A Viability Approach to Impulse Control and Hybrid Systems

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Abstract

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Given a target contained in a constrained set and an impulse control system governing the evolutions of runs *or* executions*, that are hybrids of continuous and discrete evolutions, several characterizations of the viability kernel of a subset (with a target) is provided. It is the subset of initial runs from which start at least one run viable in the constrained set forever or until it reaches the target in finite time. Algorithms and regulation rules governing the runs that reach the targets while obeying state constraints are investigated.*

Introduction

There is no reason why an arbitrary subset $K \subset X$ should be viable under a control system

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases}$$

or a differential inclusion $x' \in F(x)$ in the sense that, from any initial state $x \in K$ starts at least one evolution $x(\cdot) \in \mathcal{S}(x)$ of the differential inclusion such that $x(t) \in K$ forever.

Reinitializing States

One way to reestablish viability is to change instantaneously the initial conditions when viability is at stakes or when other requirement is asked by using a reset map **Φ mapping any state of K to a (possibly empty) set $\Phi(x) \subset X$ of new "initialized states".**

Impulse Differential Inclusion

Hence an impulse differential inclusion **is described by a pair (F,Φ), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the** continuous evolution **components $x(\cdot)$ of the system in K and where Φ, the** reset map**, governs the** discrete **impulses to new "initial conditions" when the continuous evolution is doomed to leave K.**

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called a "run" **or an** "execution"**.**

Examples

Many examples coming from different fields of knowledge fit this framework:

1. multiple-phase economic dynamics in economics, as it was proposed by the economist Richard Day back to 1995,
2. stock management in production theory,
3. viability theory, for implementing the extreme version of the "inertia principle" and in particular, evolutions with **punctuated equilibria** proposed by Eldredge and Gould in 1972 for describing biological evolution,
4. propagation of the nervous influx along axones of neurons triggering spikes in neurosciences proposed in 1907 by Lapicque ("Integrate-and-Fire" models),
5. "threshold" impulse control, when "controls jump" when the threshold is about to be trespassed,
6. demographical models, to take into account discontinuous processes such as births and deaths,

7. **dynamical "qualitative physics" in Artificial Intelligence**
8. **and, above all, in automatic control theory where a fast growing literature deals with hybrid "systems".**

Hybrid Systems

Hybrid systems are described by a family of control systems and by a family of viability (or state) constraints indexed by parameters e called "locations". Starting with an initial condition in a set associated with an initial location, the control system associated with the initial location governs the evolution of the state in this set for some time until some impulse time resets the system by imposing a new location, and thus, a new control system, a new constrained set and a new initial condition. They fit in the above class of impulse systems.

Runs or Executions

We identify $\mathcal{C}(0,0;X)$ **with** X**.**

$$\overrightarrow{x}(\cdot) \ := \ (\tau_n, x_n(\cdot))_{n\geq 0} \ \in \ \prod_{n\geq 0} \mathbf{R} \times \mathcal{C}(0,\tau_n;X)$$

made of

1. **a finite or infinite sequence** $\tau(\overrightarrow{x}(\cdot)) := \{\tau_n\}_n$ **of nonnegative** cadences $\tau_n \in [0,+\infty[$**,**
2. **a sequence of** motives $x_n(\cdot) \in \mathcal{C}(0,\tau_n;X)$**.**

Development of a Run

We associate with a run $\overrightarrow{x}(\cdot)$

1. **its impulse set** $\mathcal{T}(\overrightarrow{x}(\cdot)) := \{t_n\}_{n \geq 0}$ **of sequences of impulse times** $t_{n+1} := t_n + \tau_n$, $t_0 = 0$

2. **its development defined by**

$$\forall\, n \geq 0,\ \overrightarrow{x}(t_n) := x_n(0)$$

$$\forall\, t \in [t_n, t_{n+1}[,\ \ \overrightarrow{x}(t)\ :=\ x_n(t - t_n)$$

We say that the sequence of $x_n := x_n(0) \in X$ is the sequence of reinitialized states **of the run $\overrightarrow{x}(\cdot)$.**

A run $\overrightarrow{x}(\cdot)$ is said to be viable in K on an interval $I \subset \mathbb{R}_+$ if for any $t \in I$, $\overrightarrow{x}(t) \in K$.

Particular Ends of a Run

A run $\overrightarrow{x}(\cdot) := (\tau_n, x_n(\cdot))_{n\geq 0}$ is said to be

1. discrete **if for some p and for all $n \geq p$, $\tau_n = 0$ (the run ends with a sequence),**
2. **infinite if the sequence of cadences is infinite and finite otherwise. In this case, it stops at some τ_N, and we agree to say that *the Nth motive $x_N(\cdot)$ is taken on* $[0, +\infty[$ and to set $\tau_{N+1} = +\infty$ (the run ends with a continuous-time evolution).**

Jumps of a Run

If $t \notin \mathcal{T}(\overrightarrow{x}(\cdot))$ **is not an impulse time of the run** $\overrightarrow{x}(\cdot)$**, then we can set** $\overrightarrow{x}(t) = x(t)$ **without ambiguity. At impulse time** $t_n \in \mathcal{T}(\overrightarrow{x}(\cdot))$**, we defined** $\overrightarrow{x}(t_n) := x_n(0)$**. But we also need to define the value of the run just before it "jumps" at impulse time:**

If $t_n \in \mathcal{T}(\overrightarrow{x}(\cdot))$ **is an impulse time of the run** $\overrightarrow{x}(\cdot)$**, we set** $\overrightarrow{x}({}^{-}t_n) :=$

$$\begin{cases} \lim_{\tau \mapsto t_n -} x(\tau) & \text{if} \quad t_n > t_{n-1} \\ x_{n-1}(0) \ := \ \overrightarrow{x}({}^{-}t_{n-1}) & \text{if} \quad t_n = t_{n-1} \end{cases}$$

