Characterization of Stochastic Viability of any Nonsmooth Set

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Stochastic Differential Equations

Let us consider a σ–complete probability space $(\Omega, \mathcal{F}, P)$, an increasing family $\mathcal{F}$ of σ–sub–algebras $\mathcal{F}_t \subset \mathcal{F}$, two finite dimensional vector-spaces $X := R^n$, $Y := R^m$ and W a Y-valued Wiener measure.

An Itô process is a solution to the stochastic differential equations

$$dx \;=\; f(x(t))dt + \sigma(x(t))dW(t)$$

starting at ξ_0 if

$$\xi(t) \;:=\; \xi_0 + \int_0^t f(\xi(s))ds + \int_0^t \sigma(\xi(s))dW(s)$$

For simplicity, we shall denote by $\mathsf{S}(\xi_0)$ the Itô that is the solution to the above stochastic differential equation starting at ξ_0.

Contingent Curvature

The contingent derivative $DN_K(x,p)$ **of** $x \rightsquigarrow N_K(x)$ **at a point** $(x,p) \in \mathrm{Graph}(N_K)$ **is defined by:**

$dp \in DN_K(x,p)(dx)$ **if and only if** $dx \in T_K(x)$ **and if there exist** $h_n \to 0+$**,** $dx_n \to dx$ **and** $dp_n \to dp$ **such that**

$$\forall\, n \geq 0, \;\; p + h_n dp_n \;\in\; N_K(x + h_n dx_n)$$

The contingent curvature $\mathrm{Curv}_K(x,p)$ **of** K **at a point** $(x,p) \in \mathrm{Graph}(N_K)$ **is defined by:** $\forall\, u, v \in T_K(x)$**,**

$$\mathrm{Curv}_K(x,p)(u,v) \;:=\; \sup_{dp \in DN_K(x,p)(u)} \langle dp, v\rangle$$

When $K := \{x \in X \mid \varphi(x) \leq 0\}$ **and** $\Phi(x) = 0$**, then** $N_K(x) := \varphi'(x)R$ **and**

$$\mathrm{Curv}_K(x, \varphi'(x))(u, v) \; := \; \varphi''(x)(u, v)$$

Doss's Theorem: Assumptions

We associate the Stratonovitch drift $\widehat{s}(\gamma,\sigma)$ defined by

$$\widehat{s}(\gamma,\sigma)(x) \ := \ \gamma(x) - \frac{1}{2}\sum_{i=1}^{n} D\sigma^i(x)\sigma^i(x)$$

Let $K \subset X$ be a closed subset, $N_K(x)$ the normal cone to K at x and $DN_K(x,p)$ its contingent derivative at $p \in N_K(x)$.

Let us assume that the drift γ is Lipschitz around K and bounded on K and that the three first derivatives of the diffusions σ^i exist and are bounded on K.

Doss's Theorem: Conclusions

The following conditions are equivalent:

1. K **is invariant under the stochastic differential equation** $d\xi = \gamma(\xi)dt + \sigma(\xi)dW(t)$**,**
2. K **is invariant under the differential inclusion** $x'(t) \in \widehat{s}(\gamma,\sigma)(x(t)) + \mathrm{Im}(\sigma(x(t)))$**,**
3. K **satisfies the first-order conditions:**

$$\forall \ \in K, \ \ \forall\, p \in N_K(x),$$

$$\begin{cases} i) & \langle p, \sigma^i(x)\rangle \ = \ 0, \ \ i = 1, \ldots, n \\ ii) & \langle p, \gamma(x)\rangle - \frac{1}{2}\sum_{i=1}^{n}\langle p, D\sigma^i(x)\sigma^i(x)\rangle \leq 0 \end{cases}$$

4. K **satisfies the second-order conditions:**

$$\forall \ \in K, \ \ \forall\, p \in N_K(x),$$

$$\left\{ \begin{array}{lll} i) & \langle p, \sigma^i(x)\rangle \ = \ 0, \quad i = 1, \ldots, n \\ ii) & \langle p, \gamma(x)\rangle \\ & +\frac{1}{2}\sum_{i=1}^{n} \mathrm{Curv}_K(x,p)(\sigma^i(x), \sigma^i(x)) \leq 0 \end{array} \right.$$

