Clio Derivatives of History Functions and Applications to Valuation of Portfolios

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Outline

Motivation 1:
Dynamic Portfolio Management

The problem of dynamic portfolio valuation and management can be formulated directly in terms of viability/capturability issues. See J.-P. Aubin, P. Bernhard, H. Doss, D. Pujal, P. Saint-Pierre, etc.

However, the uncertain evolution of prices is not known, rather, regarded as uncertain (stochastic, tychastic) or forecasted: For instance

1. **discrete stochastic evolutions**
2. **continuous stochastic evolutions**
3. **history dependent (path dependent) evolutions** They involve the case when the evolution of financial asset prices is governed by an history dependent (path dependent) dynamical system as a prediction mechanism.
4. **(impulse) tychastic evolutions**, allowing to take into account payments of dividends

Motivation 2: Predictions and History Dependent Systems

We assume instead that the future evolution of the asset prices can be predicted or forecasted from its history through a convenient prediction mechanism, and based on such a prediction mechanism, valuate the portfolio and find the regulation law allowing the manager to modify his/her portfolio at each instant.

The class of prediction operators we study here is provided by history dependent (or path dependent, memory dependent, functional) control systems. A each instant, they associate with the history of the evolution up to this time and a portfolio the velocity of the price.

Therefore, the knowledge of the properties of capture basins of targets viable in a constrained set under an history dependent system can be used for obtaining the corresponding properties for portfolios.

Outline

1. Motivations: Predictions and History Dependent Systems
2. ▶ History Dependent Evolutionary Systems
3. The Viability/Capturability in the Historical Evolutionary Framework
4. History Intertemporal Optimization
5. Clio Hamilton-Jacobi-Bellman Equations
6. Historical Dynamical Valuation and Management of Portfolios

History or Path Dependent Systems

We denote by $\mathcal{H}(X) := \mathcal{C}(-\infty, 0; X)$ the history (or memory, path) space. We associate with any continuous function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ its *history (or path)* $\kappa(t)x$ *up to time* t defined by:

$$\forall\, \tau \in]-\infty, 0], \;\; (\kappa(t)x)(\tau) \; := \; x(t+\tau)$$

(often denoted by $x_t := \kappa(t)x$.) Then $\kappa(t)$ maps $\mathcal{C}(-\infty, +\infty; X)$ to $\mathcal{H}(X)$ and we observe that for any function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$, we have $x(t) = (\kappa(t)x)(0)$.

Chainings

Definition 1 ***We set***

$$\mathcal{A}(X) \ := \ \{x(\cdot) \in \mathcal{C}(0,+\infty;X) \quad \textit{such that} \ x(0) \ = \ 0\}$$

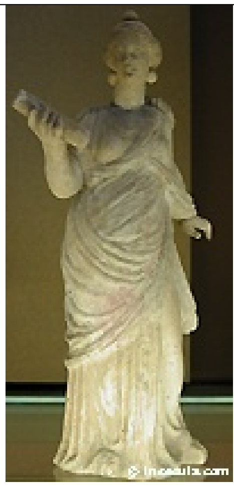
Let $h > 0$ ***be given. The*** h-chaining ***(or*** chaining ***when there are no ambiguities)*** $\varphi \diamondsuit_h \psi$ ***is the bilinear form from*** $\mathcal{H}(X) \times \mathcal{A}(X) \mapsto \mathcal{H}(X)$ ***defined by***

$$(\varphi \diamondsuit_h \psi)(\tau) \ := \begin{cases} \varphi(\tau + h) & \text{if} \ \ \tau \in]-\infty, -h] \\ \varphi(0) + \psi(\tau + h) & \text{if} \ \ \tau \in [-h, 0] \end{cases}$$

We see that $(\varphi \diamondsuit_h \psi)(0) = \varphi(0) + h\frac{\psi(h)}{h}$**.**

As an example, the chaining of $\varphi \in \mathcal{H}(X)$ **and** $u \in X$ **is defined by**

$$(\varphi \diamondsuit_h u)(\tau) \ = \begin{cases} \varphi(\tau + h) & \text{if} \ \ \tau \in]-\infty, -h] \\ \varphi(0) + (\tau + h)u & \text{if} \ \ \tau \in [-h, 0] \end{cases}$$



Definition 2 *Let* $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ *be a history dependent function. We shall that*

$$\mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(\lambda,u) := \liminf_{h\to 0+,\ \frac{\psi(h)}{h}\to u,\ \psi_h\in\mathcal{A}(X)} \frac{\mathbf{v}(t+h\lambda, \varphi\diamondsuit_h\psi_h) - \mathbf{v}(t,\varphi)}{h}$$

is the Clio epiderivative *of the function* $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ *at* (t,φ) *in the direction* (λ,u).

In other words, there exist a sequence $h_n > 0$ converging to 0, a sequence $\psi_n \in \mathcal{A}(X)$ such that $\frac{\psi_n(h_n)}{h_n}$ converges to u and a sequence v_n converging to $\mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(\lambda,u)$ such that

$$\forall\, n \geq 0, \quad v_n \;\geq\; \frac{\mathbf{v}(t + h_n\lambda, \varphi \diamondsuit_{h_n} \psi_n) - \mathbf{v}(t,\varphi)}{h_n}$$

Naturally, if the map $(\lambda, u) \mapsto \mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(\lambda,u)$ is linear (and continuous when X is a topological vector space), then we can write

$$\mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(\lambda,u) \;=\; \frac{\partial \mathbf{v}(t,\varphi)}{\partial t}\lambda + \sum_{i=1}^{n} \frac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} p_i$$

where we set

$$\begin{cases} \dfrac{\partial \mathbf{v}(t,\varphi)}{\partial t} & := \quad \mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(1,0,\ldots,0) \\ \dfrac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} & := \quad \mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(0,0,\ldots,1,\ldots,0) \end{cases}$$

and regard $\dfrac{\partial \mathbf{v}(t,\varphi)}{\partial x_i}$ as the Clio partial derivative **of $\mathbf{v}$ at (t,φ) with respect to x_i.**

History Dependent System

We introduce a map $f : \mathcal{H}(X) \mapsto X$ governing the evolution of a solution $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ to the history dependent (or path dependent, functional) differential equation

$$x'(t) \; = \; f(\kappa(t)x)$$

that associates with the history $\kappa(t)$ of the function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ up to time $t \geq 0$ the velocity $x'(t)$ of the state at time t.

