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Indices de Complexité Connexionnistes : Simplicité connexionniste sous-jacent des modèles dynamiques de décentralisation par les prix en économie

LASTRE
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Jean-Pierre Aubin

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Cet exposé présente des métaphores mathématiques de la complexité connexionniste de divers réseaux (neuronaux et/ou socio-économiques), permettant d'étudier divers problèmes liés à la régulation et à l'évolution de l'architecture de réseaux décrits par des opérateurs connexionnistes. Les opérateurs seront supposés linéaires, et représentés par des matrices de connexion. On introduit un indice de complexité connexionniste (statique) mesurant la distance entre de simples opérateurs connexionniste de référence et un indice dynamique mesurant la vitesse d'évolution des matrices connexionnistes, entre des vitesses nulles (équilibres ponctués) et de très grandes vitesses, causant l'émergence de nouveaux opérateurs connexionnistes (organisations émergentes).

On démontre que dans un cadre économique, la régulation par les matrices connexionnistes minimisant à chaque instant un indice de complexité connexionniste fournit les mêmes évolutions que les modèles décentralisés par les prix de norme minimum (modèles de planification à la Malinvaud-De la Vallée Poussin-Henry).

Dans les deux cas, l'évolution des matrices obéissent à des lois hebbiennes introduites a priori en 1949 par un neuro-physicien avant l'heure, lois qui constituent le cri de ralliement des spécialistes des réseaux de neurones formels.

The complexity of dynamic economic systems stems from a non teleological collective evolution of the set of agents, even though many individual economic agents think of themselves as being pursuing definitive and rational aims, instead of adapting permanently **to the many viability constraints (among which scarcity constraints) they face under uncertainty, either stochastic or contingent.**

This theme has been introduced and studied in the economic theory under the name of "bounded rationality". Indeed, economic agents are humans, not computers, are seldom rational, obey inertia conditions, are poor forecasters, actually very myopic. They base many decisions not on rational grounds, but on faith, beliefs and bets, rules of thumb, moods, rumors, and the like. They are more inductive than deductive in their learning processes. Actually, we can adopt Peirce terminology and look at them as "abductive", i.e., as making conjectures rather than predictions.

Even though we are far away of being able to capture mathematically the indomitable psychological, sociological, political and cultural behavior of human beings which warns their purely economic behavior set-valued analysis, differential inclusions and viability theory where introduced during the last decade for analyzing these evolutionary features in an uncertain — contingent rather than stochastic — world.

We shall investigate one aspect of complexity closely related to the decentralization issue, connectionist complexity, as one answer to adapt to more and more viability constraints, which implies the emergence of links between the agents of the dynamical economy and their evolution.

We compare it with other dynamical decentralization mechanisms, such as price decentralization.

The emergence of the cyberspace may give a new life to connectionist mechanisms, since any two agents would soon be able to communicate at each instant without the mediation of a price or other messages providing the necessary information.

Assume for instance that, if there were no scarcity constraints to comply with, the dynamical behavior of each consumer $i = 1, \ldots, n$ would be fully decentralized, and the evolution of its consumption $x_i(t) \in Y$ at time $t \geq 0$ would be governed by a differential equation of the form

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \;=\; g_i(x_i(t))$$

There is no reason why the scarcity constraints

$$\forall\, t \geq 0, \;\; \sum_{i=1}^{n} x_i(t) \;\in\; M$$

of a collective nature would be satisfied by such independent behaviors.

One first solution which comes up to the mind is to "connect" these dynamics through a connection matrix $W := (w_i^j) \in \mathcal{L}(X,X)$ and to govern the evolution of consumptions according to an evolution law of the form

$$\forall\, i = 1,\ldots,n, \;\; x_i'(t) \;=\; \sum_{j=1}^{n} w_j^i(t) g_j(x_j(t))$$

where $w_i^j \in \mathcal{L}(Y,Y)$. Naturally, we obtain a fully decentralized economy when the connection matrix is the identity matrix.