We associate with a run $\vec{x}(\cdot)$ **its sequence of** jumps $\mathrm{s}(\vec{x}(\cdot)) := (\mathrm{s}_n(\vec{x}(\cdot)))_{n \geq 1}$ **defined by**

$$\mathrm{s}_n(\vec{x}(\cdot)) := x_n(0) - x_{n-1}(\tau_{n-1}) = \overrightarrow{x}(t_n) - \overrightarrow{x}({}^{-}t_n)$$

Integral Representation of a Run

Assume that the motives $x_n(\cdot)$ of a run $\overrightarrow{x}(\cdot)$ are differentiable on the intervals $]0, \tau_n[$ for all n such that the cadence $\tau_n > 0$ is positive. Therefore, for all j,

$$\begin{cases} (i) \quad \forall\, t \in [t_{j-1}, t_j[,\ \overrightarrow{x}(t) = x_0 + \sum_{k=1}^{j-1} \mathbf{s}_k(\vec{x}(\cdot)) \\ \qquad + \int_0^t x'(\tau) d\tau \ \text{ when } t_{j-1} < t_j \\ (ii) \quad x_j \ := \ \overrightarrow{x}(t_j) \ = \ x_0 + \sum_{k=1}^{j} \mathbf{s}_k(\vec{x}(\cdot)) \\ \qquad + \int_0^{t_j} x'(\tau) d\tau \end{cases}$$

Graph of a Set-Valued Map

The graph $\mathrm{Graph}(U)$ **of a set-valued map** $U : X \rightsquigarrow Y$ **is the set of pairs** $(x, y) \in X \times Y$ **satisfying** $y \in U(x)$**.**

The *domain* of U **is the subset of elements** $x \in X$ **such that** $U(x)$ **is not empty:**

$$\mathrm{Dom}(U) \ := \ \{x \in X \ \mid \ U(x) \neq \emptyset\}$$

The *image* of U **is the union of the images (or values)** $U(x)$**, when** x **ranges over** X**:**

$$\mathrm{Im}(U) \ := \ \bigcup_{x \in X} U(x)$$

The *inverse* U^{-1} **of** U **is the set-valued map from** Y **to** X **defined by** $x \in U^{-1}(y) \Longleftrightarrow$

$$y \in U(x) \iff (x, y) \in \mathrm{Graph}(U)$$

Differential Inclusions

Consider a system (f, U) where

1. $f : X \times \mathcal{U} \mapsto X$,
2. $U : X \rightsquigarrow \mathcal{U}$.

A control system (f, U) defines the *evolutionary system* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$: for any $x \in X$, $\mathcal{S}(x)$ is the set of evolutions $x(\cdot)$ governed by

$$\begin{cases} (i) & x'(t) \; = \; f(x(t), u(t)) \\ (ii) & u(t) \; \in \; U(x(t)) \end{cases} \tag{1}$$

starting from x. Setting $F(x) := f(x, U(x))$, are governed by the differential inclusion

$$x'(t) \; \in \; F(x(t)) \tag{2}$$

Viable Evolutions

Evolutions $x(\cdot)$ are viable in a subset $K \subset X$ if

$$\forall\, t \geq 0,\;\; x(t) \;\in\; K \tag{3}$$

A "target" $C \subset K$ being given, evolutions $x(\cdot)$ capture the target C if they are viable in K until they reach C in finite time:

$$\exists\, T \geq 0 \textbf{ such that} \begin{cases} x(T) \in C \\ \forall\, t \;\in\; [0,T],\; x(t) \;\in\; K \end{cases} \tag{4}$$

An evolution is viable in K outside C if it is viable in K forever or until it reaches the target C in finite time.

Viability Kernel and Capture Basin

Let $K \subset X$ be a constrained set and $C \subset K$ be a target.

1. **The subset** $\text{Viab}(K,C) := \text{Viab}_{\mathcal{S}}(K,C)$ **of initial states** $x_0 \in K$ **such that at least one evolution** $x(\cdot) \in \mathcal{S}(x_0)$ **starting at** x_0 **is viable in** K **for all** $t \geq 0$ **or viable in** K **until it reaches** C **in finite time is called the** viability kernel of K with target C **under** $\mathcal{S}$**.**

   When the target $C = \emptyset$**, we say that** $\text{Viab}(K) := \text{Viab}(K, \emptyset)$ **is the** viability kernel **of** K**.**

2. **The subset** $\text{Capt}(K, C) := \text{Capt}_{\mathcal{S}}(K, C)$ **of initial states** $x_0 \in K$ **such that at least one evolution** $x(\cdot) \in \mathcal{S}(x_0)$ **starting at** x_0 **is viable in** K **until it reaches** C **in finite time is called the** capture basin of C viable in K **under** $\mathcal{S}$**. When** $K = X$**, we set** $\text{Capt}(C) := \text{Capt}(X, C)$**.**

1. **a subset** K **is viable under** $\mathcal{S}$ **if** $K = \text{Viab}(K)$ **and** viable outside the target $C \subset K$ **under the evolutionary system** $\mathcal{S}$ **if** $K = \text{Viab}(K, C)$**,**

2. **that** C **is** isolated **in** K **if** $C = \text{Viab}(K, C)$**, and that** K **is a** repeller **if** $\text{Viab}(K) = \emptyset$**, i.e., if the empty set is isolated in** K**.**

Characterization of Viability Kernels

The viability kernel $\text{Viab}(K, C)$ of K outside the target C is the unique subset between C and K that

1. **viable outside C (and is the largest subset $D \subset K$ viable outside C),**
2. **isolated in K (and is the smallest subset $D \supset C$ isolated in K):**

$$\begin{aligned} \text{Viab}(K, \text{Viab}(K, C)) &= \text{Viab}(K, C) \\ &= \text{Viab}(\text{Viab}(K, C), C) \end{aligned}$$

Comments

This statement is at the root of uniqueness properties of solutions to some Hamilton-Jacobi-Bellman partial differential equations whenever the epigraph of a solution is a viability kernel of the epigraph of a function outside the epigraph of another function, and of (set-valued) solutions to systems of first-order partial differential equations whenever the subsets are graphs of maps.