Doss's Lemma

Let us associate with a drift γ and continuously differentiable diffusions σ^i the Stratonovitch drift $\widehat{s}(\gamma,\sigma)$

$$\widehat{s}(\gamma,\sigma)(x) \ := \ \gamma(x) - \frac{1}{2}\sum_{i=1}^{n} D\sigma^i(x)\sigma^i(x)$$

Let K be a subset, $N_K(x)$ the normal cone to K at x and $DN_K(x,p)$ its contingent derivative at $p \in N_K(x)$. Let us assume that

$$\forall\, x \in K,\ \forall\, p \in N_K(x),\ \ \langle p, \sigma^i(x)\rangle \ = \ 0,\ \ i = 1,\ldots,n$$

Then the two following conditions are equivalent:

$$\begin{cases} 1) & \forall\, x \in K,\ \forall\, p \in N_K(x),\ \ \langle p, \widehat{s}(\gamma,\sigma)(x)\rangle \ \leq 0 \\ 2) & \forall\, x \in K,\ \forall\, p \in N_K(x), \\ & \forall\, dp_i \in DN_K(x,p)(\sigma^i(x)), \\ & \langle p, \gamma(x)\rangle + \frac{1}{2}\sum_{i=1}^{n}\langle dp_i, \sigma^i(x)\rangle \leq 0 \end{cases}$$

Curvature and Epi-Hessian

Let $v : X \mapsto R \cup \{+\infty\}$ **be a nontrivial extended function. We set**

$$\partial_-^2 v(x,p) \;:=\; D(\partial_- v)(x,p)$$

and we denote by

$$\mathrm{Hess}_{\uparrow}(v(x,p))(u,v) \;:=\; \sup_{dp \in \partial_-^2 v(x,p)(u)} \langle dp, v \rangle$$

the epi-Hessian of v **at** $(x,p) \in \mathrm{Graph}(N_K)$**.**

Therefore, $\forall\, p \in \partial_- v(x)$,

$$\mathrm{Curv}_{\mathcal{E}p(v)}((x,v(x)),(p,-1))((u,D_{\uparrow}v(x)(u)),(v,\nu))$$

is equal to

$$\begin{cases} \mathrm{Hess}_{\uparrow}(v(x,p))(u,v) & \text{if} \qquad \nu = \langle p, v \rangle \\ +\infty & \text{if not} \end{cases}$$

Stochastic Invariance of Epigraphs

Consider the system of stochastic differential equations

$$\begin{cases} i) & d\tau = -dt \\ ii) & d\xi_0 = \xi_0\rho_0(\xi)dt \\ iii) & d\xi_1 = \xi_1\rho_1(\xi)dt + \xi_1\sigma dW(t) \\ iv) & d\eta = (\xi_0\rho_0(\xi)\widetilde{p}_0(\tau,\xi) + \xi_1\rho_1(\xi)\widetilde{p}_1(\tau,\xi)dt \\ & \quad +\xi_1\sigma\widetilde{p}_1(\tau,\xi)dW(t) \end{cases}$$

governing the evolution of the prices x_0 of a nonrisky asset, the price x_1 of the risky asset and the value $y := \widetilde{p}_0(t,x)x_0 + \widetilde{p}_1(t,x)x_1$ of a feedback portfolio $(\widetilde{p}_0(t,x), \widetilde{p}_1(t,x))$.

Second-Order Black-Scholes Equation

We infer that any function v the epigraph of which is stochastic invariant under the above system of stochastic differential equations is a (generalized) solution to the Black-Scholes partial differential inequalities: $-\frac{\partial v(t,x)}{\partial t}$

$$+\left(\frac{\partial v(t,x)}{\partial x_0}x_0 + \frac{\partial v(t,x)}{\partial x_1}x_1(t) - v(t,x)\right)\rho_0(x)$$

$$+\frac{\partial^2 v(t,x)}{\partial x_1^2}(x_1\sigma)^2 \;\leq\; 0$$

In order to take into account the fact that the solution is only lower semi-continuous, we shall replace the gradient by the (regular) subgradients of the function v and the Hessian of v be a contingent epi-Hessian.