We propose mathematical metaphors designing evolution laws governing the evolution of connection matrices linking the autonomous dynamics of each agent in order to sustain the viability constraints prescribed by the environment of the isolated system.

l Our point is the following: Starting with a disconnected system — for which the nondiagonal elements of the connection matrix are all equal to zero — the viability can be maintained whenever each agent of the system is linked with other through an evolving connectionist matrix, dictating cooperation among the agents.

The larger the number of independent constraints, the more numerous the links between any agent with the others. In the mathematical version of the metaphor, a class of evolution laws governing the evolution of these connection matrices can be designed from the disconnected dynamical system and the set of constraints.

We regard here *connectionism* — a less normative and more neutral term than *cooperation* whenever the system arises in economics or biology — as an answer to adapt to more and more viability constraints, which implies the emergence of links between the components of a system and their evolution. A system is disconnected (or autonomous, free, autarchic, decentralized, etc.) **if the connection matrix is the identity (or unit) matrix. In some loose words, the distance between the** connection matrix **— which is the matrix linking each component of the system to the others — and the identity matrix should capture and measure the concept of an** index of (connectionist) complexity**. The larger such a connectionist complexity index, the more complex — or labyrinthine or intricate — the connectionist feature of the system.**

The velocity $W'(t)$ of a time dependent connection matrix $W(t)$ starting from the identity matrix as we shall do in the case of heavy evolution can be used as another measure of connectionist complexity.

One can also measure other features of connectionnist complexity through the sparsity of the connection matrix**, i.e., the number — or the position — of entries which are equal to zero or "small". The sparser such a connectionist matrix, the less complex the system.**

We shall single out for instance the problem of maximal decentralization which requires to find connecting matrices $W(t)$ as close as possible to the identity matrix 1

— slow evolution —

or which evolve as slowly as possible while sustaining the viability of the allocation of scarce resources

— heavy evolution —

in such a way that the connection matrix remains constant as long as the viability of the system is not at stakes.

We advocate the latter mode, which has the property of locking in the connection matrices which can regulate forever a nonempty subset of states, called the viability niche of the connection matrix.

Physicists have attempted to measure **"complexity" in various ways, through the concept of Clausius's entropy, Shannon's information, the degree of regularity instead of randomness, "hierarchical complexity" in the display of level of interactions, "grammatical complexity" measuring the language to describe it, temporal or spatial computational, measuring the computer time or the amount of computer memory needed to describe a system, etc.**

The problem is that living systems being open, there is no way to describe them entirely and consequently, to demonstrate the truth of any proposition. We are left with metaphors, and among them, mathematical ones, to validate a consensual definition of one meaning — actually, one submeaning, so to speak — of complexity.

Therefore, scientific activity begins by dividing a system into two classes, the system under study and its environment. This division is always arbitrary, but often justified by the scientists in quest of explanation. Once conceptually isolated from its environment, a living system fuels itself in the last analysis on solar energy through the consumptions of wastes of the other components of the open system set apart in the description of the environment. Each component of the system which can evolve independently in the absence of constraints, must interact each other in order to maintain the viability of the system imposed by its environment.

This attempt to sustain the viability of the system by connecting the dynamics of its agents may be a general feature of "complex systems".

Is not complexity meaning in the day-to-day language the labyrinth of connections between the components of a living organism or organization or system ? Is not the purpose of complexity to sustain the constraints set by the environment and its growth parallel to the increase of the web of constraints ?

Economic history has shown an everlasting trend toward a highly connected network of labyrinthine — connectionist — complexity. In this sense, complexity arose with the apparition of life and seems necessary to the pursuit of its evolution at all levels of organization.

Compounded with this issue of connectionist complexity — that is naturally involved in neural networks — is the question of whether these links relating one agent to another can be subsumed by regulation parameters — which we call in short regulees — and whether and when they provide the same evolutions of the state of the system. Such regulees are messages sent to the consumers, and the simplest ones are the prices.