Uniqueness of Bilateral Fixed Points

Let us consider a map $(K,C) \mapsto \mathcal{A}(K,C)$ satisfying

$$\begin{cases} (i) \quad C \subset \mathcal{A}(K,C) \subset K \\ (ii) \quad (K,C) \mapsto \mathcal{A}(K,C) \text{ is increasing} \end{cases} \tag{5}$$

1. **If $\mathcal{A}(K,C) = \mathcal{A}(\mathcal{A}(K,C),C)$, it is the largest fixed point of the map $D \mapsto \mathcal{A}(D,C)$ between C and K,**

2. **If $\mathcal{A}(K,C) = \mathcal{A}(K,\mathcal{A}(K,C))$, it is the smallest fixed point of the map $E \mapsto \mathcal{A}(K,E)$ between C and K.**

Then, any subset D between C and K satisfying

$$D \subset \mathcal{A}(D, C) \text{ and } \mathcal{A}(K, D) = D$$

is the unique **bilateral fixed point D between C and K of the map $\mathcal{A}$ in the sense that:**

$$\mathcal{A}(K, D) = D = \mathcal{A}(D, C)$$

and is equal to $\mathcal{A}(K, C)$.

Reset Map

The reset map $\Phi : X \rightsquigarrow X$ **governs the resetting of initial conditions. We set**

$$\mathbf{Viab}^1_\Phi(K) := K \cap \Phi^{-1}(K)$$

$$\mathbf{Egress}_\Phi(K) \ := \ K \backslash \Phi^{-1}(K)$$

The graphical restriction $\Phi|_{(K \to K)} : K \rightsquigarrow K$ **of** Φ **to** $K \times K$ **defined by**

$$\Phi|_{(K \rightsquigarrow K)}(x) \ := \begin{cases} \Phi(x) \cap K & \text{if} \ \ x \in \mathbf{Viab}^1_\Phi(K) \\ \emptyset & \text{if} \ \ x \in \mathbf{Egress}_\Phi(K) \end{cases}$$

$$\mathrm{Graph}(\Phi_{(K \rightsquigarrow K)}) \ = \ \mathrm{Graph}(\Phi) \cap (K \times K)$$

Impulse Evolutionary Systems

Let $\Phi : X \rightsquigarrow X$ and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$. We associate with any initial state $x \in X$ the subset $\mathcal{R}(x) := \mathcal{R}_{(\mathcal{S},\Phi)}(x)$ of runs $\overrightarrow{x}(\cdot)$ satisfying

$$\forall\, n \geq 0, \begin{cases} (i) & x_n(\cdot) \in \mathcal{S}(x_n(0)) \\ (ii) & x_{n+1}(0) \in \Phi(x_n(\tau_n)) \end{cases} \tag{6}$$

The set-valued map $\mathcal{R} : X \rightsquigarrow \prod_{n\geq 0} \mathbf{R} \times \mathcal{C}(0,\tau_n;X)$ is called the impulse evolutionary system associated with the pair $(\mathcal{S},\Phi)$.

Impulse Viability

A subset K is said to be viable outside a target $C \subset K$ under the impulse evolutionary system $\mathcal{R}$ is from every $x \in K$ starts at least one run $\overrightarrow{x}(\cdot) \in \mathcal{R}(\cdot)$ viable in K forever or until it reaches the target C in finite time.

Characterization of Impulse Viability

We set

$$\mathrm{Viab}^1_\Phi(K,C) \ := \ C \cup (K \cap \Phi^{-1}(K))$$

A subset $K \subset X$ **is viable outside the target** C **under the impulse evolutionary system** $\mathcal{R}$ **if and only if** K **is viable outside** $\mathrm{Viab}^1_\Phi(K,C)$ **under the evolutionary system** $\mathcal{S}$**:**

$$K \ = \ \mathrm{Viab}_{\mathcal{S}}(K, \mathrm{Viab}^1_\Phi(K,C))$$

Hybrid Differential Inclusions

An hybrid differential inclusion (K, F, Φ) is defined by

1. **a finite dimensional vector space E of states e called locations or events,**
2. **a set-valued map $K : E \rightsquigarrow X$ associating with any e a subset $K(e) \subset X$,**
3. **a set-valued map $F : \mathrm{Graph}(K) \rightsquigarrow X$ defining $x'(t) \in F(e, x(t))$,**
4. **a reset map $\Phi : \mathrm{Graph}(K) \rightsquigarrow E \times X$.**

It is a qualitative differential inclusion if $\Phi(e, x) := \Phi_E(e, x) \times \{x\}$.

Runs Hybrid Differential Inclusions

A run

$$\overrightarrow{x}(\cdot) := (\tau_n, e_n, x_n(\cdot))_n \in \prod_{n \geq 0} \mathbb{R}_+ \times E \times \mathcal{C}(0, \tau_n, X)$$

is a solution to such a hybrid differential inclusion if for every n**,**

1. **the motives** $x_n(\cdot)$ **are solutions to differential inclusion** $x_n'(t) \in F(e_n, x_n(t))$ **viable in** $K(e_n)$ **on the interval** $[0, \tau_n]$**,**
2. $(e_n, x_n(0)) \in \Phi(e_{n-1}, x_{n-1}(\tau_{n-1}))$

Hybrid systems are particular cases of impulse differential inclusions in the following sense:

A run $\overrightarrow{x}(\cdot) := (\tau_n, e_n, x_n(\cdot))_n$ **is a solution of the hybrid differential inclusions** (K, F, Φ) **if and only if**

$$(\overrightarrow{e}(\cdot), \overrightarrow{x}(\cdot)) := (\tau_n, (e_n(\cdot), x_n(\cdot)))_n$$

where $e_n := e(\tau_n)$ **is a run of the auxiliary system of impulse differential inclusions** $\mathcal{R}_{(G,\Phi)}$ **where** $G : E \times X \rightsquigarrow E \times X$ **defined by** $G(e,x) := \{0\} \times F(e,x)$ **governs the differential inclusion**

$$\begin{cases} i) & e'(t) \; = \; 0 \\ ii) & x'(t) \; \in \; F(e(t), x(t)) \end{cases}$$

viable in $\mathrm{Graph}(K)$**.**

Existence of Solutions to Impulse Differential Inclusions

Consider the set-valued map $K_1 : E \rightsquigarrow X$ defined by

$$\text{Graph}(K_1) \ := \ Viab^1_\Phi(\text{Graph}(K))$$

Then the hybrid differential inclusion has a solution for every initial state if and only if $\forall\ e \in E,\ \ K(e)$, $K(e)$ is viable under the differential inclusion$x'(t) \in F(e, x(t))$ outside $K_1(e)$.

