t > 0,\ \ \mathbf{c}(t,x)\ :=\ +\infty$$

In this case, $t^* = T$ and condition (9) boils down to

$$\begin{cases} i) & \forall\, t \in [0,T],\ \ y(t) \geq b(T-t,x(t)) \\ ii) & y(T) \geq c(0,x(T)) \end{cases}$$

Indeed, since $y(t^*)$ is finite and since $\mathbf{c}(T-t^*,x(t^*))$ is infinite whenever $T-t^*>0$, we infer from inequality (9)ii) that $T-t^*$ must be equal to 0. □

Allowing the characteristic functions to take infinite values (i.e., to be extended), allows us to acclimate many examples.

For example, *the three rules (2) associated with a same characteristic function* $\mathbf{u}:[0,1]^n \mapsto \mathbf{R}\cup\{+\infty\}$ *can be written in the form (9) by adequate choices of pairs* $(\mathbf{b},\mathbf{c})$ *of functions associated with* $\mathbf{u}$. Indeed, denoting by u_∞ the function defined by

$$\mathbf{u}_\infty(t,x)\ :=\ \begin{cases} \mathbf{u}(x) & \text{if} \quad t=0 \\ +\infty & \text{if} \quad t>0 \end{cases}$$

and by $\mathbf{0}$ the function defined by

$$\mathbf{0}(t,x)\ =\ \begin{cases} 0 & \text{if} \quad t\geq 0, \\ +\infty & \text{if not} \end{cases}$$

we can recover the three rules of the game

1. We take $\mathbf{b}(t,x) := \mathbf{0}(t,x)$ and $\mathbf{c}(t,x) = \mathbf{u}_\infty(t,x)$, we obtain the prescribed final time rule (2)i).
2. We take $\mathbf{b}(t,x) := \mathbf{u}(x)$ and $\mathbf{c}(t,x) := \mathbf{u}_\infty(t,x)$, we obtain the span time rule (2)ii).
3. We take $\mathbf{b}(t,x) := \mathbf{0}(t,x)$ and $\mathbf{c}(t,x) = \mathbf{u}(x)$, we obtain the first winning time rule (2)iii).

Using a pair $(\mathbf{b},\mathbf{c})$ of time-dependent extended characteristic functions for describing rules allows to consider more general fuzzy dynamical cooperative games than the ones using time independent characteristic functions. The problem of cooperative dynamical games is now that at each instant $t\in[0,t^*]$, the payoff $y(t) := \langle p(t), x(t)\rangle$ of the fuzzy coalition at time t is larger than or equal to the characteristic function of the dynamical game associating with any time t and any coalition $x(t)$ a lower bound $\mathbf{b}(T-t,x(t))$ on the payoff that the fuzzy coalition $x(t)$ may accept. Furthermore, one can impose final constraint at the end of the game an other lower bound $\mathbf{c}(T-t^*,x(t^*))$ on the payoff that the fuzzy coalition $x(t^*)$ when the game can stop.

1.2 The Valuation Function

By now, we have all the elements for setting the problem we shall study:

Definition 1.1 *Let us consider the dynamical game (3) governing the evolution of the coalitions, the allotment and the payoff.*

1. *the first problem is to find the* guaranteed valuation *subset* $\mathcal{V}^{\sharp}_{(,\mathbf{c})} \subset \mathbf{R}_{+} \times \mathbf{R}^{n} \times \mathbf{R}_{+}$ *of triples* (T, x, y) *made of the final time* T*, the initial fuzzy coalition* x *and the initial payoff* y *such that there exists a feedback* $x \mapsto \widetilde{p}(x) \in P(x)$ *such that, for all perturbations* $t \in [0, T] \mapsto v(t) \in Q(x(t))$*, for all solutions to system (4) of differential equations satisfying* $x(0) = x$*,* $y(0) = y$*, there exists a time* $t^{*} \in [0, T]$ *such that conditions (9):*

$$\begin{cases} i) & \forall\, t \in [0, t^{*}], \;\; y(t) \geq \mathbf{b}(T - t, x(t)) \\ ii) & y(t^{*}) \geq \mathbf{c}(T - t^{*}, x(t^{*})) \end{cases}$$

   are satisfied

2. *associate with any final time* T *and initial coalition* x *the smallest payoff* $V^{\sharp}(T, x)$*:*

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T, x) \;:=\; \inf_{(T,x,y)\in\mathcal{V}^{\sharp}_{(,\mathbf{c})}} y \tag{10}$$

   The function $(T, x) \mapsto V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T, x)$ *is called the* guaranteed valuation function *of the allotment, i.e., the minimal initial payoff* y *satisfying the two constraints (9).*

Formulas (5), (6) and (7) for the valuation functions for each of the three rules of the game (2) that we mentioned in the preceding section are particular cases of the valuation function $V_{(\mathbf{b},\mathbf{c})}$ that we instanced above.

1.3 Formula for the Valuation Function

We shall associate with the objective function $\mathbf{c}$ the functional

$$\begin{cases} J_{\mathbf{c}}(t; (x(\cdot), v(\cdot)); \widetilde{p})(T, x) \;:=\; e^{\int_{0}^{t} \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds}\mathbf{c}(T - t, x(t)) \\ -\displaystyle\int_{0}^{t} e^{\int_{0}^{\tau} \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds} \langle \widetilde{p}(x(\tau)), f(x(\tau), v(\tau))\rangle d\tau \end{cases}$$

(where t ranges over $[0, T]$),

$$I_{\mathbf{b}}(t; (x(\cdot), v(\cdot)); \widetilde{p})(T, x) \;:=\; \sup_{s\in[0,t]} J_{\mathbf{b}}(s; (x(\cdot), v(\cdot)); \widetilde{p})(T, x)$$

and

$$\begin{cases} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \\ := \max\left(J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x), I_{\mathbf{b}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x)\right) \end{cases}$$

We shall prove the

Theorem 1.2 *The guaranteed valuation function* $(T,x,p) \mapsto V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x)$ *is equal to*

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x) \ = \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} \ \inf_{t\in[0,T]} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x)$$

satisfies the initial condition

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(0,x) \ = \ \mathbf{c}(0,x)$$

and inequalities

$$\forall\ (T,x) \in \mathbf{R}_{+}\times\mathbf{R}^{n}\times\mathbf{R}^{n}, \ \ 0 \ \le \ \mathbf{b}(T,x) \ \le \ V^{\sharp}_{(,\mathbf{c})}(T,x) \ \le \ \mathbf{c}(T,x)$$