In a decentralized mechanism **— actually, we should say in an "a-centralized mechanism", since we do not assume the existence of a "center" — the information on the alloaction problem is split and mediated by, say, a "message" which summarizes part of the information. In our case, we use the "price"** p **as a main example of message (actually, the message with the smallest dimension). Knowing the price** p**, consumers are supposed to know how to choose their consumption bundle,** without

- **knowing the behavior of their fellow consumers**
- **knowing the set of scarce resources**

There are many other decentralized models, such as "rationing" mechanisms which involve shortages (and lines), or "frustration" of consumers, or "monetary" mechanisms, or others.

Naturally, there is no "pure" decentralization, since the choice of the decentralization message is in some sense centralized. The prices help consumer to make their choice in a decentralized way, but the difficulty is postponed to explain the evolution of price.

In order to comply to viability constraints, we investigate successively

1. **the regulation by subtracting prices to the original dynamical behavior**

$$x'(t) \; = \; g(x(t)) - p(t)$$

2. **the regulation by connecting the agents of the original dynamics through connecting matrices**

$$x'(t) \; = \; W(t)g(x(t))$$

3. **a combination of both regulation procedures**

$$x'(t) \; = \; W(t)g(x(t)) - p(t)$$

We also need to "differentiate" viability constraints, and for that purpose, we need to implement the concept of tangency to any subset.

When K **is a subset of** X **and** x **belongs to** K**, the** contingent cone $T_K(x)$ **to** K **at** x **is the closed cone of elements** v

$$\begin{cases} v \in T_K(x) \ \textbf{if and only if} \ \exists\, h_n \to 0+ \ \text{ and } \ \exists\, v_n \to v \\ \textbf{such that} \ \forall\, n, \ \ x + h_n v_n \in K \end{cases}$$

We recall that a function $x(\cdot) : I \mapsto X$ **is said to be** viable in K **if and only if**

$$\forall\, t \geq 0, \;\; x(t) \; \in \; K$$

Then a differentiable viable function in K **satisfies**

$$\forall\, t \geq 0, \;\; x'(t) \; \in \; T_K(x(t))$$

If $K := h^{-1}(M)$ **where** $h : X \mapsto Y$ **is a continuously differentiable map such that** $h'(x)$ **is surjective and** M **is closed and convex (or, more generally, sleek), then**

$$T_K(x) \; = \; h'(x)^{-1} T_M(h(x))$$

As the simplest example, we look for ways of coordinating the decisions of the consumers by subtracting prices to their dynamical priceless behavior

$$\forall\ i = 1, \ldots, n,\ \ x_i'(t) \ = \ g_i(x_i(t)) - p_i(t)$$

in order that viability constraints of the form

$$\forall\ t \geq 0,\ \ h(x_1(t), \ldots, x_n(t)) \ \in \ M$$

are always satisfied.

We define the Regulation Map Π_M **by**

$$\Pi_M(x) \ := \ \{p \mid h'(x)(g(x) - p) \in T_M(h(x))\}$$

The prices $p(t) := (p_1(t), \ldots, p_n(t))$ **regulating viable solutions are given by the regulation law**

$$p(t) \ \in \ \Pi_M(x_1(t), \ldots, x_n(t))$$

In the case of the problem of allocations of scarce resources

$$\forall\, t \geq 0, \;\; \sum_{i=1}^{n} x_i(t) \;\in\; M$$

for instance, we saw that

$$\Pi_M(x) \;:=\; \frac{1}{n}\left(T_M\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} g_i(x_i)\right)$$

For building feedback prices, we can for instance think of explicitly selecting some prices of the regulation map, for instance, the price $\varpi^{\circ}(x) \in \Pi_M(x)$ with minimal norm. Viable solutions obtained with this feedback price are called slow viable solutions.

When $B \in \mathcal{L}(X, Y)$ is surjective, its orthogonal right inverse is equal to

$$B^+ \;=\; B^\star (BB^\star)^{-1}$$

that we can supply Y with the final norm μ^B defined by $\mu^B(z) := \|B^+ z\|$ and that we denote by π_K^B the projector of best approximation onto the closed subset K for this final norm.

In this case, the unique solution $\bar{x}$ to the minimization problem

$$\inf_{Bx \in K+v} \|x - u\|$$

is equal to

$$\bar{x} \;=\; u - B^+(1 - \pi_M^B)(Bu - v)$$

When M is convex (or more generally, sleek), then its tangent cones $T_M(y)$ are convex.