Backward Discrete System

The sequence $(x_n)_{n\in\{0,\ldots,N\}}$ is the sequence of reinitialized states $x_n := x_n(0)$ of a run $\overrightarrow{x}(\cdot) := (\tau_n, x_n(\cdot))_{n\geq 0}$ of an impulse evolutionary system $\mathcal{R} := \mathcal{R}_{(\mathcal{S},\Phi)}$ viable in K outside C if and only if it satisfies the backward discrete dynamical system

$$x_n \; \in \; \text{Capt}_{\mathcal{S}}(K, K \cap \Phi^{-1}(x_{n+1}))$$

1. **for every $n \geq 0$ when the run is infinite and viable in K,**
2. **or until some $N \geq 0$ when**

$$x_N \; \in \; \text{Viab}_{\mathcal{S}}(K, C)$$

The inverse $\mathbf{I} := \mathbf{I}_{(\mathcal{S},\Phi)} : K \rightsquigarrow K$ of the map $y \rightsquigarrow \mathrm{Capt}_{\mathcal{S}}(K, K \cap \Phi^{-1}(y))$ is called the reinitialization map of the impulse evolutionary system restricted to K.

A subset $K \subset X$ is viable under the impulse evolutionary system $\mathcal{R}$ if and only if K is viable outside $\mathrm{Viab}_{\mathcal{S}}(K, C)$ under the reinitialization map $\mathbf{I}_{(\mathcal{S},\Phi)}$:

$$K \;=\; \mathrm{Viab}_{\mathbf{I}_{(\mathcal{S},\Phi)}}(K, \mathrm{Viab}_{\mathcal{S}}(K, C))$$

Therefore the behavior of a run is "summarized" by the reinitialization map $\mathbf{I}_{(\mathcal{S},\Phi)}$. It is a discrete dynamical system governing the sequence of reinitialized states of infinite runs of the impulse evolutionary system viable in K forever or until it reaches the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C)$ of K with target C under the the continuous-time evolutionary system, when the last motive of the run is viable forever or reaches the target C in finite time.

In other words,

1. **either** $\mathbf{I}_{(\mathcal{S},\Phi)}(x_n) \cap \text{Viab}_{\mathcal{S}}(K,C) = \emptyset$**, and then** $\mathbf{I}_{(\mathcal{S},\Phi)}(x_n)$ **is the set of new reinitialized conditions** $x_{n+1} \in \Phi(x_n(\tau_n)) \cap K$ **when** $x_n(\cdot) \in \mathcal{S}(x_n)$ **ranges over the set of evolutions starting at** $x_n \in K$ **viable in** K **until they reach** $\Phi^{-1}(K)$ **at time** $\tau_n \geq 0$ **at** $x_n(\tau_n) \in \Phi^{-1}(K)$**,**

2. **or** $\forall\ x_{N+1} \in \mathbf{I}_{(\mathcal{S},\Phi)}(x_N) \cap \text{Viab}_{\mathcal{S}}(K,C)$**, and thus a motive** $x_N(\cdot) \in \mathcal{S}(x_N)$ **such that** $x_{N+1} \in \Phi(x_N)(\tau_N) \cap K$**. From** x_{N+1} **starts a motive** $x_{N+1}(\cdot) \in \mathcal{S}(x_N)$ **which is viable in** K **forever or until it reaches the target** C**.**

Characterization of the Reinitialization Map

We introduce the auxiliary system

$$\begin{cases} (i) & x'(t) \;=\; f(x(t),u(t)), \;\; u(t) \in U(x(t)) \\ (ii) & y'(t) \;=\; 0 \end{cases} \tag{7}$$

Then $\mathrm{Graph}(\mathbf{I}_{(\mathcal{S},\Phi)}) \;=$

$$\mathrm{Viab}_{(7)}\left(K \times K, \mathrm{Graph}(\Phi|_{(K \leadsto K)})\right)$$

and coincides with the capture basin.

Hence the graph of the reinitialization map inherits the properties of the viability kernels and capture basins.

Substratum of an Impulse System

We denote by

$$(t,x) \mapsto \vartheta_{\mathcal{S}}^{K}(t,x) \ := \bigcup_{x(\cdot)\in\mathcal{S}^K(x)} \{x(t)\}$$

the reachable map of $x \in K$.

The **substratum** o on K is the set-valued map $\Gamma := \Gamma_{(\mathcal{S},\Phi)} : \mathbb{R}_+ \times K \rightsquigarrow K$ associating with any (t,x) the subset

$$\Gamma_{(\mathcal{S},\Phi)}(t,x) = \Phi(\vartheta_{\mathcal{S}}^{K}(t,x) \cap \Phi^{-1}(K)) \cap K$$

The set-valued map $\mathbf{T}_{(\mathcal{S},\Phi)}$ defined by

$$\mathbf{T}_{(\mathcal{S},\Phi)}(x) \ := \ \{\tau \geq 0 \ \mid \ \Gamma_{(\mathcal{S},\Phi)}(\tau,x) \ \neq \ \emptyset\}$$

is called the **cadence map.**

The inverse of the substratum $x \rightsquigarrow \Gamma_{(\mathcal{S},\Phi)(t,x)}$ is given by

$$\Gamma^{-1}_{(\mathcal{S},\Phi)}(t,y) \;=\; \vartheta^{K}_{\overleftarrow{\mathcal{S}}}(t, K \cap \Phi^{-1}(y))$$

The reinitialization map **is linked to the substratum and the cadence map by the formula:**

$$\mathbf{I}_{(\mathcal{S},\Phi)}(x) \;=\; \bigcup_{t \in \mathbf{T}_{(\mathcal{S},\Phi)}(x)} \Gamma_{(\mathcal{S},\Phi)}(t,x)$$