We denote by $\mathcal{G}(\varphi)$ the set of solutions $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ of this history dependent differential equation starting at time 0 at the given initial history $\varphi \in \mathcal{H}(X)$.

This set-valued map $\mathcal{G} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ can be regarded as a prediction **or an extrapolation system:**

Prediction Process

Definition 3 ***A general*** prediction process ***is a set-valued map*** $\mathcal{G} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ ***associating with any*** $\varphi \in \mathcal{H}(X)$ ***a set*** $\mathcal{G}(\varphi)$ ***of future evolutions*** $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ ***satisfying*** $\kappa(x) = \varphi$ ***that is required to satisfy the two following properties***

1. ***Let*** $x(\cdot) \in \mathcal{G}(\varphi)$***. Then for all*** $T \geq 0$***, the function*** $y(\cdot)$ ***defined by*** $y(t) := x(t+T)$ ***is a prediction*** $y(\cdot) \in \mathcal{G}(\kappa(T)x)$ ***associated with*** $\kappa(T)x$***,***

2. ***Let*** $x(\cdot) \in \mathcal{G}(\varphi)$ ***and*** $T \geq 0$***. Then for every*** $y(\cdot) \in \mathcal{G}(\kappa(T)x)$***, the function*** $z(\cdot)$ ***defined by***

$$z(t) \; := \begin{cases} x(t) & \text{if} \;\; t \in]-\infty, T] \\ y(t-T) & \text{if} \;\; t \geq T \end{cases}$$

belongs to $\mathcal{G}(\varphi)$***.***

We denote from now on by φ an element of $\mathcal{H}(X)$ and by $\varphi(\cdot) : \mathbf{R}_+ \mapsto \mathcal{H}(X)$ a function associating with every nonnegative time $t \geq 0$ an history $\varphi(t) \in \mathcal{H}(X)$. Hence $\varphi(\cdot)$ can also be regarded as a function

$$(t, \tau) \in [0, +\infty[\times] - \infty, 0] \mapsto \varphi(t)(\tau)$$

This is the case of histories $\varphi(\cdot) := \kappa(\cdot)x$ of functions $x(\cdot) \in \mathcal{C}(0, +\infty; X)$.

It is more convenient to associate with a prediction process $\mathcal{CG}$ mapping $\mathcal{H}(X)$ to $\mathcal{C}(-\infty, +\infty; X)$ an evolutionary system **on the history space $\mathcal{H}(X)$:**

Evolutionary System

Definition 4 ***An evolutionary system on $\mathcal{H}(X)$ is a set-valued map $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; \mathcal{H}(X))$ satisfying***

1. ***the* translation property*: Let $\varphi(\cdot) \in \mathcal{B}(\varphi)$. Then for all $T \geq 0$, the function $\psi(\cdot)$ defined by $\psi(t) := \varphi(t+T)$ belongs to $\mathcal{B}(\varphi(T))$,***

2. ***the* concatenation property*: Let $\varphi(\cdot) \in \mathcal{B}(\varphi)$ and $T \geq 0$. Then for every history $\psi(\cdot) \in \mathcal{B}(\varphi(T))$, the history $\xi(\cdot)$ defined by***

$$\xi(t) \; := \begin{cases} \varphi(t) & \text{if} \;\; t \in [0, T] \\ \psi(t-T) & \text{if} \;\;\; t \geq T \end{cases}$$

belongs to $\mathcal{B}(\varphi)$.

Therefore, we associate with a prediction process $\mathcal{G}$ the evolutionary system $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0,\infty;\mathcal{H}(X))$ defined by

$$\mathcal{B}(\varphi) \ := \ \{t \mapsto \varphi(t) := \kappa(t)x\}_{x(\cdot)\in\mathcal{G}(\varphi)}$$

associating with φ the histories of evolutions $t \mapsto \varphi(t) := \kappa(t)x$ associated with the predictions $x(\cdot) \in \mathcal{G}(\varphi)$ of prediction process.

Indeed, it satisfies:

1. **the** translation property: **Let** $\varphi(\cdot) \in \mathcal{B}(\varphi)$. **Then for all** $T \geq 0$, **the function** $\psi(\cdot)$ **defined by** $\psi(T) := \varphi(t+T)$ **is the history** $\psi(\cdot) := \kappa(\cdot)y \in \mathcal{B}(\kappa(s)x)$ **of the prediction** $y(\cdot)$ **from** $\kappa(s)x$,

2. **the** concatenation property: **Let** $\varphi(\cdot) \in \mathcal{B}(\varphi)$ **be the history and** $T \geq 0$. **Then for every history** $\psi(\cdot) \in \mathcal{B}(\kappa(s)x)$ **of a prediction** $y(\cdot)$ **from** $\kappa(s)x$, **the function** $\xi(\cdot)$ **defined by**

$$\xi(t) \; := \begin{cases} \varphi(t) := \kappa(t)x & \text{if} \;\; t \in [0,T] \\ \psi(t-T) := \kappa(t-T)y & \text{if} \;\; t \geq T \end{cases}$$

is the history of the solution $z(\cdot)$ **defined by**

$$z(t) \; := \begin{cases} x(t) & \text{if} \;\; t \in [0,s] \\ y(t-T) & \text{if} \;\; t \geq s \end{cases}$$

to the history dependent differential inclusion starting at the initial path φ, **and thus, belongs to** $\mathcal{B}(\varphi)$.