In this case, since the polar cone to the contingent cone $T_M(y)$ is the normal cone $N_M(y)$, we also know that we can write the solution to the minimization problem

$$\inf_{Bx \in T_M(y)+v} \|x-u\|$$

is equal to

$$\bar{x} \;=\; u - B^{+}(1-\pi^{B}_{T_M(y)})(Bu-v)$$

It is also equal to

$$\bar{x} \;=\; u - B^{\star}\pi^{B^{\star}}_{N_M(y)}(BB^{\star})^{-1}(Bu-v)$$

where $\pi^{B^{\star}}_{N_M(y)}$ denotes the projector onto the normal cone $N_M(y)$ when the dual $Y^{\star}$ is supplied with the "dual final scalar product".

Let us assume that

$$\forall\, x \in K, \quad h'(x) \text{ is surjective} \tag{1}$$

and that M **is closed convex (or more generally, sleek). Then the slow solution of the dynamical economy**

$$x'(t) \;=\; g(x(t)) - p(t) \tag{2}$$

subjected to the viability constraints

$$\forall\, t \geq 0, \quad h(x(t)) \;\in\; M$$

is the solution to the differential equation

$$x'(t) \;=\; g(x(t)) - \varpi^{\circ}(x(t))$$

where

$$\left\{\begin{array}{l} \varpi^{\circ}(x) \\ = \; h'(x)^{+}\left(1 - \pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x) \\ = \; h'(x)^{\star}\pi_{N_M(h(x))}^{h'(x)^{\star}}(h'(x)h'(x)^{\star})^{-1}h'(x)g(x) \end{array}\right. \tag{3}$$

We look for the problem of sustaining viability of the evolution by connecting each consumer with the other ones through a connection matrix $W := (w_i^j) \in \mathcal{L}(X,X)$**:**

$$\forall\ i = 1,\dots,n,\ \ x_i'(t) \ = \ \sum_{j=1}^{n} w_i^j(t) g_j(x_j(t))$$

where $w_i^j(t) \in \mathcal{L}(Y,Y)$**. We write it in the form**

$$x'(t) \ = \ W(t)g(x(t)) \tag{4}$$

Naturally, we obtain a fully decentralized economy when the connection matrix is the identity matrix. But this is not always the case, and consumers may take into account the behavior of the other consumers in their dynamics. Then the new regulation parameter is no longer a price $p \in X^\star$, but a connection matrix W**.**

We introduce the new regulation map

$$R_M(x) \;:=\; \{W \in \mathcal{L}(X, X)$$

such that

$$h'(x) W g(x) \;\in\; T_M(h(x))\}$$

Whenever $p \in X^\star$ and $y \in Y$, we denote by $p \otimes y \in \mathcal{L}(X, Y)$ the rank one linear operator defined by

$$x \mapsto (p \otimes y)(x) := \langle p, x \rangle y$$

the matrix of which is

$$\left(p^i y_j\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$$

A connection matrix of the form

$$W(x) := 1 - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi(x)$$

the entries of which are equal to

$$w_{ij}(x) \ = \ \delta_{i,j} - \frac{g(x)_i \varpi(x)_j}{\|g(x)\|^2}$$

belongs to $R_M(x)$ if and only if $\varpi(x)$ belongs to $\Pi_M(x)$ and the viable solutions to the differential equations $x' = W(x)g(x)$ and $x' = g(x) - \varpi(x)$ do coincide.

Indeed, we observe that

$$W(x)g(x) \;=\; g(x) - \frac{\langle g(x), g(x)\rangle}{\|g(x)\|^2}\varpi(x) \;=\; g(x) - \varpi(x)$$

so that the two differential equations $x' = W(x)g(x)$ **and** $x' = g(x) - \varpi(x)$ **are the same. Furthermore, to say that** $W(x) \in R_M(x)$ **amounts to saying that**

$$h'(x)W(x)g(x) \;=\; h'(x)g(x) - h'(x)\varpi(x) \;\in\; T_M(h(x))$$

so that the viability conditions are the same. $\square$

In other words, the regulation by connection matrices of the form

$$W(x) := 1 - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi(x)$$

is equivalent to the price decentralization mechanism.