1.4 Examples of Valuation Functions

Let us consider a given time-independent function $\mathbf{u} : \mathbf{R}^{n} \mapsto \mathbf{R}\cup\{+\infty\}$, with which we associate the functional

$$\begin{cases} J_{\mathbf{u}}(t;(x(\cdot),v(\cdot));\widetilde{p})(x) \ := \ e^{\int_0^t \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds}\mathbf{u}(x(t)) \\ -\displaystyle\int_0^t e^{\int_0^{\tau} \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds}\langle\widetilde{p}(x(\tau)), f(x(\tau),v(\tau))\rangle d\tau \end{cases}$$

We shall associate with it three pairs of time-dependent functions $(\mathbf{b},\mathbf{c})$ and obtain the valuation functions for the three rules of the game:

1. We assume that $f(x,v) \le 0$ and we take $\mathbf{b}(t,x) := 0$ and $\mathbf{c}(t,x) := \mathbf{u}_{\infty}(t,x)$. In this case, we obtain

$$V^{\sharp}_{(\mathbf{0},\mathbf{u}_{\infty})}(T,x) \ := \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} J_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

2. We take $\mathbf{b}(t,x) := \mathbf{u}(x)$ and $\mathbf{c}(t,x) = \mathbf{u}_{\infty}(t,x)$. In this case, we obtain

$$V^{\sharp}_{(\mathbf{u},\mathbf{u}_{\infty})}(T,x) \ := \inf_{\widetilde{p}(x)\in P(x)} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{p}}(x)} \ \sup_{t\in[0,T]} J_{\mathbf{u}}(t;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

3. We assume that $f(x,v) \leq 0$ and we take $\mathbf{b}(t,x) := 0$ and $\mathbf{c}(t,x) = \mathbf{u}(x)$ In this case, we obtain

$$V^{\sharp}_{(\mathbf{0},\mathbf{u})}(T,x) \; := \; \inf_{\widetilde{p}(x)\in P(x)} \; \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} \; \inf_{t\in[0,T]} J_{\mathbf{u}}(t;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

Indeed, when $\mathbf{b} = 0$ and $f(x,v) \leq 0$, we observe that

$$J_{\mathbf{0}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; = \; -\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds} \langle \widetilde{p}(x(\tau)), f(x(\tau),v(\tau))\rangle d\tau$$

so that,

$$\left\{ \begin{array}{l} I_{\mathbf{0}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \\ = \; -\sup_{s\in[0,t]} \int_0^s e^{\int_0^\tau \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds} \langle \widetilde{p}(x(\tau)), f(x(\tau),v(\tau))\rangle d\tau \end{array} \right.$$

If $f(x,v) \leq 0$, we infer that

$$I_{\mathbf{0}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; = \; -\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),\widetilde{p}(x(s)),v(s))ds} \langle \widetilde{p}(x(\tau)), f(x(\tau),v(\tau))\rangle d\tau$$

Therefore, for any nonnegative cost function $\mathbf{c}$, we have $L_{(\mathbf{0},\mathbf{c})} = J_{\mathbf{c}}$ and thus,

$$V^{\sharp}_{(\mathbf{0},\mathbf{c})}(T,x) \; = \; \inf_{\widetilde{p}(x)\in P(x)} \; \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} \; \inf_{t\in[0,T]} J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \tag{11}$$

When $\mathbf{c}(t,x) := \mathbf{u}(x)$, we find the example of the first winning time problem.

When $\mathbf{b}(t,x) := \mathbf{0}(t,x)$ and $\mathbf{c}(t,x) := \mathbf{u}_\infty(t,x)$, the two above remarks imply that

$$V^{\sharp}_{(\mathbf{0},\mathbf{u}_\infty)}(T,x) \; = \; \inf_{\widetilde{u}(x)\in P(x)} \; \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{u}}(x)} J_{\mathbf{u}_\infty}(T;(x(\cdot),v(\cdot));\widetilde{u})(x)$$

When we take $\mathbf{c}(t,x) := \mathbf{u}_\infty(t,x)$ that takes infinite values for $t > 0$, we have seen that

$$J_{\mathbf{u}_\infty}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; := \left\{ \begin{array}{ll} J_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(T,x) & \text{if} \quad t = T \\ +\infty & \text{if} \quad t \in [0,T[\end{array} \right.$$

so that

$$\inf_{t\in[0,T]} J_{\mathbf{u}_\infty}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; := \; J_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

When $\mathbf{b}(t,x) := \mathbf{u}(x)$ and $\mathbf{c}(t,x) := \mathbf{u}_\infty(t,x)$, we infer that

$$L_{(\mathbf{u},\mathbf{u}_\infty)}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; := \left\{ \begin{array}{ll} I_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x) & \text{if} \quad t = T \\ +\infty & \text{if} \quad t < T \end{array} \right.$$

so that

$$\inf_{t\in[0,T]} L_{(\mathbf{u},\mathbf{u})_\infty}(t;(x(\cdot),v(\cdot));\widetilde{p})(T,x) \; = \; I_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

Consequently, we deduce that

$$V^{\sharp}_{(\mathbf{u},\mathbf{u})_\infty}(T,x) \; := \; \inf_{\widetilde{p}(x)\in P(x)} \; \sup_{(x(\cdot),v(\cdot))\in \mathcal{C}_{\widetilde{p}}(x)} I_{\mathbf{u}}(T;(x(\cdot),v(\cdot));\widetilde{p})(x)$$

1.5 Hamilton-Jacobi-Isaacs Variational Inequalities

Let us associate with a nonnegative extended function $\mathbf{v}$ the subset

$$\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}) \ := \ \{(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \quad \text{such that} \\ \mathbf{b}(t,x) \ \leq \ \mathbf{v}(t,x) \ < \ \mathbf{c}(t,x)\}$$

which depends of the function $\mathbf{v}$.