The Doss Theorem

Hence the epigraph of v is stochastic viable outside the epigraph of c if and only if it is viable under the equivalent "tychastic system"

$$\left\{\begin{array}{lll} i) & \tau' & = -1 \\ ii) & x_0' & = x_0\rho_0(x) \\ iii) & x_1' & = x_1\rho_1(x) - \frac{1}{2}x_1\sigma^2 + x_1\sigma u \\ iv) & y' & = -\mu(x)y + \widetilde{p}_0(\tau,x)x_0\rho_0(x) \\ & & -\dfrac{1}{2}x_1\sigma^2\left(\widetilde{p}_1(\tau,x) + \dfrac{\partial\widetilde{p}_1(t,x)}{\partial x_1}x_1\right) \\ & & +\widetilde{p}_1(\tau,x)x_1\rho_1(x) + \widetilde{p}_1(\tau,x)x_1\sigma v \end{array}\right.$$

where the tyches (u,v) range over R^2.

Contingent Second-Order Black-Scholes Equations

We deduce that v is a solution on

$$\Omega_c(v) \;:=\; \{(t,x) \in R_+ \times X \mid v(t,x) \;<\; c(t,x)\}$$

to the second-order partial differential equation in the sense that for every $x \in \Omega_c(v)$,

$$\begin{cases} i) \quad \widetilde{p}_1(x) \;=\; \frac{\partial v(t,x)}{\partial x_1} \\ ii) \quad \left(\dfrac{\partial v(t,x)}{\partial x_0}x_0 + \dfrac{\partial v(t,x)}{\partial x_1}x_1 - v(t,x)\right)\rho_0(x) \\ \qquad + \dfrac{1}{2}\dfrac{\partial \widetilde{p}_1(t,x)}{\partial x_1}x_1^2\sigma^2 - \dfrac{\partial v(t,x)}{\partial t} \;\leq\; 0 \end{cases}$$

Taking into account the first equation, we obtain

$$-\frac{\partial v(t,x)}{\partial t} + \rho_0(x)\left(\frac{\partial v(t,x)}{\partial x_0}x_0 + \frac{\partial v(t,x)}{\partial x_1}x_1 - v(t,x)\right)$$

$$+\frac{1}{2}\frac{\partial \widetilde{p}_1(t,x)}{\partial x_1}x_1^2\sigma^2 \ \leq \ 0$$

Contingent Second-Order Black-Scholes Equations

We deduce that v is a solution on

$$\Omega_c(v) \;:=\; \{(t,x) \in R_+ \times X \mid v(t,x) \;<\; c(t,x)\}$$

to the second-order partial differential equation in the sense that for every x, for every $\forall\,(p_t,p_x) \in \partial_- v(t,x)$,

$$\begin{cases} i) & \widetilde{p}_1(x) \;=\; p_{x_1} \\ ii) & -p_t + (p_{x_0}x_0 + p_{x_1}x_1 - v(t,x))\,\rho_0(x) \\ & +\frac{1}{2}\mathrm{Hess}_{\uparrow x_1}(v)(t,x,p_t,p_{x_0},p_{x_1}))(x_1\sigma,x_1\sigma) \;\le\; 0 \end{cases}$$

Contingent Stratonovitch First-Order Black-Scholes Equations

We deduce that v is a solution on

$$\Omega_c(v) \ := \ \{(t,x) \in R_+ \times X \mid v(t,x) \ < \ c(t,x)\}$$

to the first-order partial differential equation in the sense that: for every x, for every $\forall \ (p_t, p_x) \in \partial_- v(t,x)$,

$$\begin{cases} i) & \widetilde{p}_1(t,x) \ = \ p_{x_1} \\ ii) & -p_t + p_{x_0} x_0 \rho_0(x) - x_0 \rho_0(x) \widetilde{p}_0(t,x) \\ & +\frac{1}{2}\frac{\partial \widetilde{p}_1(t,x)}{\partial x_1} x_1^2 \sigma^2 \ \leq \ 0 \end{cases}$$

Taking into account the first equation, we obtain

$$-p_t + \rho_0(x) \left(p_{x_0} x_0 + p_{x_1} x_1 - v(t,x)\right)$$

$$+\frac{1}{2}\frac{\partial \widetilde{p}_1(t,x)}{\partial x_1}x_1^2\sigma^2 \ \leq \ 0$$