Let us assume that

$$\forall\, x \in K, \quad h'(x) \text{ is surjective} \tag{5}$$

and that M is closed convex (or sleek). The solution with maximal decentralization is governed by the differential equation

$$x'(t) \;=\; W^{\circ}(x(t))g(x(t))$$

where

$$\begin{cases} W^{\circ}(x) \\ = \; 1 - \dfrac{g(x)}{\|g(x)\|^2} \otimes h'(x)^{+} \\ \left(1 - \Pi^{h'(x)}_{T_M(h(x))}\right) h'(x)g(x) \\ = \; 1 - \dfrac{g(x)}{\|g(x)\|^2} \otimes h'(x)^{\star} \\ \pi^{h'(x)^{\star}}_{N_M(h(x))}(h'(x)h'(x)^{\star})^{-1}(h'(x)g(x)) \end{cases} \tag{6}$$

Furthermore, the slow viable solutions to dynamical economy regulated by prices and the viable solutions to the connection economy under maximal decentralization coincide.

Let us consider the mixed regulation mode using both connection matrices W **and regulees** p**:**

$$x'(t) \; = \; Wg(x(t)) - p(t)$$

subjected to the same viability constraints

$$\forall \, t \geq 0, \;\; h(x(t)) \; \in \; M$$

Then the regulation map takes the form

$$R_M(x) \; := \; \{(W, p) \in \mathcal{L}(X, X) \times X$$

such that

$$h'(x)Wg(x) - h'(x)p \; \in \; T_M(x)\}$$

We supply the space $\mathcal{L}(X, X) \times X$ **with the norm** $\nu_{\alpha,\beta}$ **defined by**

$$\nu_{\alpha,\beta}(W, p)^2 \; := \; \alpha \|W\|^2 + \beta \|p\|^2$$

Then the slow solutions are regulated by the pair $(W_{\alpha,\beta}(x), \varpi_{\alpha,\beta}(x))$ **in the regulation map which minimizes the distance**

$$\nu_{\alpha,\beta}(W-1,p)$$

in $R_M(x)$ **The solution is given by**

$$\begin{cases} W_{\alpha,\beta}(x) \ = \\ 1 - \dfrac{\beta g(x)}{\alpha+\beta\|g(x)\|^2} \otimes h'(x)^+ \\ \left(1-\pi^{h'(x)}_{T_M(h(x))}\right) h'(x)g(x) \\ \\ \varpi_{\alpha,\beta}(x) \ = \ \dfrac{\alpha}{\alpha+\beta\|g(x)\|^2} h'(x)^+ \\ \left(1-\pi^{h'(x)}_{T_M(h(x))}\right) h'(x)g(x) \end{cases}$$

We then observe that, setting

$$\varpi^\circ(x) := h'(x)^+\left(1 - \Pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x)$$

$$\left\{\begin{array}{l} W_{\alpha,\beta}(x)g(x) - \varpi_{\alpha,\beta}(x) \\ = \;\; g(x) - \dfrac{\beta\|g(x)\|^2}{\alpha+\beta\|g(x)\|^2}\varpi^\circ(x) - \dfrac{\alpha}{\alpha+\beta\|g(x)\|^2}\varpi^\circ(x) \\ = \;\; g(x) - \varpi^\circ(x) \end{array}\right.$$

Therefore, for every pair (α,β)**, the slow viable solutions, which are solutions to the differential equations**

$$x'(t) \; = \; W_{\alpha,\beta}(x(t))g(x(t)) - \varpi_{\alpha,\beta}(x(t))$$

are the slow solutions to

$$x'(t) \; = \; g(x(t)) - \varpi^\circ(x(t)) \;\; \Box$$

Using a pricing map $P : K \rightsquigarrow X$, we consider a dynamical economy

$$\left\{ \begin{array}{ll} i) & x'(t) \;=\; g(x(t)) - p(t) \\ \\ ii) & p(t) \in P(x(t)) \\ \\ \end{array} \right. \tag{7}$$

subject to the scarcity constraints

$$\forall\, t \geq 0, \;\; h(x(t)) \;\in\; M$$

Instead of connecting the dynamics, we connect the constraints by allowing transactions of the form

$$\forall\, t \geq 0, \;\; h(W(t)x(t)) \;\in\; M$$

and we set a bound on the velocity of the connection matrix:

$$\|W'(t)\| \;\leq\; \varphi(x(t), W(t))$$

where B denotes the unit ball of the space $\mathcal{L}(X, X)$.