Remark

The solution map $\mathcal{B}$ has the advantage of mapping histories $\varphi \in \mathcal{H}(X)$ into time-dependent functions $t \mapsto \varphi(t) := \kappa(t)x$ taking their values in the same history space $\mathcal{H}(X)$, even though the intuitive view is to handle predictions that are functions $t \mapsto x(t)$ taking their values in the state space X. But the introduction of these definitions is justified by the fact that the main viability/capturability theorems hold true for evolutionary systems defined on any metric space, such as $\mathcal{H}(X)$.

Anticipation Operator

This format allows us to associate a prediction process with and anticipation operator, that is a map $\phi : \mathcal{H}(X) \mapsto \mathcal{C}(0,\infty;X)$ from the history space to the future space $\mathcal{C}(0,\infty;X)$. Therefore, a differential equation with anticipation is defined from both an anticipation operator ϕ and a single-valued map $g : \mathcal{C}(0,\infty;X) \mapsto X$ by taking $f := g \circ \phi$ as the dynamics governing the evolution through the history dependent differential equation

$$x'(t) = g(\phi(\kappa(t)x))$$

Examples

By the way, many examples of history dependent differential inclusion equations are governed by dynamics $f := g \circ \alpha$ where $\alpha : \mathcal{H}(X) \mapsto Y$ and $g : Y \mapsto X$ where Y is an intermediate space.

By taking $Y := X^{p+1}$, delays $\theta_0 := 0 > \theta_1 > \cdots > \theta_p$ and $\alpha(\varphi) := (\varphi(0), \varphi(\theta_1), \ldots, \varphi(\theta_p),)$, we obtain delay equations

$$x'(t) \ = \ g\left(\varphi(t), \varphi(t+\theta_1), \ldots, \varphi(t+\theta_p), \right)$$

governed by the dynamics f defined by

$$f(\varphi) \ := \ g\left(\varphi(0), \varphi(\theta_1), \ldots, \varphi(\theta_p)\right)$$

The classical example is provided by Volterra dynamics defined through a "kernel" $k :] -\infty, 0] \times X \mapsto Y$, the associated integral operator $\alpha : \mathcal{H}(X) \mapsto Y$ defined by

$$\alpha(\varphi) \ := \ \int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s)$$

and a single-valued map $g : Y \rightsquigarrow X$ by

$$f(\varphi) \ := \ g\left(\int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s)\right)$$

where the measure $d\mu$ can be the Lebesgue measure dx or a discrete measure $\sum_{i=0}^{p} \delta_{\theta_i}$ where δ_{θ_i} are Dirac measures at times $\theta_0 := 0 > \theta_1 > \cdots > \theta_p$. $\square$

History Dependent Control System

A history dependent (or path dependent) control system (U, f) is defined by

- **a set-valued map $U : \mathcal{H}(X) \rightsquigarrow \mathcal{U}$ assigning to every history $\varphi \in \mathcal{H}(X)$ a subset $U(\varphi) \subset \mathcal{U}$ of controls,**
- **and a single-valued map $f : \mathrm{Graph}(U) \mapsto X$ associating with every history-control pair (φ, u) the velocity $f(\varphi, u) \in X$ of the state.**

It governs the evolution of the system according to the history dependent control system (or a path dependent control system, a functional control system)

$$\begin{cases} i) & x'(t) \; = \; f(\kappa(t)x, u(t)) \\ ii) & \text{where } u(t) \; \in \; U(\kappa(t)x) \end{cases}$$

The initial condition is an history $\varphi \in \mathcal{H}(X)$ and we require that at initial time 0,

$$\kappa(0)x \; := \; \varphi$$

is satisfied.

History Dependent Differential Inclusions

When we do not need the explicit use of the controls, they boil down to history dependent differential inclusions

$$x'(t) \in F(\kappa(t)x)$$

where $F : \mathcal{H}(X) \rightsquigarrow X$ **is the set-valued map defined by**

$$\forall \, \varphi \in \mathcal{H}(X), \;\; F(\varphi) := \{f(\varphi, u)\}_{u \in U(\varphi)}$$

We associate with it the prediction process $\mathcal{C} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ **associating with any** $\varphi \in \mathcal{H}(X)$ **the set of solutions** $x(\cdot)$ **to the history dependent differential equation starting at** φ **and the evolutionary system** $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; \mathcal{H}(X))$ **defined by** $\mathcal{B}(\varphi) := \{\kappa(\cdot)x\}_{x(\cdot) \in \mathcal{G}(\varphi)}$**.**

Outline

Viable-Capture Basins

We recall the following definitions of viability theory:

Definition 5 ***Let*** $\mathbf{C} \subset \mathbf{K} \subset \mathbf{X}$ ***be two subsets,*** $\mathbf{C}$ ***being regarded as a target,*** $\mathbf{K}$ ***as a constrained set and*** $\mathcal{S} : \mathbf{X} \rightsquigarrow \mathcal{C}(0, \infty; \mathbf{X})$ ***be an evolutionary system.***

1. ***The subset*** $\mathrm{Capt}(\mathbf{K}, \mathbf{C})$ ***of initial states*** $\mathbf{x} \in \mathbf{K}$ ***such that*** $\mathbf{C}$ ***is reached in finite time before possibly leaving*** $\mathbf{K}$ ***by at least one evolution*** $\mathbf{x}(\cdot) \in \mathcal{S}(\mathbf{x})$ ***starting at*** $\mathbf{x}$ ***is called the*** viable-capture basin ***of*** $\mathbf{C}$ ***in*** $\mathbf{K}$ ***under the evolutionary system*** $\mathcal{S}$ ***and*** $\mathrm{Capt}(\mathbf{C}) := \mathrm{Capt}(\mathbf{X}, \mathbf{C})$ ***is said to be the*** capture basin ***of*** $\mathbf{C}$***.***