Chronector

We set

$$\varpi^{\flat}_{(\mathcal{S},\Phi)}(x) \ := \ \inf \ \mathbf{T}_{(\mathcal{S},\Phi)}(x)$$

The chronector of the impulse evolutionary system on K **is the set-valued map** $\mathbf{I}^{\flat}_{(\mathcal{S},\Phi)} : K \rightsquigarrow K$ **defined by**

$$\mathbf{I}^{\flat}_{(\mathcal{S},\Phi)}(x) \ := \ \Gamma_{(\mathcal{S},\Phi)}(\varpi^{\flat}_{(\mathcal{S},\Phi)}(x), x)$$

(where we set $\mathbf{I}^{\flat}_{(\mathcal{S},\Phi)}(x) = \emptyset$ **when** $\mathbf{T}_{(\mathcal{S},\Phi)}(x) = \emptyset$**).**

Using chronectors, the state of the system ***must be reinitialized as soon a motive*** $x(\cdot) \in \mathcal{S}(x)$ ***reaches*** $(K \cap \Phi^{-1}(K)) \backslash C$**.**

Construction of Runs with the Substratum

Given the cadence τ_n and the initial state x_n, we take

1. **the next cadence $\tau_{n+1} \in \mathbf{T}_{(\mathcal{S},\Phi)}(x_n)$,**
2. **the next reinitialized state**

$$x_{n+1} \in \Gamma_{(\mathcal{S},\Phi)}(\tau_{n+1}, x_n) \subset \mathbf{I}_{(\mathcal{S},\Phi)}(x_n)$$

3. **the next motive $x_n(\cdot) := x(\cdot + t_n) \in \mathcal{S}(x_n)$ satisfying $x_n(0) = x_n$ and $x_n(\tau_{n+1}) \in \Phi^{-1}(x_{n+1})$.**

Characterization of the Substratum

Introduce the auxiliary system

$$\left\{\begin{array}{ll} (i) & \tau'(t) \;=\; -1 \\ (ii) & x'(t) = f(x(t), u(t)), \;\; u(t) \in U(x(t)) \\ (iii) & y'(t) \;=\; 0 \end{array}\right. \tag{8}$$

The graph of the substratum $\Gamma_{(\mathcal{S},\Phi)}$ **of** $\mathcal{R}_{(\mathcal{S},\Phi)}$ **is the viability kernel of** $\mathbf{R}_+ \times K \times K$ **with target** $\{0\} \times \mathrm{Graph}(\Phi|_{(K \rightsquigarrow K)})$**:** $\mathrm{Graph}(\Gamma_{(\mathcal{S},\Phi)}) \;=$

$$\mathrm{Viab}_{(8)}\left(\mathbf{R}_+ \times K \times K, \{0\} \times \mathrm{Graph}(\Phi|_{(K \rightsquigarrow K)})\right)$$

and coincides with the capture basin.

Impulse Viability Kernel

The impulse viability kernel **of** K

$$\text{ImpViab}(K,C) \ := \ \text{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$$

with target C under the impulse evolutionary system $(\mathcal{S},\Phi)$ is the subset of initial states $x \in K$ from which starts at least **one run viable in K forever or until it reaches the target C in finite time.**

Fundamental Characterization of Impulse Viability Kernels

The impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ **of** K **with a target** $C \subset K$ **is**

1. **the** largest **subset** D **satisfying** $C \subset D \subset K$ **and** $D \subset \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(D,C)$**,**
2. **the** smallest **subset** D **satisfying** $C \subset D \subset K$ **and** $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,D) \subset D$**,**
3. **the** unique **subset** D **satisfying** $C \subset D \subset K$ **and**

$$D = \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,D) = \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(D,C)$$

Fixed Point Characterization of the Impulse Viability Kernel

The impulse viability kernel $\text{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ **is the largest of the fixed points of the pre-opening** $D \mapsto \text{Viab}_{\mathcal{S}}(D, \text{Viab}^1_{\Phi}(D,C)$**:**

$$\text{Viab}_{\mathcal{S}}(\text{ImpViab}_{(\mathcal{S},\Phi)}(K,C), \text{Viab}^1_{\Phi}(\text{ImpViab}_{(\mathcal{S},\Phi)}(K,C)), C)$$

$$= \text{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$$

Regulation of Viable Runs

Starting from $x_0 \in \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K, C)$ **a run** $\overrightarrow{x}(\cdot) = (\tau_n, x_n(\cdot))_{n \geq 0}$ **viable in** K **forever or until it reaches the target** C **in finite time is regulated in the following way: Assume that at stage** n**,** x_n **belong to the impulse viability kernel** $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K, C)$ **of** K **outside** C**.**

1. if $x_n \in C$**, then the run may stop,**

2. if $x_n \in \mathrm{Viab}^1_{\Phi}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)\backslash C$**, then we may take** $\tau_n = 0$ **and** $x_{n+1} \in \Phi(x_n) \cap \mathrm{ImpViab}_{(\mathcal{S},\Phi)}$ **as the next reinitialization state,**

3. if $x_n \in \mathrm{Viab}^1_\Phi(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)\backslash C$**, then we must take one motive motive** $x_n(\cdot) \in \mathcal{S}(x_n)$ **viable in** $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ **forever (and the run may stop) or until the first time** $\tau_n > 0$ **when**
$x_n(\tau_n) \in \mathrm{Viab}^1_\Phi(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)$**. In this case,**

(a) either $x_n(\tau_n) \in C$**, and the run may stop,**

(b) or $x_n(\tau_n) \in \Phi^{-1}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C))$**, and then, we take** $x_{n+1} \in \Phi(x_n(\tau_n) \cap \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ **as the next reinitialization state.**

The Impulse Viability Kernel Algorithm

Assume that K is a compact repeller under $\mathcal{S}$, that F is Marchaud and that $C \subset K$ and the graph of Φ are closed.