Let us assume that

$$\forall\, x \in K,\ \forall\, W,\ h'(Wh(x)) \text{ is surjective} \tag{8}$$

and that M **is convex (or sleek).**

Then from any initial $(x_0, W_0, p_0) \in \mathcal{K}$ **starts one heavy solution** $x(\cdot)$ **to the connected system starting at** x_0 **and a solution** $W(\cdot)$ **which are governed by the system of differential equations**

$$\begin{cases} x' \ = \ g(x) - p \\ W' \ = \ -\dfrac{x}{\|x\|^2} \otimes h'(Wx)^+ \left(1 - \pi_{T_M(h(Wx))}^{h'(Wx)}\right) \\ h'(Wx)W(g(x) - p) \end{cases}$$

sustaining the viability of the system in the sense that the viability constraints

$$\forall\, t \geq 0,\ \ h'(x(t))(W(t)g(x(t)) - p(t)) \ \in \ T_M^{\square}(h(x(t)))$$

are always met.

We consider the simple dynamical economy

$$x'(t) \; = \; g(x(t)) - p(t)$$

subject to viability constraints of the form

$$\forall \, t \geq 0, \;\; h(x(t)) \; \in \; M$$

which is regulated by

$$p(t) \; \in \; \Pi_M(x(t))$$

where

$$\Pi_M(x) \; := \; \{p \in X^\star \mid h'(x(t))p(t) \; \in \; h'(x(t))g(x(t)) - T_M(x)\}$$

We now set a bound on the velocity of the prices:

$$\|p'(t)\| \; \leq \; \varphi(x(t), p(t))$$

where B denotes the unit ball of the space $X^\star$.

If the graph of the contingent cone is not closed, we introduce
a closed graph selection $T_M^{\square}(\cdot)$
of $T_M(\cdot)$**,**
such as the Maderner selection $T_M^{\diamond}(\cdot)$ **in the case of inequality constraints,**
or the set-valued map whose graph is the viability kernel of the graph of $T_M(\cdot)$**.**

Therefore, we solve the system of differential inclusions which governs the initial system together with the price which controls it:

$$\begin{cases} i) \quad x'(t) \ = \ g(x(t)) - p(t) \\ \\ ii) \ \ \|p'(t)\| \ \leq \ \varphi(x(t), p(t)) \end{cases} \tag{9}$$

subject to the new viability constraints

$$\mathcal{K} \ := \ \{(x,p) \mid h'(x)p \ \in \ h'(x)g(x) - T_M^{\square}(h(x))\}$$

When g is continuously differentiable and h twice continuously differentiable, we infer that (v,q) belongs to the contingent cone $T_{\mathcal{K}}(x,p)$ if and only if

$$\left\{ \begin{array}{l} h'(x)q \ \in \ h''(x)(v,g(x)-p)+h'(x)g'(x)v \\ -DT_M^{\Box}(h(x),h'(x)(g(x)-p))(h'(x)v) \end{array} \right.$$

Hence, when $v := g(x)-p$, we set

$$\left\{ \begin{array}{l} H_M(x,p) \ := \ h''(x)(g(x)-p,g(x)-p) \\ +h'(x)g'(x)(g(x)-p) \\ -DT_M^{\Box}(h(x),h'(x)(g(x)-p))(h'(x)g(x)-p) \end{array} \right.$$