Example When for all $t > 0$, $\mathbf{c}(t,x) := +\infty$, and when $\mathbf{b}(0,x) := \mathbf{c}(0,x)$, we observe that

$$\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}) \ := \ \{(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \quad \text{such that} \ \ t > 0 \ \& \ \mathbf{b}(t,x) \ \leq \ \mathbf{v}(t,x)\} \ \square$$

Then the guaranteed value-function $V^{\sharp}_{(\mathbf{b},\mathbf{c})}$ is a "generalized" solution $\mathbf{v}$ to the Hamilton-Jacobi-Isaacs variational inequality: for every $(t,x) \in \Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$,

$$-\frac{\partial \mathbf{v}(t,x)}{\partial t} + \inf_{p \in P(x)} \sup_{v \in Q(x)} \left(\sum_{i=1}^{n} \left(\frac{\partial \mathbf{v}(t,x)}{\partial x_i} - p_i \right) f_i(x,v) + \mathbf{m}(x,p,v)\mathbf{v}(t,x) \right) \ = \ 0$$

satisfying the initial condition

$$\mathbf{v}(0,x) \ = \ \mathbf{c}(0,x)$$

This is a free boundary problem, well studied in mechanics and physics: the domain $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$ on which we look for a solution $\mathbf{v}$ to the Hamilton-Jacobi partial differential equation depends upon the unknown solution $\mathbf{v}$.

Observe that the Hamilton-Jacobi partial differential equation itself depends only upon the dynamic of the system (f,P,Q) and the map $\mathbf{m}$, whereas the domain $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$ depends only upon the pair $(\mathbf{b},\mathbf{c})$ describing the characteristic functions of the fuzzy dynamical cooperative game. Changing them, the valuation function is a solution of the same Hamilton-Jacobi partial differential equation, but defined on different "free set" $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$ depending on $\mathbf{v}$.

The usefulness and relevance of the Hamilton-Jacobi-Isaacs variational inequality is that it provides the dynamical core of the game — through dynamical feedbacks — that we are looking for. Indeed, we introduce the dynamical core map Γ associating with any $(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n$ the subset $\Gamma(t,x)$ of allotments $p \in P(x)$ satisfying

$$\sup_{v \in Q(x)} \left(\left\langle \frac{\partial V^{\sharp}_{(\mathbf{b},\mathbf{c})}(t,x)}{\partial x} - p, f(x,v) \right\rangle + \mathbf{m}(x,p,v) V^{\sharp}_{(\mathbf{b},\mathbf{c})}(t,x) \right) \ \leq \ \frac{\partial V^{\sharp}_{(\mathbf{b},\mathbf{c})}(t,x)}{\partial t}$$

Namely, knowing the guaranteed valuation function and its derivatives, a guaranteed evolution is obtained in the following way: Starting from an initial fuzzy coalition x_0 such that $\mathbf{b}(T,x_0) \leq V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x_0) < c(T,x_0,p_0)$, solutions to the new system

$$\begin{cases} i) & \forall\, i = 0,\ldots,n,\ \ x_i'(t) \ = \ f_i(x(t),v(t)) \\ ii) & y'(t) \ = \ -y(t)\mathbf{m}(x(t),p(t),v(t)) + \langle p(t), f(x(t),v(t))\rangle \\ iii) & p(t) \ \in \ \Gamma(T-t,x(t)) \end{cases}$$

regulate the guaranteed solutions of the cooperative dynamical game until the first time $t^* \in [0,T]$ when

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T-t^*,x(t^*)) \ = \ \mathbf{c}(T-t^*,x(t^*))$$

Actually, the guaranteed valuation function is seldom differentiable, but, generally only lower semicontinuous: This happens whenever state constraints are involved, i.e., whenever the cost function takes infinite values. However, one can define generalized directional derivatives — contingent epiderivatives — of any lower semicontinuous function. Replacing the classical derivatives by contingent epiderivatives in the Hamilton-Jacobi-Isaacs variational inequalities above and in the definition of the regulation map, the same conclusions hold true for Frankowska's episolutions[14]. In particular, we can still build the dynamical core of the fuzzy dynamic cooperative game. By duality, one can formulate and prove equivalent statements involving subdifferentials and superdifferentials in the "viscosity solution format".

We shall derive from Theorem 4.2 the following

Theorem 1.3 *Let us assume that the maps f, & $\mathbf{m}$ are Lipschitz, that the set-valued maps P and Q are Lipschitz and bounded and that the functions $\mathbf{b}$ and $\mathbf{c}$ are lower semicontinuous. Then*

1. *the dynamic core $\Gamma_{(\mathbf{b},\mathbf{c})}$ of the fuzzy dynamical cooperative with rules defined by $(\mathbf{b},\mathbf{c})$ is equal to*

$$\begin{cases} \Gamma_{(\mathbf{b},\mathbf{c})}(t,x) \ := \ \{p \in P(x) \ \ \text{such that} \\ \sup_{v\in Q(x)} \Big(D_{\uparrow} V^{\sharp}_{(\mathbf{b},\mathbf{c})}(t,x)(-1,f(x,v)) - \langle p, f(x,v)\rangle + \mathbf{m}(x,p,v) V^{\sharp}_{(\mathbf{b},\mathbf{c})}(t,x)\Big) \leq 0\} \end{cases}$$

   defined on $\Omega_{(\mathbf{b},\mathbf{c})}(V_{(\mathbf{b},\mathbf{c})})$,

[14] Hélène Frankowska proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is invariant and backward viable under a (natural) auxiliary system. Furthermore, when it is continuous, she proved that its epigraph is viable and its hypograph invariant ([78, 79, 81, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions. See also [41, Barron & Jensen] and [37, Bardi & Capuzzo-Dolcetta] for more details.