2. ***We say that the subset*** $\mathrm{Viab}(\mathbf{K})$ ***of elements*** $\mathbf{x} \in \mathbf{K}$ ***from which starts at least one evolution*** $\mathbf{x}(\cdot) \in \mathcal{S}(\mathbf{x})$ ***viable in*** $\mathbf{K}$ ***forever is the*** viability kernel ***of*** $\mathbf{K}$ ***under the evolutionary system*** $\mathcal{S}$***. A subset*** $\mathbf{K}$ ***is a*** repeller ***under*** $\mathcal{S}$ ***if its viability kernel is empty.***

Upper Semicompact Maps

To say that $\mathcal{S}_{\mathbf{F}}$ is upper semicompact means that whenever $\varphi_n \in \mathcal{H}(X)$ converge uniformly on compact intervals to φ in $\mathcal{H}(X)$ and any history $\varphi_n(\cdot) := \kappa(\cdot)x_n \in \mathcal{S}_{\mathbf{F}}(\varphi_n)$ associated to a solution $x_n(\cdot)$ to the history-dependent differential inclusion $x'(t) \in \mathbf{F}(\kappa(t)x)$ starting at φ_n, there exists a subsequence (again denoted by) $\varphi_n(\cdot)$ converging uniformly on compact intervals to the history $\varphi(\cdot) := \kappa(\cdot)x$ of a solution $x(\cdot)$ to the path-dependent differential inclusion starting at φ.

The Aubin & Catté' Characterization

The Aubin & Catté' Double Fixed-Point Property states that the capture basin $\mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{C})$ is the UNIQUE subset D between C and K satisfying the double fixed point property

$$\mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{D}) \;=\; \mathbf{D} \;=\; \mathrm{Capt}_{\mathcal{S}}(\mathbf{D},\mathbf{C})$$

Theorem 6 ***Let us assume that an evolutionary system*** $\mathcal{S}$ ***is upper semicompact and that the subsets*** $\mathbf{C} \subset \mathbf{K}$ ***and*** $\mathbf{K}$ ***are closed. If*** $\mathbf{K}\backslash\mathbf{C}$ ***is a repeller (this is the case when*** $\mathbf{K}$ ***itself is a repeller), then the viable-capture basin*** $\mathrm{Capt}(\mathbf{K},\mathbf{C})$ ***of the target*** $\mathbf{C}$ ***under*** $\mathcal{S}$ ***is the*** largest ***closed subset satisfying*** $\mathbf{C} \subset \mathbf{D} \subset \mathbf{K}$ ***such that*** $\mathbf{D}\backslash\mathbf{C}$ ***is*** locally viable ***under*** $\mathcal{S}$***.***

Marchaud Maps

Definition 7 *We shall say that* $\mathbf{F} : \mathcal{H}(X) \rightsquigarrow X$ *is a* Marchaud map *if*

$$\begin{cases} i) & \mathbf{F} \textit{ is upper semicontinuous} \\ ii) & \textit{the values } \mathbf{F}(\varphi) \textit{ of } \mathbf{F} \textit{ are convex} \\ iii) & \textit{the growth of } \mathbf{F} \textit{ is linear: } \exists\, c > 0 \mid \forall\, \varphi \in \mathcal{H}(X), \\ & \|\mathbf{F}(\varphi)\| := \sup_{v \in \mathbf{F}(\varphi)} \|v\| \ \leq\ c(\|\varphi(0)\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(\varphi, u) \mapsto f(\varphi, u)$ is continuous, affine with respect to the controls u and with linear growth and when $U : \mathcal{H}(X) \rightsquigarrow \mathcal{U}$ is Marchaud.

We supply the history space $\mathcal{H}(X)$ with the compact convergence topology. We denote by $\mathcal{H}_\lambda$ the subset of Lipschitz functions with Lipschitz constant λ.

A *closed subset* $\mathbf{K} \subset \mathcal{H}(X)_\lambda$ *is compact if and only if* $\mathbf{K}(0) := \{\varphi(0)\}_{\varphi \in \mathbf{K}}$ *is bounded.*

The Main Stability Theorem

Theorem 8 ***Assume that $\mathbf{F} : \mathcal{H}(X) \rightsquigarrow X$ is Marchaud. Then its solution map $\mathcal{S}_{\mathbf{F}}$ is upper semicompact with nonempty values on each $\mathcal{H}_\lambda(X)$.***

Contingent Directions

Definition 9 ***Let*** $\mathbf{K} \subset \mathcal{H}(X)$ ***be a subset of histories and*** $\varphi \in \mathcal{H}$***. We denote by*** $\mathcal{D}_{\mathbf{K}}(\varphi)$ ***the set of vectors*** $v \in X$ ***such that there exist a sequence*** $h_n > 0$ ***converging to*** 0 ***and a sequence*** $\psi_n \in \mathcal{A}(X)$ ***satisfying***

$$\begin{cases} i) \quad \forall\, n \geq 0, \;\; \varphi \diamond_{h_n} \psi_n \; \in \; \mathbf{K} \\ ii) \quad \dfrac{\psi_n(h_n)}{h_n} \to v \end{cases} \tag{1}$$

Theorem 10 ***Let us assume that*** $\mathbf{F}$ ***is Marchaud, that*** $\mathbf{K} \subset \mathcal{H}_\lambda(X)$ ***is closed and that a closed subset*** $\mathbf{C}$ ***satisfies*** $\mathrm{Viab}_{\mathbf{F}}(\mathbf{K}\backslash\mathbf{C}) = \emptyset$***. Then the viable-capture basin*** $\mathrm{Capt}_{\mathbf{F}}^{\mathbf{K}}(\mathbf{C})$ ***is the*** **largest** ***closed subset*** $\mathbf{D}$ ***satisfying*** $\mathbf{C} \subset \mathbf{D} \subset \mathbf{K}$ ***and***

$$\forall\, \varphi \in \mathbf{D}\backslash\mathbf{C}, \;\; \mathbf{F}(\varphi) \cap \mathcal{D}_{\mathbf{D}}(\varphi) \;\neq\; \emptyset \tag{2}$$