The sequence of subsets K_j starting at $K_0 := K$ and defined recursively by

$$K_{j+1} := \text{Viab}_{\mathcal{S}}(K_j, \text{Viab}^1_{\Phi}(K_j, C))$$

The impulse viability kernel $\text{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ **is obtained through the "Impulse Viability Kernel Algorithm":**

$$\text{ImpViab}_{(\mathcal{S},\Phi)}(K,C) \ = \ \bigcap_{j=0}^{+\infty} K_j$$

Impulse Capture Basins

The impulse capture basin **of** C

$$\text{ImpCapt}(K, C) \ := \ \text{ImpCapt}_{(\mathcal{S},\Phi)}(K, C)$$

viable in K **under the impulse evolutionary system** $(\mathcal{S}, \Phi)$ **is the subset of initial states** $x \in K$ **from which starts** at least **one run viable in** K **until it reaches the target** C **in finite time.**

Fixed Point Characterization of the Impulse Capture Basins

The impulse capture basin $\text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ **is the largest of the fixed points of the preopening** $D \mapsto \text{Capt}_{\mathcal{S}}(D, \text{Viab}_{\Phi}^{1}(D,C))$**:**

$$\text{Viab}_{\mathcal{S}}(\text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C), \text{Viab}_{\Phi}^{1}(\text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)), C)$$

$$= \text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$$

Second Fixed-Point Characterization

The impulse capture basin $\text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ **is the smallest fixed point containing** C **of the map** $D \mapsto \text{Capt}_{\mathcal{S}}(K, K \cap \Phi^{-1}(D))$**:**

$$\text{Capt}_{\mathcal{S}}(K, K \cap \Phi^{-1}(\text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C))) = \text{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$$

Impulse Capture Basin Algorithm

Consider the increasing sequence of subsets C_j starting at $C_0 := C$ and defined recursively by

$$C_{j+1} \ := \ \mathrm{Capt}_{\mathcal{S}}(K, K \cap \Phi^{-1}(C_j))$$

The impulse capture basin $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ **is obtained through the "Impulse Capture Basin Algorithm":**

$$\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C) \;=\; \bigcup_{j}^{+\infty} C_j$$

Zeno Runs

A run $\overrightarrow{x}(\cdot)$ is said to be

1. a Zeno run **if its life expectation**

$$\lambda(\overrightarrow{x}(\cdot)) := \sum_{n=0}^{+\infty} \tau_n = \lim_{n \to +\infty} t_n < +\infty$$

is finite, and nonZeno **in the opposite case,**

2. strict **is all the cadences $\tau_n > 0$ are strictly positive, i.e., if the sequence of impulse times t_n is strictly increasing,**

3. proper **if it is strict, infinite and nonZeno.**

We observe that *discrete runs are Zeno and that finite runs are nonZeno*.

Infinite and Nondiscrete Systems

We now observe that

1. **if K is a repeller under the evolutionary system $\mathcal{S}$, then any viable impulse evolutionary system $\mathcal{R}_{(\mathcal{S},\Phi)}$ is infinite (no run viable in K can end with a continuous-time evolution). Furthermore, if K is compact, all the cadences τ_n are bounded by $\max_{x\in K} \tau^{\sharp}_{K}(x) < +\infty$.**

2. **if** K **is a repeller under the reset map** Φ**, then any viable impulse evolutionary system** $\mathcal{R}_{(\mathcal{S},\Phi)}$ **is nondiscrete (no run can end with a discrete-time evolution).**

High Order Egress Sets

Let $\Phi : X \rightsquigarrow X$ **and** $K \subset X$ **be given. We define recursively the subsets** discrete egress set $\mathbf{Egress}^0_\Phi(K) := \mathbf{Egress}_\Phi(K)$ **and, for** $N \geq 1$**, the** discrete egress set **of order** N

$$\mathbf{Egress}^N_\Phi(K) \; := \; \mathbf{Egress}_\Phi(\mathbf{Viab}^N_\Phi(K))$$

$$\mathbf{Viab}^N_\Phi(K) \backslash \mathbf{Viab}^{N+1}_\Phi(K)$$

We observe that

$$x \in \mathbf{Egress}^N_\Phi(K) \text{ iff } \Phi(x) \cap \mathbf{Viab}^N_\Phi(K) \; = \; \emptyset$$

Properties of Egress Sets

Let $\Phi : X \rightsquigarrow X$ and $K \subset X$ be given. Then

$$K \; = \; \mathbf{Viab}_{\Phi}(K) \cup \bigcup_{N \geq 0} \mathbf{Egress}_{\Phi}^{N}(K)$$

is covered by the partition made of the viability kernel of K under Φ and the egress subsets of order $N \geq 1$.

The graphical restriction

$$\Phi_N := \Phi|_{(\mathbf{Viab}_{\Phi}^{N}(K) \rightsquigarrow \mathbf{Viab}_{\Phi}^{N}(K))}$$

of Φ to $\mathbf{Viab}_{\Phi}^{N}(K) \times \mathbf{Viab}_{\Phi}^{N}(K)$ maps the egress set $\mathbf{Egress}_{\Phi}^{N}(K)$ into $\mathbf{Egress}_{\Phi}^{N-1}(K)$.

Nondiscrete Impulse Systems

If K is a repeller under Φ, then

$$\mathbf{Viab}^1_\Phi(K) \;=\; \bigcup_{N\geq 1} \mathbf{Egress}^N_\Phi(K)$$

so that for any state $x \in \mathbf{Egress}^N_\Phi(K)$, there exists a run starting from x made of a sequence of at most N elements $x_{j+1} \in \Phi(x_j) \cap \mathbf{Egress}^{N-j}_\Phi(K)$ starting at x until $x_N \in \mathbf{Egress}_\Phi(K)$ form which must start a motive $x_N(\cdot) \in \mathcal{S}(x_N)$.