2. *the guaranteed valuation function* $V^{\sharp}_{(\mathbf{b},\mathbf{c})}$ *is the smallest of the lower semicontinuous "episolutions"* $\mathbf{v}$ *to the Hamilton-Jacobi-Isaacs contingent inequalities*

$$\left\{\begin{array}{ll} i) & \mathbf{b}(t,x) \ \leq \ \mathbf{v}(t,x) \ \leq \ \mathbf{c}(t,x) \\ ii) & \text{if } \mathbf{v}(t,x) < \mathbf{c}(t,x), \\ & \inf_{p\in P(x)} \sup_{v\in Q(x)} \left(D_{\uparrow}\mathbf{v}(t,x)(-1,f(x,v)) - \langle p, f(x,v)\rangle + \mathbf{m}(x,p,v)\mathbf{v}(t,x)\right) \leq 0 \end{array}\right. \tag{12}$$

satisfying the initial condition $\mathbf{v}(0,x) = \mathbf{c}(0,x)$ *and such that there exists a Lipschitz selection* $\widetilde{p}$ *of the set-valued map* Γ *defined by*

$$\left\{\begin{array}{l} \Gamma(t,x) \ := \ \{p \in P(x) \quad \textit{such that} \\ \sup_{v\in Q(x)} \left(D_{\uparrow}\mathbf{v}(t,x)(-1,f(x,v)) - \langle p, f(x,v)\rangle + \mathbf{m}(x,p,v)\mathbf{v}(t,x)\right) \leq 0\} \end{array}\right.$$

Remark — We describe only the equivalent dual version of episolutions to the above Hamilton-Jacobi-Isaacs partial differential equation. We introduce the Hamiltonian H defined by

$$H(t,x,p_t,p_x,y) \ := \ -p_t + \inf_{u\in P(x)} \sup_{v\in Q(x)} \left(\langle p_x - u, f(x,v)\rangle + \mathbf{m}(x,u,v)y\right)$$

We recall that the subdifferential $\partial_-\mathbf{v}(t,x)$ of the extended function $\mathbf{v}$ at (t,x) is the set of pairs (p_t,p_x) such that

$$\forall\ (\lambda,v) \in \mathbf{R}\times X, \ \ p_t\lambda + \langle p_x, v\rangle \ \leq \ D_{\uparrow}\mathbf{v}(t,x)(\lambda,v)$$

Hence, the function $\mathbf{v}$ is an episolution of (12) if and only if $\mathbf{v}$ satisfies

$$\forall\ (t,x) \in \Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}), \ \ \forall\ (p_t,p_x) \in \partial_-\mathbf{v}(t,x), \ \ H(t,x,p_t,p_x,\mathbf{v}(t,x)) \ \leq \ 0 \ \ \square$$

The proofs of the above results require a more abstract geometric approach that we are now describing.

2 The Viability/Capturability Strategy

2.1 Epigraphs of Extended Functions

The *epigraph* of an extended function $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty$ is defined by

$$\mathcal{E}p(\mathbf{v}) \ := \ \{(x,\lambda) \in X \times \mathbf{R} \mid \mathbf{v}(x) \leq \lambda\}$$

We recall that *an extended function* $\mathbf{v}$ *is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone)* and that the epigraph of $\mathbf{v}$ is closed if and only if $\mathbf{v}$ is lower semicontinuous:

$$\forall\ x \in X, \ \ \mathbf{v}(x) \ = \ \liminf_{y\to x} \mathbf{v}(y)$$

The definition of the guaranteed valuation function $V^{\sharp}_{(\mathbf{b},\mathbf{c})}$ from the guaranteed valuation subset $\mathcal{V}^{\sharp}_{(,\mathbf{c})}$ fits the following definition:

Definition 2.1 *We associate with a subset $\mathcal{V} \subset X \times \mathbf{R}_+$ the function $\mathbf{v}_{\mathcal{V}} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\mathbf{v}_{\mathcal{V}}(x) \;:=\; \inf_{(x,w)\in\mathcal{V}} w \;\in\; \overline{\mathbf{R}}$$

that we shall call its southern border*.*

We shall say that $[\mathcal{V}]^{\uparrow} := \mathcal{E}p(\mathbf{v}_{\mathcal{V}})$ is the southern closure[15] *of $\mathcal{V}$.*

We recall the convention $\inf(\emptyset) := +\infty$.

We observe that

$$\mathcal{V} + \{0\} \times \mathbf{R}_+ \;\subset\; \mathcal{E}p(\mathbf{v}_{\mathcal{V}}) \;\subset\; \overline{\mathcal{V} + \{0\} \times \mathbf{R}_+}$$

and that if $\mathcal{V} \subset X \times \mathbf{R}_+$ is a closed subset, then its southern closure $\mathbf{v}_{\mathcal{V}}$ is lower semicontinuous and the three above sets are closed and equal.

We shall need the following

Lemma 2.2 *Let $\mathcal{V}_{i\in I}$ be a family of subsets $\mathcal{V}_i \subset X \times \mathbf{R}$. Then the southern border of the union of the $\mathcal{V}_i$ is the infimum of the southern borders of the set $\mathcal{V}_i$:*

$$\mathbf{v}_{\bigcup_{i\in I}\mathcal{V}_i} \;=\; \inf_{i\in I} \mathbf{v}_{\mathcal{V}_i}$$

or, equivalently,

$$\mathcal{E}p(\inf_{i\in I} \mathbf{v}_{\mathcal{V}_i}) \;=\; \left[\bigcup_{i\in I} \mathcal{V}_i\right]^{\uparrow}$$

In particular, the epigraph of the pointwise infimum $\inf_{i\in I} \mathbf{v}_i$ of a family of functions $\mathbf{v}_i$ is the southern closure of the union of their epigraphs:

$$\mathcal{E}p(\inf_{i\in I} \mathbf{v}_i) \;=\; \left[\bigcup_{i\in I} \mathcal{E}p(\mathbf{v}_i)\right]^{\uparrow}$$

Proof — Indeed,

$$\mathbf{v}_{\bigcup_{i\in I}\mathcal{V}_i}(x) \;=\; \inf_{(x,y)\in\bigcup_{i\in I}\mathcal{V}_i} y \;=\; \inf_{i\in I} \inf_{(x,y)\in\mathcal{V}_i} y \;=\; \inf_{i\in I} \mathbf{v}_{\mathcal{V}_i}(x)$$

[15] When $\mathcal{V} = \mathcal{V} + \{0\} \times \mathbf{R}_+$, this is the vertical closure introduce in [108, Rockafellar & Wets].