Outline

Consider the history dependent control system

$$\begin{cases} i) & x'(t) = f(\kappa(t)x, u(t)) \\ ii) & y'(t) = -\mathbf{m}(\kappa(t)x, u(t))y(t) - \mathbf{l}(\kappa(t)x, u(t)) \\ iii) & \text{where } u(t) \in U(\kappa(t)x) \end{cases} \quad (3)$$

We denote now by $\mathcal{G}(\varphi) \subset \mathcal{C}(0,\infty;X)$ **the set of solutions** $x(\cdot)$ **to the history dependent control system**

$$\begin{cases} i) & x'(t) = f(\kappa(t)x, u(t)) \\ ii) & \text{where } u(t) \in U(\kappa(t)x) \end{cases} \quad (4)$$

satisfying $\kappa(0)x = \varphi$ **and by** $\mathcal{B}(\varphi) \subset \mathcal{C}(0,\infty;\mathcal{H}(X))$ **the subset of associated evolutions** $\kappa(\cdot)x$**.**

Valuation Subset

Definition 11 ***Given the history dependent control system (16) and two functions*** $\mathbf{b} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ ***(constraint function) and*** $\mathbf{c} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ ***(objective function) satisfying***

$$\forall\, (t,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X),\;\; 0 \;\leq\; \mathbf{b}(t,\varphi) \;\leq\; \mathbf{c}(t,\varphi) \;\leq\; +\infty$$

the valuation ***subset*** $\mathcal{V}_{(\mathbf{b},\mathbf{c})} \subset \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+$ ***is the subset of triples*** (T,φ,y) ***such that there exist*** at least ***one evolution*** $(x(\cdot),u(\cdot)) \in \mathcal{G}(\varphi)$ ***starting from*** φ ***and a time*** $t^\star \in [0,T]$ ***such that conditions (13):***

$$\begin{cases} i) & \forall\, t \in [0,t^\star],\;\; y(t) \geq \mathbf{b}(T-t,\kappa(t)x,u(t)) \\ & \text{(dynamical constraint)} \\ ii) & y(t^\star) \geq \mathbf{c}(T-t^\star,\kappa(t^\star)x,u(t^\star)) \\ & \text{(final objective)} \end{cases}$$

are satisfied.

Intertemporal History Functionals

In order to compute the southern border of the valuation set subset $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$, we associate with any history dependent function $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ the functional

$$\left\{ \begin{array}{l} \mathbf{J_v}(t; (x(\cdot), u(\cdot)))(T, \varphi) \ := \ e^{\int_0^t \mathbf{m}(\kappa(s)x, u(s))ds} \mathbf{v}(T-t, \kappa(t)x) \\ + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x, u(s))ds} \mathbf{l}(\kappa(\tau)x, u(\tau)) d\tau \end{array} \right.$$

where $\varphi := \kappa(0)x$. We next associate with b the functional

$$\mathbf{I_b}(t; (x(\cdot), u(\cdot)))(T, \varphi) \ := \ \sup_{s \in [0,t]} \mathbf{J_b}(s; (x(\cdot), u(\cdot)))(T, \varphi)$$

We integrate this cumulated cost together with the cost $\mathbf{J_c}(t; (x(s), u(s)))(T, \varphi)$ associated with the function c by introducing the new cost functions

$$\mathbf{L}_{(\mathbf{b},\mathbf{c})}(t; (x(\cdot), u(\cdot)))(T, \varphi) \ := \ \max(\mathbf{I_b}(x(\cdot), u(\cdot))(T, \varphi), \mathbf{J_c}(t; (x(\cdot), u(\cdot)))(T, \varphi))$$

History Valuation Function

We define the history dependent functional

$$\mathbf{V}_{(\mathbf{b},\mathbf{c})}(x(\cdot),u(\cdot))(T,\varphi) \;=\; \inf_{t\in[0,T]} \mathbf{L}_{(\mathbf{b},\mathbf{c})}(t;(x(\cdot),u(\cdot)))(T,\varphi)$$

We shall prove that the southern border

$$V_{(\mathbf{b},\mathbf{c})}(T,\varphi) \;:=\; \inf_{(T,\varphi,y)\in\mathcal{V}_{(\mathbf{b},\mathbf{c})}} y \tag{5}$$

of the valuation set $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ is equal to the valuation of the history dependent minimization problem

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V}_{(\mathbf{b},\mathbf{c})}(x(\cdot),u(\cdot))(T,\varphi)$$

and gather other characterizations:

Theorem 12 ***Let us assume that the extended functions*** b ***and*** c ***are nontrivial and non negative.***

Then the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ ***satisfies***

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \ := \inf_{(T,\varphi,y)\in\mathcal{V}_{(\mathbf{b},\mathbf{c})}} y$$

When $(T,\varphi,\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi))$ ***belongs to the valuation set*** $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$***, there exists at least one solution*** $(x(\cdot),u(\cdot)) \in \mathcal{G}(\varphi)$ ***starting from*** φ ***satisfying the inequality:***

$$\begin{cases} \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \\ \geq \ e^{\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x) + \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{l}(\kappa(\kappa)x,u(\tau))d\tau \end{cases} \quad (6)$$

until the first time $t^* \in [0,T]$ ***when***

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t^*,\kappa(t^*)x) = \mathbf{c}(T-t^*,\kappa(t^*)x)$$

and any such evolution is an optimal evolution for the optimal time t^****.***

Furthermore, if $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) > \mathbf{b}(T,\varphi)$***, any such evolution actually satisfies***

$$\left\{ \begin{array}{l} \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \\ = \; e^{\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x) + \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{l}(\kappa(\kappa)x,u(\tau))d\tau \end{array} \right. \quad (7)$$

until the first time $t^{**} \in [0,t^*]$ ***when***

$$V^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \; = \; \mathbf{I}_{\mathbf{b}}(t^{**};(x(\cdot),u(\cdot))))(T,\varphi)$$