In particular, if $\text{Viab}^2_\Phi(K) = \emptyset$, then the reset map $\Phi|_{(K \rightsquigarrow K)}$ maps $\text{Viab}^1_\Phi(K)$ to $\text{Egress}_\Phi(K)$, so that the runs have *at most on jump* whenever they have to be reinitialized:

Let us assume that F is Marchaud, that K is compact and that the subsets $\Phi(K)$ and $\Phi^{-1}(K)$ are closed. If K is a repeller under $\mathcal{S}$ and if $\text{Viab}^2_\Phi(K) := K \cap \Phi^{-1}(K \cap \Phi^{-1}(K)) = \emptyset$, then the cadences τ_n of the viable runs $\overrightarrow{x}(\cdot) \in \mathcal{R}(x)$

range in an interval $[\overline{\tau}, \overline{T}]$ **where** $\overline{\tau} > 0$ **and** $\overline{T} < +\infty$**.**

“Summarizing” the Reset Map

We associate with $\Phi : X \rightsquigarrow X$ **the set-valued maps** ${}^{N}\Phi : \mathbf{Egress}_{\Phi}^{N}(K) \rightsquigarrow \mathbf{Egress}_{\Phi}(K) = K \backslash \Phi^{-1}(K)$ **defined by:** $\forall\, x \in \mathbf{Egress}_{\Phi}^{N}(K)$,

$$ {}^{N}\Phi(x) \ = \ \Phi_0(\Phi_1(\cdots(\Phi_{N-2}(\Phi_{N-1}(x))))) $$

and the map $\vec{\Phi} : K \backslash \mathbf{Viab}_{\Phi}^{1}(K) \rightsquigarrow X$ **defined by** $\forall\, x \in \mathbf{Egress}_{\Phi}^{N}(K)$,

$$ \vec{\Phi}(x) \ := \ {}^{N}\Phi(x) \ \subset \ \mathbf{Egress}_{\Phi}(K) $$

If K is a repeller under Φ, we can replace it by $\vec{\Phi}$ for which $\textbf{Viab}^2_{\vec{\Phi}}(K) = \emptyset$.

Continuous Dependence on Initial States

Let us assume that the subset K is closed, that F is Marchaud, that the graph of Φ is closed, that $K \cap \Phi(K)$ is compact and that

$$\mathrm{Viab}_{\mathcal{S}}(K) \cap \Phi(K) = \emptyset$$

holds true. Then the solution map $\mathcal{R}^K_{(F,\Phi)}$ is upper semicompact on $K\backslash\text{Viab}_{\mathcal{S}}(K)$: If x_0^ε converges to x_0 and if $\overrightarrow{x}^\varepsilon(\cdot) \in \mathcal{R}_{(\mathcal{S},\Phi)}(x_0^\varepsilon)$ is a solution to the impulse evolutionary system starting at x_0^ε, a subsequence (again denoted by) $\overrightarrow{x}^\varepsilon(\cdot)$ converges to a run $\overrightarrow{x}(\cdot) \in \mathcal{R}_{(\mathcal{S},\Phi)}(x_0)$.

Cadenced Runs

Assume that F is Marchaud, that the graph of the reset map Φ is closed, that $\mathrm{Viab}_{\mathcal{S}}(K) = \emptyset$, that $\mathrm{Viab}^2_{\Phi}(K, C) = \emptyset$ and that K is compact. Let $\overrightarrow{x}(\cdot) := (\tau_n, x_n(\cdot))_{n \geq 0} \in \mathcal{R}(x)$ be a run viable in K associated with a sequence $\mathcal{T}(x(\cdot))$ of impulse times t_n viable in K.

If the sequence of reinitialized states $\overrightarrow{x}(t_n)$ of the run $\overrightarrow{x}(\cdot)$ converges to some $\overline{x}$, then a subsequence of the motives $x_n(\cdot)$ converges to

the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ and viable in K (the cadences and the motives are constant: for every $n \geq 0$, $\tau_n = \overline{\tau}$ and $x_n(\cdot) = \overline{x}(\cdot)$)

Dynamical Games

A dynamical game (P, Q, F) **is defined by**

- **a "set-valued feedback map"** $P : X \rightsquigarrow \mathcal{P}$
- **a "tychastic" (or perturbation, disturbance) set-valued map** $Q : X \rightsquigarrow \mathcal{Q}$

- **a set-valued map** $F : X \times \mathcal{P} \times \mathcal{Q} \leadsto X$ **describing the dynamics of the dynamical game:**

$$\left\{ \begin{array}{lll} (i) & x'(t) & \in & F(x(t), u(t), v(t))) \\ (ii) & u(t) & \in & P(x(t)) \\ (iii) & v(t) & \in & Q(x(t)) \end{array} \right.$$

Evolutionary Games

A dynamical game generates the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ where $\widetilde{\mathcal{V}}$ is the set of continuous selections of the set-valued map Q (feedbacks) and where $\mathcal{S}_{\widetilde{v}}(x)$ is the set of solutions to the differential inclusion

$$x'(t) \in F(x(t), P(x(t)), \widetilde{v}(x(t)))$$

starting at x.

Impulse Dynamical Games

Consider a family of set-valued maps $\Phi_{\widetilde{v}} : X \rightsquigarrow X$, regarded as reset maps **indexed by $\widetilde{v}$.**

$$\mathcal{G} : (x, \widetilde{v}) \in X \times \widetilde{\mathcal{Q}} \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x) \in \mathcal{C}(0, \infty; X)$$

be an evolutionary dynamical game**.**

We denote by $\mathcal{R}_{\widetilde{v}} := \mathcal{R}_{(\mathcal{S}_{\widetilde{v}}, \Phi_{\widetilde{v}})}$ the impulse evolutionary system indexed by $\widetilde{v}$. An impulse evolutionary game is a family $\mathcal{R}_{\widetilde{v}} := (\mathcal{R}_{(\mathcal{S}_{\widetilde{v}}, \Phi_{\widetilde{v}})})_{\widetilde{v}}$ of impulse evolutionary systems.