2.2 The Epigraphical Approach

With these definitions, we can translate the viability/capturability conditions (9) in the following geometric form:

$$\left\{\begin{array}{ll} i) & \forall\, t \in [0, t^*],\ \ (T-t, x(t), y(t)) \in \mathcal{E}p(\mathbf{b}) \\ & \text{(viability constraint)} \\ ii) & (T-t^*, x(t^*), y(t^*)) \in \mathcal{E}p(\mathbf{c}) \\ & \text{(capturability of a target)} \end{array}\right. \tag{13}$$

This "epigraphical approach" proposed by J.-J. Moreau and R.T. Rockafellar in convex analysis in the early 60's [16], has been used in optimal control by H. Frankowska in a series of papers [78, 79, 81, Frankowska] and [28, Aubin & Frankowska] for studying the value function of optimal control problems and characterize it as generalized solution (episolutions and/or viscosity solutions) of (first-order) Hamilton-Jacobi-Bellman equations, in [6, 22, 8, 12, Aubin] for characterizing and constructing Lyapunov functions, in [62, 63, 64, 65, Cardaliaguet] for characterizing the minimal time function, in [102, Pujal] and [34, Aubin, Pujal & Saint-Pierre] in finance and other authors since. This is this approach that we adopt and adapt here, since the concepts of "capturability of a target" and of "viability" of a constrained set allows us to study this problem under a new light (see for instance [12, Aubin] and [13, Aubin] for economic applications) for studying the evolution of the state of a tychastic control system subjected to viability constraints in control theory and in dynamical games against nature or robust control (see [103, Quincampoix], [62, 63, 64, 65, Cardaliaguet], [66, Cardaliaguet, Quincampoix & Saint-Pierre]. Numerical algorithms for finding viability kernels have been designed in [110, Saint-Pierre] and adapted to our type of problems in [102, Pujal].

The properties and characterizations of the valuation function are thus derived from the ones of guaranteed viable-capture basins, that are easier to study — and that have been studied — in the framework of plain constrained sets K and targets $C \subset K$ (see [18, 19, Aubin] and [21, Aubin & Catté] for recent results on that topic).

2.3 Introducing Auxiliary Dynamical Games

We observe that the evolution of $(T-t, x(t), y(t))$ made up of the backward time $\tau(t) := T-t$, of fuzzy coalitions $x(t)$ of the players, of allotments and of the payoff $y(t)$ is governed by the dynamical game

$$\left\{\begin{array}{ll} i) & \tau'(t) \ = \ -1 \\ ii) & \forall\, i = 0, \ldots, n,\ \ x_i'(t) \ = \ f_i(x(t), v(t)) \\ iii) & y'(t) \ = \ -y(t)\mathbf{m}(x(t), p(t), v(t)) + \langle p(t), f(x(t), v(t))\rangle \\ & \text{where } p(t) \ \in \ P(x(t)) \ \& \ v(t) \ \in \ Q(x(t)) \end{array}\right. \tag{14}$$

[16] see for instance [27, Aubin & Frankowska] and [108, Rockafellar & Wets] among many other references.

starting at (T, x, y). We summarize it in the form of the dynamical game

$$\begin{cases} i) & z'(t) \in g(z(t), u(t), v(t)) \\ ii) & u(t) \in P(z(t)) \; \& \; v(t) \in Q(z(t)) \end{cases}$$

where $z := (\tau, x, y) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}$, where the controls $u := p$ are the allotments, where the map $g : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \rightsquigarrow \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ is defined by $g(z, v)$

$$= (-1, f(x, v), u, -\mathbf{m}(x, u, v)y + \langle u, f(x, v)\rangle)$$

where u ranges over $P(z) := P(x)$ and v over $Q(z) := Q(x)$.

We say that a selection $z \mapsto \widetilde{p}(z) \in P(z)$ is a feedback, regarded as a strategy. One associates with such a feedback chosen by the decision maker or the player the evolutions governed by the tychastic differential equation

$$z'(t) \;=\; g(z(t), \widetilde{p}(z(t)), v(t))$$

starting at time 0 at z.

2.4 Introducing Guaranteed Capture Basins

We now define the guaranteed viable-capture basin that are involved in the definition of guaranteed valuation subsets.

Definition 2.3 *Let K and $C \subset K$ be two subsets of Z.*

The guaranteed viable-capture basin *of the target C viable in K is the set of elements $z \in K$ such that there exists a continuous feedback $\widetilde{p}(z) \in P(z)$ such that for every $v(\cdot) \in Q(z(\cdot))$, for every solutions $z(\cdot)$ to $z' = g(z, \widetilde{p}(z), v)$, there exists $t^\star \in \mathbf{R}_+$ such that the viability/capturability conditions*

$$\begin{cases} i) & \forall\, t \in [0, t^\star], \quad z(t) \in K \\ ii) & \qquad\qquad z(t^\star) \in C \end{cases}$$

are satisfied.

We thus observe that

Proposition 2.4 *The guaranteed valuation subset $\mathcal{V}^\sharp_{(,\mathbf{c})}$ defined in Definition 1.1 is the southern border of the guaranteed viable-capture basin under the dynamical game (14) of the epigraph of the function $\mathbf{c}$ viable in the epigraph of the function $\mathbf{b}$.*

The characterization of this subset and the study of its properties are one of the major topics of the viability approach to dynamical games theory that we summarize in the two next sections.

2.5 The Strategy

Since we have related the guaranteed valuation problem to the much simpler — although more abstract — study of guaranteed viable-capture basin of a target and other guaranteed viability/capturability issues for dynamical games,

1. we first "solve" these "viability/capturability problems" for dynamical games at this general level, and in particular, study the tangential conditions enjoyed by the guaranteed viable-capture basins (see Theorem 4.1 below),

2. and use set-valued analysis and nonsmooth analysis for translating the general results of viability theory to the corresponding results of the auxiliary dynamical game, in particular translating tangential conditions to give a meaning to the concept of a generalized solution (Frankowska's episolutions or, by duality, viscosity solutions) to Hamilton-Jacobi-Isaacs variational inequalities (see Theorems 3.2 and 4.2 below).

3 Guaranteed Viability/Capturability under Dynamical Games

3.1 Guaranteed Viable-Capture Basins

We summarize the main results on guaranteed viability/capturability of a target under dynamical games that we need to prove the results stated in the preceding section.

We denote by X, $\mathcal{U}$ and $\mathcal{V}$ three finite dimensional vector spaces, and we introduce a set-valued map $F : X \times \mathcal{U} \times \mathcal{V} \rightsquigarrow X$, a set-valued map $P : X \rightsquigarrow \mathcal{U}$ and a set-valued map $Q : X \rightsquigarrow \mathcal{V}$.

We consider a dynamical game described by

$$\left\{ \begin{array}{lll} i) & x'(t) & \in & F(x(t), u(t), v(t))) \\ ii) & u(t) & \in & P(x(t)) \\ iii) & v(t) & \in & Q(x(t)) \end{array} \right. \tag{15}$$

which is, so to speak, a control system regulated by two parameters, $u(t)$ and $v(t)$, the first one regarded as a regulating parameter, controlled by a player, the second one regarded as a perturbation, or a disturbance, or a tyche, chosen in a unknown way by "Nature".