Finally, the valuation function is the unique ***solution*** $\mathbf{v}$ ***to the system of two following functional equations stating that the functions*** $\mathbf{V}_{(\mathbf{b},\mathbf{v})}$ ***and*** $\mathbf{V}_{(\mathbf{v},\mathbf{c})}$ ***have the same infimum than*** $\mathbf{V}_{(\mathbf{b},\mathbf{c})}$***:***

$$\left\{ \begin{array}{l} \mathbf{V}^{\inf}_{(\mathbf{v},\mathbf{c})}(T,\varphi) \; := \; \inf_{(x(\cdot),u(\cdot))\in\mathcal{G}(\varphi)} \mathbf{V}_{(\mathbf{v},\mathbf{c})}((x(\cdot),u(\cdot)))(T,\varphi) \\ = \; \mathbf{v}(T,\varphi) \\ = \; \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{v})}(T,\varphi) \; := \; \inf_{(x(\cdot),u(\cdot))\in\mathcal{G}(\varphi)} \mathbf{V}_{(\mathbf{b},\mathbf{v})}((x(\cdot),u(\cdot)))(T,\varphi) \end{array} \right. \quad (8)$$

Outline

Clio Hamilton-Jacobi-Bellman Equation

The next theorem characterizes the valuation functions as a generalized solution v to the following Clio Hamilton-Jacobi-Bellman partial differential equation

$$-\frac{\partial \mathbf{v}(t,\varphi)}{\partial t} + \inf_{u \in U(\varphi)} \left(\sum_{i=1}^{n} \frac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} f_i(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi) + \mathbf{l}(\varphi,u) \right) \;\leq\; 0$$

on the subset

$$\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}) \;:=\; \{(t,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X) \mid \mathbf{b}(t,\varphi) \;\leq\; \mathbf{v}(t,\varphi) \;<\; \mathbf{c}(t,\varphi)\}$$

Theorem 13 ***Let us assume that the control system*** $(U, f, \mathbf{l}, \mathbf{m})$ ***is Marchaud in the sense that***

$$\left\{\begin{array}{ll} i) & f \text{ and } U \text{ are Marchaud} \\ ii) & \sup_{\varphi} \|U(\varphi)\| \ < \ +\infty \\ iii) & \mathbf{m} \text{ and } \mathbf{l} \text{ are continuous with linear growth and convex with respect to } u \end{array}\right. \tag{9}$$

and that the functions b ***and*** c ***are nontrivial, non negative and lower semicontinuous.***

Then the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ ***is lower semicontinuous and is characterized as the smallest of the nonnegative lower semicontinuous functions*** $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ ***satisfying for every*** $(t, \varphi) \in]0, \infty[\times X$

$$\left\{\begin{array}{ll} i) & \mathbf{b}(t,\varphi) \ \leq \ \mathbf{v}(t,\varphi) \ \leq \ \mathbf{c}(t,\varphi) \\ ii) & \forall \ (t,\varphi) \in \Omega_{(\mathbf{b},\mathbf{c})}\mathbf{v}), \\ & \inf_{u \in U(\varphi)} \left(\mathbf{D}_{\uparrow}\mathbf{v}(t,\varphi)(-1, f(\varphi,u)) + \mathbf{l}(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi)\right) \ \leq \ 0 \end{array}\right. \tag{10}$$

Let us set

$$\mathbf{R}(t,\varphi) \ := \ \{u \in U(\varphi) \mid \mathbf{D}_{\uparrow}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi)(-1, f(\varphi,u)) + \mathbf{l}(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi) \leq 0\}$$

Knowing the valuation function, an optimal solution is obtained in the following way: Starting from φ_0 ***such that*** $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi_0) < \mathbf{c}(T,\varphi_0)$***, any solution*** $(x(\cdot),u(\cdot))$ ***to the control system***

$$\left\{ \begin{array}{lll} i) & x'(t) \ = \ f(\kappa(t)x, u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}(T-t, \kappa(t)x) \\ & \text{(regulation law)} \end{array} \right. \quad (11)$$

is an optimal solution, and the first time $t^* \geq 0$ ***when***

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t^*, \kappa(t^*)x) \ = \ \mathbf{c}(T-t^*, \kappa(t^*)x)$$

is the optimal time.

Outline

Portfolios

Let $n+1$ financial assets $i = 0, 1, \ldots, n$, 0 denoting a non-risky asset and the n other risky ones. The components of the state variable

$$x := (x_0, x_1, \ldots, x_n) \in R^{n+1}$$

are the prices. A portfolio is an element $p := (p_0, p_1, \ldots, p_n) \in R^{n+1}$ describing the number of shares of assets $i = 0, 1, \ldots, n$. They range over a subset $D(t, x) = D(t, x_0, x_1, \ldots, x_n)$.

A portfolio feedback is a (continuous) selection $\widetilde{p}$ of the set-valued map P in the sense that

$$\forall\, x \in R^{n+1}, \;\; \widetilde{p}(t, x) \;\in\; D(t, x)$$

The associated capital (or the value of the portfolio) y (usually denoted by W in the financial literature) can be written

$$y \;:=\; \langle p, x \rangle \;=\; \sum_{i=0}^{n} p_i x_i$$

Contingent Claims and Financial products

Let us also consider a given history-dependent (but time-independent) function $\mathbf{u} : \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$, known under the name of contingent claim in the financial literature.