Conditional Impulse Viability Kernels

Consider an impulse evolutionary game $(\mathcal{S}_{\widetilde{v}}, \Phi_{\widetilde{v}})_{\widetilde{v}}$ **where** $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ **and two subsets** K **and** $C \subset K$**. We set**

$$\text{ImpViab}_{\widetilde{v}}(K, C) \ := \ \text{ImpViab}_{(\mathcal{S}_{\widetilde{v}}, \Phi_{\widetilde{v}})}(K, C)$$

and say that the subset

$$\text{CondImpViab}_{\mathcal{G}}(K, C) \ := \ \bigcap_{\widetilde{v}} \text{ImpViab}_{\widetilde{v}}(K, C)$$

is the conditional impulse viability kernel with a target under the impulse evolutionary game $(\mathcal{S}, \Phi)$.

We say that K is conditionally viable outside C **if**

$$K \subset \mathrm{CondImpViab}_{\mathcal{G}}(K, C)$$

i.e., if for any state $x_0 \in K$, for all feedbacks $\widetilde{v}$**,** there exists at least **one run $\overrightarrow{x}(\cdot) \in \mathcal{R}_{\widetilde{v}}(x_0)$ viable in K forever or until it reaches C.**

Sufficient Condition

If a subset D between C and K satisfies

$$\begin{cases} (i) \quad \forall\, \widetilde{v} \in \widetilde{\mathcal{Q}}, \quad D \;\subset\; \mathrm{ImpViab}_{\widetilde{v}}(K, D) \\ (ii)\ \exists\, \widetilde{v}_0 \in \widetilde{\mathcal{Q}} \ \textbf{ such that } \mathrm{ImpViab}_{\widetilde{v}_0}(D, C) \;\subset\; D \end{cases} \tag{9}$$

then D is equal to $\mathrm{CondImpViab}_{\mathcal{G}}(K, C)$ and is the unique bilateral fixed point **of** $\bigcap_{\widetilde{v}} \mathrm{ImpViab}_{\widetilde{v}}(\cdot, \cdot)$**:**

$$\mathrm{CondImpViab}_{\mathcal{G}}\left(\mathrm{CondImpViab}_{\mathcal{G}}(K, C), C\right)$$

$$
\begin{aligned}
&= \mathrm{CondImpViab}_{\mathcal{G}}(K, C) \\
&= \mathrm{CondImpViab}_{\mathcal{G}}\big(K, \mathrm{CondImpViab}_{\mathcal{G}}(K, C)\big)
\end{aligned}
$$

Fixed-Point Characterization

The conditional viability kernel

$$\text{CondImpViab}_{\mathcal{G}}(K,C)$$

of a subset K **with target** $C \subset K$ **is the** smallest **subset** $D \in \mathcal{D}(K,C)$ **satisfying**

$$\text{CondImpViab}_{\mathcal{G}}(K,D) \ \subset \ D$$

and actually is a fixed point

$$\text{CondImpViab}_{\mathcal{G}}(K,C)$$

$$= \bigcap_{\widetilde{v}} \mathrm{ImpViab}_{\widetilde{v}}\big(K, \mathrm{CondImpViab}_{\mathcal{G}}(K, C)\big)$$

The Conditional Basin is not Conditionally viab s not Conditionally viable

Unfortunately, the conditional impulse viability kernel $\mathrm{CondImpViab}_{\mathcal{G}}(K, C)$ **of** K **outside the target** $C \subset K$ **is not itself necessarily conditionally viable outside the target** C**, i.e., it is not a fixed point of the map** $K \mapsto \mathrm{CondImpViab}_{\mathcal{G}}(K,$

As in the case of dynamical games, we follow the idea of Pierre Cardaliaguet by introducing

the discriminating impulse kernel $\mathrm{ImpDisc}(K, C)$ **of** K **with target** C**:**

Impulse Discriminating Kernel

Let us consider an impulse evolutionary game $(\mathcal{S}, \Phi)$ and two subsets K and $C \subset K$.

We say that the largest subset $D \in \mathcal{D}(K, C)$ conditionally viable in K outside C under the impulse evolutionary game is the discriminating impulse kernel of K with target C under the evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$**, denoted by** $\mathrm{ImpDisc}(K, C)$**.**

Fixed Point Property

The impulse Discriminating Kernel $\mathrm{ImpDisc}(K, C$ **is the largest fixed point of the map**

$$K \in \mathcal{D}(K, C) \mapsto \mathrm{CondImpViab}_{\mathcal{G}}(K, C):$$

$$\mathrm{ImpDisc}(\mathrm{ImpDisc}(K, C), C) \;=\; \mathrm{ImpDisc}(K, C)$$

Remarks on Impulse Discriminating Kernels

Let us consider an evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$**.**

Then

$$\left\{ \begin{array}{ll} (i) & C \mapsto \mathrm{ImpDisc}(K, C) \text{ is increasing} \\ (ii) & \mathrm{ImpDisc}(K, \mathrm{ImpDisc}(K, C)) \\ & = \ \mathrm{ImpDisc}(K, C) \end{array} \right. \quad (10)$$

Characterization of the Impulse Discriminating Kernel

The discriminating impulse kernel $\mathrm{ImpDisc}(K, C)$ **of a subset** K **with target** $C \subset K$ **is**

1. **the** largest **subset** $D \in \mathcal{D}(K, C)$ **conditionally viable outside the target** C**,**
2. **the** smallest **subset** $D \in \mathcal{D}(K, C)$ **satisfying** $\mathrm{ImpDisc}(K, D) = D$**,**

3. the unique **minimax** $D \in \mathcal{D}(K, C)$ **in the sense that**

$$D = \mathrm{ImpDisc}(K, D) = \mathrm{ImpDisc}(D, C)$$

The Discriminating Impulse Viability Kernel Algorithm

We set $K_0 := K$ **and**

$$\forall\, i \geq 0, \quad K_i \;:=\; \mathrm{CondImpViab}_{\mathcal{G}}(K_{i-1}, C)$$

$$\bigcap_{\widetilde{v} \in \widetilde{\mathcal{Q}}} \mathrm{ImpViab}_{\widetilde{v}}(K_{i-1}, C)$$

Let us assume that the evolutionary game $(x, \widetilde{v}) \sim$
$\mathcal{S}_{\widetilde{v}}(x)$ **is upper semicompact. Then**

$$\mathrm{ImpDisc}(K, C) \; = \; \bigcap_{i=1}^{\infty} K_i$$