We introduce a class $\widetilde{\mathcal{P}}$ of continuous selections $x \mapsto \widetilde{u}(x) \in P(x)$, that are used as feedbacks or strategies by the player controlling the parameters u.

We associate with such a feedback $\widetilde{u}(x) \in P(x)$ the set $\mathcal{C}_{\widetilde{u}}(x)$ of solutions $(x(\cdot), v(\cdot)) \in \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ to the parametrized system

$$\left\{ \begin{array}{llll} i) & x'(t) & \in & F(x(t), \widetilde{u}(x(t))), v(t)) \\ ii) & v(t) & \in & Q(x(t)) \end{array} \right. \tag{16}$$

starting at x.

We may identify the above dynamical game with the set-valued map $(x, \widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$, that we regard as an evolutionary game.

Definition 3.1 *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set.*

The subset $\mathrm{Abs}_{\widetilde{u}}(K, C)$ of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by all solutions to (16) starting at x_0 is called the invariance-absorption basin *of C in K.*

The subset

$$[\mathrm{Capt}_P \mathrm{Abs}_Q](K, C) \ := \ \bigcup_{\widetilde{u} \in \widetilde{\mathcal{P}}} \mathrm{Abs}_{\widetilde{u}}(K, C)$$

of elements $x \in K$ such that there exists a feedback $\widetilde{u} \in \widetilde{\mathcal{P}}$ such that for every solutions $(x(\cdot), v(\cdot)) \in \mathcal{C}_{\widetilde{u}}(x)$, there exists $t^* \in \mathbf{R}_+$ satisfying the viability/capturability conditions

$$\begin{cases} i) & \forall\, t \in [0, t^*], \quad x(t) \in K \\ ii) & \qquad\qquad\qquad x(t^*) \ \in \ C \end{cases}$$

is called the guaranteed viable-capture basin of a target under the evolutionary game $(x, \widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$ *defined on $X \times \widetilde{\mathcal{P}}$ (that, naturally, depends upon the choice of the family $\widetilde{\mathcal{P}}$ of feedbacks).*

3.2 Intertemporal Games

We introduce the following four features:

1. a discount factor
$$\mathbf{m} : (x, u, v) \in X \times \mathcal{U} \times \mathcal{V} \mapsto \mathbf{m}(x, u, v) \in \mathbf{R}$$

2. a "Lagrangian"
$$\mathbf{l} : (x, u, v) \in X \times \mathcal{U} \times \mathcal{V} \mapsto \mathbf{l}(x, u, v) \in \mathbf{R}_+$$

3. two nonnegative extended cost functions $\mathbf{b}$ and $\mathbf{c}$ from $\mathbf{R}_+ \times X$ to $\mathbf{R}_+ \cup \{+\infty\}$ satisfying
$$\forall\ (t, x) \in \mathbf{R}_+ \times X, \ \ 0 \ \le \ \mathbf{b}(t, x) \ \le \ \mathbf{c}(t, x) \ \le \ +\infty$$
that we shall extend to cost functions (apayoff denoted by) $\mathbf{b}$ (constrained function) and $\mathbf{c}$ (objective function) from $\mathbf{R} \times X$ to $\mathbf{R}_+ \cup \{+\infty\}$ by setting
$$\mathbf{b}(t, x) \ = \ \mathbf{c}(t, x) \ := \ +\infty \ \text{ whenever } \ t \ < \ 0$$

We next fix an horizon or an final time T and associate with it the cost functionals

$$\begin{cases} J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \ := \ e^{\int_0^t \mathbf{m}(x(s),\widetilde{u}(x(s)),v(s))ds}\mathbf{c}(T-t,x(t)) \\ + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),\widetilde{u}(x(s)),v(s))ds}\mathbf{l}(x(\tau),\widetilde{u}(x(\tau)),v(\tau))d\tau \end{cases}$$

(where t ranges over $[0,T]$),

$$I_{\mathbf{b}}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \ := \ \sup_{s\in[0,t]} J_{\mathbf{b}}(s;(x(\cdot),v(\cdot));\widetilde{u})(T,x)$$

and

$$L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \ := \ \max\left(J_{\mathbf{c}}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x), I_{\mathbf{b}}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x)\right)$$

We associate with it the guaranteed valuation function

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x) \ := \ \inf_{\widetilde{u}\in\widetilde{\mathcal{P}}} \ \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{u}}(x)} \ \inf_{t\in[0,T]} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \tag{17}$$

The function $V^{\sharp}_{(\mathbf{b},\mathbf{c})}$ is called the guaranteed valuation function associated with $\mathbf{l}$, $\mathbf{m}$ and the cost functions $\mathbf{b}$ and $\mathbf{c}$.

Let us consider the extended dynamical game of the form:

$$\begin{cases} i) & \tau'(t) \ = \ -1 \\ ii) & x'(t) \ \in F(x(t),u(t),v(t)) \\ iii) & y'(t) \ = \ -y(t)\mathbf{m}(x(t),u(t),v(t)) - \mathbf{l}(x(t),u(t),v(t)) \\ & \text{where } u(t) \ \in \ P(x(t)) \ \& \ v(t) \ \in \ Q(x(t)) \end{cases} \tag{18}$$

We associate with such a feedback $\widetilde{u}(x)\in P(x)$ the set $\mathcal{B}_{\widetilde{u}}(T,x,y)$ of solutions $(T-\cdot,x(\cdot),v(\cdot),y(\cdot))$ to the auxiliary system

$$\begin{cases} i) & \tau'(t) \ = \ -1 \\ ii) & x'(t) \ \in F(x(t),\widetilde{u}(x(t)),v(t)) \\ iii) & y'(t) \ = \ -y(t)\mathbf{m}(x(t),\widetilde{u}(x(t)),v(t)) - \mathbf{l}(x(t),\widetilde{u}(x(t)),v(t)) \\ & \text{where } v(t) \ \in \ Q(x(t)) \end{cases}$$

Theorem 3.2 *Let us assume that the extended functions* $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial and non negative.*