For instance, we can single out history dependent contingent claims of the form

$$\mathbf{u}(\varphi) \;:=\; \mathbf{w}\left(\int_{-\infty}^{0} k(-s,\varphi(s))d\mu(s)\right)$$

or

$$\mathbf{u}(\varphi) \;:=\; \mathbf{w}\left(\varphi(0),\varphi(\theta_1),\ldots,\varphi(\theta_p)\right)$$

Given an exercise time $T > 0$ and an initial history $\varphi \in \mathcal{H}(X)$ of prices, the general problem of dynamic portfolio valuation is to find an initial capital y such that there exists a portfolio $u(\cdot)$ satisfying one of the three following option rules[1] are satisfied:

$$\left\{\begin{array}{ll} i) & \sum_{i=0}^{n} u_i(T)x_i(T) \ \geq \ \mathbf{u}(\kappa(T)x) \\ & \text{(European Options)} \\ ii) & \forall\, t \in [0,T], \ \sum_{i=0}^{n} u_i(t)x_i(t) \ \geq \ \mathbf{u}(\kappa(t)x) \\ & \text{(American Options)} \\ iii) & \exists\, t^* \in [0,T] \ \textbf{ such that } \sum_{i=0}^{n} u_i(t^*)x_i(t^*) \ \geq \ \mathbf{u}(\kappa(t^*)x) \\ & \text{(First Time Options)} \end{array}\right. \tag{12}$$

[1] These rules, among many other ones, are applied in finance theory for determining the portfolios replicating options when the function $\mathbf{u}$ is regarded as a claim function.

A General Framework

Actually, in order to treat the three option rules (12) as particular cases of a more general framework, we introduce two functions b : $\mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ **(constraint function) and c** : $\mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ **(objective function) satisfying**

$$\forall\, (t,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X), \;\; 0 \;\leq\; \mathbf{b}(t,\varphi) \;\leq\; \mathbf{c}(t,\varphi) \;\leq\; +\infty$$

with which we require that there exist a portfolio $u(\cdot)$ **and a time** $t^\star \in [0,T]$ **satisfying conditions:**

$$\begin{cases} i) & \forall\, t \in [0,t^\star], \;\; y(t) \geq \mathbf{b}(T-t,\kappa(t)x) \\ & \text{(dynamical constraint)} \\ ii) & y(t^\star) \geq \mathbf{c}(T-t^\star,\kappa(t^\star)x) \\ & \text{(final objective)} \end{cases} \tag{13}$$

are satisfied.

Examples

The three option rules (12) associated with contingent function $\mathbf{u} : \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ ***can be recovered from (13) by adequate choices of pairs*** $(\mathbf{b}, \mathbf{c})$ ***of functions associated with*** $\mathbf{u}$**: denoting by** $\mathbf{u}_\infty$ **the function defined by**

$$\mathbf{u}_\infty(t, \varphi) \; := \; \mathbf{u}(\varphi) \text{ if } t = 0 \; \& \; +\infty \text{ if not} \tag{14}$$

and by $\mathbf{0}$ **the function defined by**

$$\mathbf{0}(t, \varphi) \; = 0 \text{ if } t \geq 0 \; \& \; +\infty \text{ if not}$$

1. **by taking** $\mathbf{b}(t, \varphi) := \mathbf{0}(t, \varphi)$ **and** $\mathbf{c}(t, \varphi) = \mathbf{u}_\infty(t, \varphi)$**, we obtain the rule for the** European option **(12)i),**
2. **by taking** $\mathbf{b}(t, \varphi) := \mathbf{u}(\varphi)$ **and** $\mathbf{b}(t, \varphi) := \mathbf{u}_\infty(t, \varphi)$**, we obtain the rule for the** American option **(12)ii),**
3. **by taking** $\mathbf{b}(t, \varphi) := \mathbf{0}(t, \varphi)$ **and** $\mathbf{c}(t, \varphi) = \mathbf{u}(\varphi)$**, we obtain the rule for the** first time option**(12)iii).**

History Dependent Interest Rates

Instead of studying the evolution of prices under tychastic or stochastic uncertainty, we assume that we have a history dependent interest rate $\rho : \mathcal{H}(X) \mapsto X$ providing the interest rates $\rho_i(\varphi)$ in function of the history of the prices of the assets, using prediction or other extrapolation operators that are assumed to be given here. The evolution of the prices being governed by the history dependent tychastic system

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \;=\; x_i(t)\rho_i(\kappa(t)x)$$

We denote by $\mathcal{G}(\varphi)$ the set of solutions to the history dependent differential equation system satisfying $\kappa(0)x = \varphi$.

The Viability/Capturability Strategy

An evolution $t \mapsto (T - t, \kappa(t)x))$ is viable in $\mathcal{E}p(\mathbf{b})$ until it captures the target $\mathcal{E}p(\mathbf{c})$ if there exists a finite time $t^\star \geq 0$ such that

$$\begin{cases} (i) & \quad (T - t^\star, \kappa(t^*)x, y(t^\star)) \in \mathcal{E}p(\mathbf{c}) \\ (ii) \quad \forall\, t \in [0, t^\star], & \quad (T - t, \kappa(t)x), y(t)) \in \mathcal{E}p(\mathbf{b}) \end{cases} \tag{15}$$

Portfolio Dynamics

The dynamics of the portfolio is assumed to be constrained by the dynamical inequalities

$$\langle p'(t), x(t)\rangle \;=\; -\mathbf{m}(\kappa(t)x, p(t))\langle p(t), x(t)\rangle$$

It states that the cost $\langle p'(t), x(t)\rangle$ of an instantaneous exchange $p'(t)$ of the portfolio $p(t)$ at price $x(t)$ is proportional to the value (or the capital) $\langle p(t), x(t)\rangle$ of the portfolio. Usually, the cost m is assumed to be equal to 0 (self-financing assumption). Here, the function m could be regarded as a very specific example of "transaction costs". However, real transaction costs depend upon the absolute value of the instantaneous exchange p_i' of shares, and in this case, these instantaneous exchanges p_i' have to be used as components of the state of an augmented system.