The guaranteed valuation function $V^{\sharp}_{(\mathbf{b},\mathbf{c})}$ *defined by (17) is the southern border of the guaranteed viable-capture basin* $[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})$ *of the epigraph* $\mathcal{E}p(\mathbf{c})$ *of* $\mathbf{c}$ *under the dynamical game (18) viable in the epigraph* $\mathcal{E}p(\mathbf{b})$ *of* $\mathbf{b}$*:*

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x) \ := \ \inf_{(x,T,y)\in[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})} y$$

In other words, Theorem 3.2 states that

$$\mathcal{E}p(V^{\sharp}_{(\mathbf{b},\mathbf{c})}) \;=\; [[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})]^{\uparrow}$$

Since the guaranteed viable-capture basin

$$[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}) \;:=\; \bigcup_{\widetilde{u}\in\widetilde{\mathcal{P}}} \mathrm{Abs}_{\widetilde{u}}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))$$

Lemma 2.2 implies that the southern border

$$V^{\sharp}_{(\mathbf{b},\mathbf{c})}(T,x) \;:=\; \inf_{(x,T,y)\in[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})} y$$

of $[\mathrm{Capt}_P\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})$ is the pointwise infimum

$$V^{\sharp}(T,x) \;=\; \inf_{\widetilde{u}\in\widetilde{\mathcal{P}}} U_{(\mathbf{b},\mathbf{c});\widetilde{u})}(T,x)$$

of the southern borders

$$U_{(\mathbf{b},\mathbf{c};\widetilde{u})}(T,x) \;:=\; \inf_{(x,T,y)\in\mathrm{Abs}_{\widetilde{u}}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c})} y$$

It remains to prove that

$$U_{(\mathbf{b},\mathbf{c};\widetilde{u})}(T,x) \;=\; \sup_{(x(\cdot),v(\cdot))\in\mathcal{C}_{\widetilde{u}}(x)} \inf_{t\in[0,T]} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \tag{19}$$

to derive the Theorem 3.2.

This is purpose of

Theorem 3.3 *Let us assume that the extended functions* $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial and non negative.*

The valuation function $U_{(\mathbf{b},\mathbf{c};\widetilde{u})}$ *is equal to the southern border of the invariant-absorption basin* $\mathrm{Abs}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))$ *of* $\mathcal{E}p(\mathbf{c})$*:*

$$U_{(\mathbf{b},\mathbf{c};\widetilde{u})}(T,x) \;:=\; \inf_{(x,T,y)\in\mathrm{Abs}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))} y$$

We refer to [34, Aubin, Pujal & Saint-Pierre] for the proof of this theorem.

4 Hamilton-Jacobi-Isaacs Equations

4.1 Lipschitz Dynamical Games

We shall assume that the dynamical game (15) is Lipschitz in the sense that the set-valued maps P and Q are Lipschitz with compact values and that the set-valued map F is Lipschitz with closed values.

Let $\widetilde{\mathcal{P}}_\lambda$ be the set of Lipschitz selections with constant λ of the set-valued map P: for every x, $\widetilde{u}(x) \in P(x)$.

The subset

$$[\mathrm{Capt}_{P_\lambda}\mathrm{Abs}_Q](K,C) \;:=\; \bigcup_{\widetilde{u}\in\widetilde{\mathcal{P}}_\lambda} \mathrm{Abs}_{\widetilde{u}}(K,C)$$

is called the λ-guaranteed viable-capture basin of a target under the evolutionary game $(x,\widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$.

One can prove that when the game is Lipschitz, the set-valued map $(x,\widetilde{u}) \in X \times \widetilde{\mathcal{P}}_\lambda \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x) \subset \mathcal{C}(0,\infty;X)$ is lower semicontinuous and consequently, that the λ-guaranteed viable-capture basin is closed.

We recall that the contingent cone to a subset K at a point $x \in K$, introduced in the early thirties independently by Bouligand and Severi, adapts to any subset the concept of tangent space to manifolds: A direction $v \in X$ belongs to $T_K(x)$ if there exist sequences $h_n > 0$ and $v_n \in X$ converging to 0 and v respectively such that

$$\forall\, n \geq 0,\;\; x + h_n v_n \;\in\; K$$

Using the Viability and the Invariance Theorems, one can prove the following tangential properties of guaranteed viability kernels with targets:

Theorem 4.1 *Let us assume that the dynamical game (P,Q,F) is Lipschitz, that $C \subset K$ and K are closed subsets of X and that $K\backslash C$ is a repeller under all the maps $(x,\widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$.*

Then the λ-guaranteed viable-capture basin $[\mathrm{Capt}_{P_\lambda}\mathrm{Abs}_Q](K,C)$ of target C viable in K is the largest of the closed subsets D satisfying $C \subset D \subset K$ and

1. *the tangential property*[17]

[17]or, the equivalent dual formulation,

$$\forall\, x \in D\backslash C,\; \forall\, p \in N_D(x),\;\; \inf_{u\in P(x)} \sup_{v\in Q(x)} \sigma(F(x,u,v),p) \;\leq\; 0$$

where the (regular) normal cone $N_D(x) := T_D(x)^-$ is the polar cone to the contingent cone $T_D(x)$ and where

$$\forall\, p \in X^\star,\;\; \sigma(F,p) \;:=\; \sup_{x\in F}\langle p,x\rangle$$

is the support function of F.

$$\forall\, x \in D\backslash C,\ \exists\, u \in P(x) \ \ \textit{such that}\, \forall\, v \in Q(x), \ \ F(x,u,v) \subset T_D(x) \qquad (20)$$

2. there exists a λ-Lipschitz selection of the guaranteed regulation map Γ_D defined by

$$\forall\, x \in D\backslash C, \ \ \Gamma_D(x) \ := \ \{u \in P(x) \mid F(x,u,Q(x)) \subset T_D(x)\}$$

This theorem is a restatement of Theorems 9.2.14 and 9¿2.18 of [13, Aubin, Chapter 9].

4.2 Hamilton-Jacobi-Isaacs Variational Inequalities

Let us recall that the contingent epiderivative $D_\uparrow \mathbf{v}(t,x)$ of $\mathbf{v}$ at (t,x) satisfies the property:

$$\mathcal{E}p(D_\uparrow \mathbf{v}(t,x)) \ = \ T_{\mathcal{E}p(\mathbf{v})}(t,x,\mathbf{v}(t,x))$$

Since the λ-guaranteed viable-capture basin is closed under Lipschitz equations, then its southern border, which is the λ-guaranteed valuation function

$$V^\sharp_{(\mathbf{b},\mathbf{c})_\lambda}(T,x) \ := \ \inf_{\widetilde{u} \in \widetilde{\mathcal{P}}_\lambda} \ \sup_{(x(\cdot),v(\cdot)) \in \mathcal{C}_{\widetilde{u}}(x)} \ \inf_{t \in [0,T]} L_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),v(\cdot));\widetilde{u})(T,x) \qquad (21)$$

is lower semicontinuous and thus, its epigraph coincides with the λ-guaranteed viable-capture basin.

Theorem 4.2 *Let us assume that the dynamical game (18) is Lipschitz and that the cost functions $\mathbf{b}$ and $\mathbf{c}$ from $\mathbf{R}_+ \times X$ to $\mathbf{R}_+ \cup \{+\infty\}$ are nontrivial, nonnegative and lower semicontinuous. Then the λ-guaranteed valuation function $V^\sharp_{(\mathbf{b},\mathbf{c})_\lambda}$ under the dynamical game (15) is the smallest of the nonnegative lower semicontinuous solutions $\mathbf{v}$ to the Hamilton-Jacobi-Isaacs contingent inequalities*

$$\begin{cases} i) & \mathbf{b}(t,x) \ \le \ \mathbf{v}(t,x) \ \le \ \mathbf{c}(t,x) \\ ii) & \text{if } \mathbf{v}(t,x) < \mathbf{c}(t,x), \\ & \inf_{u \in P(x)} \sup_{w \in F(x,u,Q(x))} \left(D_\uparrow \mathbf{v}(t,x)(-1,w) + \mathbf{l}(x,u,v) + \mathbf{m}(x,u,v)\mathbf{v}(t,x)\right) \\ & \le \ 0 \end{cases}$$

such that there exists a λ-Lipschitz selection $\widetilde{u}$ of the guaranteed regulation map Γ defined by

$$\begin{cases} \Gamma(t,x) \\ := \ \{u \in P(x) \mid \sup_{w \in F(x,u,Q(x))} \left(D_\uparrow \mathbf{v}(t,x)(-1,w) + \mathbf{l}(x,u,v) + \mathbf{m}(x,u,v)\mathbf{v}(t,x)\right) \ \le \ 0\} \end{cases}$$

Proof — It is a consequence of Theorem 4.1 when $K := \mathcal{E}p(\mathbf{b})$, $C := \mathcal{E}p(\mathbf{c})$ and when the dynamical game is the extended dynamical game (18).

Theorem 4.1 states that the λ-guaranteed viable-capture basin

$$[\mathrm{Capt}_{P_\lambda}\mathrm{Abs}_Q](\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))$$

under (18) of the epigraph $\mathcal{E}p(\mathbf{c})$ of $\mathbf{c}$ viable in the epigraph $\mathcal{E}p(\mathbf{b})$ of $\mathbf{b}$ is the largest of the closed subsets $\mathcal{U}$ satisfying $\mathcal{E}p(\mathbf{b}) \subset \mathcal{U} \subset \mathcal{E}p(\mathbf{c})$, the tangential conditions

$$\left\{ \begin{array}{l} \forall\,(t,x,y) \in \mathcal{U}\backslash\mathcal{E}p(\mathbf{c}), \exists\, u \in P(x) \text{ such that } \forall\, w \in F(x,u,Q(x)), \\ (-1,w,-\mathbf{m}(x,u,v)y - \mathbf{l}(x,u,v)) \;\in\; T_{\mathcal{U}}(t,x,y) \end{array} \right. \tag{22}$$

and such that there exists a λ-Lipschitz selection of the guaranteed regulation map $\Gamma_{\mathcal{U}}$ defined by

$$\Gamma_{\mathcal{U}}(t,x) \;:=\; \{u \in P(x) \mid \{-1\} \times F(x,u,v) \times \{-\mathbf{m}(x,u,v)y - \mathbf{l}(x,u,v)\} \cap T_{\mathcal{U}}(t,x,y) \neq \emptyset\}$$

Let $(t,x) \mapsto \mathbf{v}(t,x)$ be the southern border of $\mathcal{U}$, that satisfies $\mathcal{U} = \mathcal{E}p(\mathbf{v})$ since $\mathcal{U}$ is closed. When $y := \mathbf{v}(t,x)$, the above condition (22) reads

$$D_\uparrow \mathbf{v}(t,x)(-1,w) \;\le\; -\mathbf{m}(x,u,v)\mathbf{v}(t,x) - \mathbf{l}(x,u,v)$$

because

$$T_{\mathcal{U}}(t,x,\mathbf{v}(t,x)) \;=\; \mathcal{E}p(D_\uparrow \mathbf{v}(t,x))$$

Conversely, this condition implies the tangential condition (22) for $y := \mathbf{v}(t,x)$ whenever $(t,x,\mathbf{v}(t,x))$ belongs to $\mathcal{U}$. Otherwise, let $(t,x,y) \in \mathcal{U}$ with $y > \mathbf{v}(t,x)$ and set $\lambda := D_\uparrow \mathbf{v}(t,x)(-1,w)$.

By definition of $\lambda := D_\uparrow \mathbf{v}(t,x)(-1,w)$, there exist sequences $h_n > 0$ converging to 0, w_n converging to w and λ_n converging to λ such that $(t-h_n, x+h_n w_n, \mathbf{v}(t,x)+h_n\lambda_n)$ belongs to $\mathcal{E}p(\mathbf{v})$. Therefore, for $\mu \in \mathbf{R}$ and h_n small enough,

$$(t-h_n, x+h_n w_n, y+h_n\mu) \;=\; (t-h_n, x+h_n w_n, \mathbf{v}(t,x)+h_n\lambda_n) + (0,0,y-\mathbf{v}(t,x)+h_n(\mu-\lambda_n))$$

belongs to $\mathcal{E}p(\mathbf{v})$ because $y - \mathbf{v}(t,x)$ is strictly positive. This implies that $(-1,w,\mu)$ belongs to the contingent cone to the epigraph $\mathcal{U}$ of $\mathbf{v}$ at (t,x,y), so that tangential condition (22) is satisfied with $\mu := -\mathbf{m}(x,u,v)y - \mathbf{l}(x,u,v)$. $\square$

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