We deduce that the evolution of price-capital pair $(x(t), y(t))$ is governed by the history dependent control system

$$\left\{\begin{array}{ll} i) & \forall\, i=1,\ldots,n,\;\; x_i'(t) \;=\; x_i(t)\rho_i(\kappa(t)x) \\ ii) & y'(t) \;=\; \displaystyle\sum_{i=0}^{n} p_i(t)x_i(t)\rho_i(\kappa(t)x) - \mathbf{m}(\kappa(t)x, p(t))y(t) \\ & \text{where } p(t) \;\in\; P(\kappa(t)x) \end{array}\right. \tag{16}$$

regulated by the portfolio.

The Questions Raised

1. **find the valuation subset $\mathcal{V}_{(\mathbf{b},\mathbf{c})} \subset \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+$ of triples (T, φ, y) made of the exercise time T, the initial price history φ and the initial capital y such that there exist at least a portfolio $p(\cdot)$ and a time $t^* \in [0, T]$ for which conditions (13) are satisfied,**

2. **associate with any exercise time T and initial price history φ the smallest capital $V(T, \varphi)$:**

$$V_{(\mathbf{b},\mathbf{c})}(T, \varphi) := \inf_{(T,\varphi,y) \in \mathcal{V}_{(\mathbf{b},\mathbf{c})}} y$$

   The function $(T, \varphi) \mapsto V_{(\mathbf{b},\mathbf{c})}(T, \varphi)$ — called the southern border of $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ — is said to be the valuation function, i.e., the minimal initial capital y satisfying the two constraints (13).

3. find the regulation map $\Gamma_{(\mathbf{b},\mathbf{c})}$ **associating with any** $(T, \varphi) \in \mathbf{R}_+ \times \mathcal{H}(X)$ **a subset** $\Gamma_{(\mathbf{b},\mathbf{c})}(T, \varphi)$ **of portfolios** $p \in P(\varphi)$ **that are** optimal **in the sense that the optimal capital is governed by the history dependent control system**

$$\begin{cases} i) & \forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; x_i(t)\rho_i(\kappa(t)x) \\ ii) & y'(t) \; = \; \displaystyle\sum_{i=0}^{n} p_i(t)x_i(t)\rho_i(\kappa(t)x) - \mathbf{m}(\kappa(t)x, p(t))y(t) \\ iii) & \text{where } p(t) \; \in \; \Gamma_{(\mathbf{b},\mathbf{c})}(T - t, \kappa(t)x) \\ & \text{(regulation law)} \end{cases} \tag{17}$$

This regulation law **is the main answer the theory brings: it tells the manager at each instant how to change his portfolio in terms to time** $T - t$ **left to exercise time and the prices of the assets.**

We denote by $(\mathbf{b}, \mathbf{c})$ one of the three pairs $(\mathbf{0}, \mathbf{u}_\infty)$, $(\mathbf{u}, \mathbf{u}\infty)$ and $(\mathbf{0}, \mathbf{u})$ involved in the three option rules studied in this section. When the valuation function $V_{(\mathbf{b},\mathbf{c})}$ is Clio differentiable, it is actually the largest solution $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ between the functions b and c to the nonlinear Hamilton-Jacobi-Isaacs partial differential inequalities (that play the role of Black-Scholes partial differential equations when the evolution of prices is governed by a stochastic differential equation):

$$-\frac{\partial \mathbf{v}(t,\varphi)}{\partial t} + \inf_{p \in P(\varphi)} \left(\sum_{i=0}^{n} \left(\frac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} - p_i \right) \varphi_i(0)\rho_i(\varphi) + \mathbf{m}(\varphi, u)\mathbf{v}(t,\varphi) \right) \;\leq\; 0$$

satisfying the initial condition

$$\mathbf{v}(0,\varphi) \;=\; \mathbf{u}(\varphi)$$

on each of the subsets

1. European case:

$$\Omega_{(\mathbf{0},\mathbf{u}_\infty)}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t > 0 \;\&\; \mathbf{v}(t,\varphi) \;\geq\; 0\}$$

2. American Case

$$\Omega_{(\mathbf{u},\mathbf{u}\infty)}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t > 0 \;\&\; \mathbf{v}(t,\varphi) \;\geq\; \mathbf{u}(\varphi)\}$$

3. First time case

$$\Omega_{(\mathbf{0},\mathbf{u})}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t > 0 \;\&\; \mathbf{u}(\varphi) \;>\; \mathbf{v}(t,\varphi) \;\geq\; 0\}$$

The Regulation Map

We shall prove that the regulation map $\Gamma_{(\mathbf{b},\mathbf{c})}$ is defined on the corresponding set $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$ by

$$\begin{cases} \Gamma_{(\mathbf{b},\mathbf{c})}(t,\varphi) \;:=\; \Big\{ p \in P(\varphi) \;\; \textbf{such that } \mathbf{D}_{\uparrow}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi)(-1,\varphi_1(0)\rho_1(\varphi),\ldots,\varphi_n(0)\rho_n(\varphi)) \\ +\mathbf{m}(\varphi,p)\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi) - \displaystyle\sum_{i=1}^{n} p_i\varphi_i(0)\rho_i(\varphi) \leq 0 \Big\} \end{cases}$$

hence the regulation law providing the portfolios is

$$p(t) \;\in\; \Gamma_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x)$$

Merci pour votre
Thanks for your
Attention

See details in AUBIN J.-P. & HADDAD G. (2002) *History (Path) Dependent Optimal Control and Portfolio Valuation and Management*, **J. Positivity, 6, 331-358.**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