The differential equation governing the evolution of the heavy evolution of the connecting matrix is given by the system

$$\begin{cases} x' \;=\; g(x) - p \\ p' \;=\; h'(x)^{+} v(x,p) \\ \text{where } \; v(x,p) \;:= \\ (1 - \pi^{h'(x)}_{DT^{\square}_{M}(h(x),h'(x)(g(x)-p))(h'(x)g(x)-p)}) \\ (h''(x)(g(x)-p, g(x)-p) + h'(x)g'(x)(g(x)-p) \end{cases}$$

Then for any initial state $x_0 \in K$ **and** p_0**, there exist a solution** $(x(\cdot), p(\cdot))$ **to the above system of differential equations starting at** (x_0, p_0) **sustaining the viability of the system in the sense that the viability constraints**

$$\forall\, t \geq 0, \;\; h'(x(t))(g(x(t)) - p(t)) \;\in\; T^{\square}_{M}(h(x(t)))$$

are always met.

We still consider the initial — priceless — dynamical economy

$$x'(t) \; = \; g(x(t))$$

subjected to scarcity constraints of the form

$$\forall \, t \geq 0, \;\; h(x(t)) \; \in \; M$$

When the necessary and sufficient viability condition $h'(x)g(x) \; \in \; T_M(h(x))$ **is not met by every** $x \in K$**, we already suggested to "correct" the priceless dynamical economy by the connected system**

$$x'(t) \; = \; W(t)g(x(t))$$

The connection matrices regulating viable solutions obey the regulation law

$$W(t) \in R_M(x(t))$$

where

$$R_M(x) := \{W \in \mathcal{L}(X,X) \mid h'(x)Wg(x) \in T_M(h(x))\}$$

The disconnected system is obtained when the connection matrix is equal to the identity matrix $\mathbb{1}$.

We now set a bound on the velocity of the connection matrix:

$$\|W'(t)\| \ \leq \ \varphi(x(t), W(t))$$

where B **denotes the unit ball of the space** $\mathcal{L}(X, X)$**.**

We still introduce a closed graph selection $T_M^{\square}(\cdot)$ **of** $T_M(\cdot)$ **whenever the graph of the contingent cone is not closed, such as the Maderner selection.**

Therefore, we want to solve the system of differential inclusions which governs the initial system together with the connection matrix which guides it:

$$\left\{\begin{array}{ll} i) & x'(t) \;=\; W(t)g(x(t)) \\ \\ ii) & \|W'(t)\| \;\leq\; \varphi(x(t), W(t)) \end{array}\right. \tag{10}$$

subject to the viability constraints

$$\mathcal{K} \;:=\; \{(x, W) \mid h'(x)Wg(x) \;\in\; T_M^{\square}(h(x))\}$$

We set

$$\left\{\begin{array}{l} G_M(x, W) \;:=\; DT_M^{\square}(h(x), h'(x)Wg(x))(h'(x)Wg(x)) \\ -h'(x)Wg'(x)Wg(x) - h''(x)(Wg(x), Wg(x)) \end{array}\right.$$

The differential equation governing the evolution of the heavy evolution of the connecting matrix is given by the system

$$\begin{cases} x' \;=\; Wg(x) \\ W' \;=\; \dfrac{g(x)}{\|g(x)\|^2} \otimes q(x,W) \\ \text{where } q(x,W) \;:= \\ h'(x)^{+}(\pi^{h'(x)}_{DT^{\square}_{M}(h(x),h'(x)Wg(x))(h'(x)Wg(x))} - 1) \\ (h'(x)Wg'(x)Wg(x) - h''(x)(Wg(x),Wg(x))) \end{cases}$$

Then from any initial $(x_0, W_0) \in K$ **starts one solution** $(x(\cdot), W(\cdot))$ **to the above system of differential equations sustaining the viability of the system in the sense that the viability constraints**

$$\forall \;\; t \geq 0, \;\; h'(x(t))W(t)g(x(t))) \;\in\; T^{\square}_{M}(h(x(t)))$$

are always met.

We now mix these two modes of regulation to solve the system of differential inclusions which governs the initial system together with the connection matrix and the price which guide it:

$$\left\{\begin{array}{lrcl} i) & x'(t) & = & W(t)g(x(t)) - p(t) \\ ii) & \|W'(t)\| & \leq & \varphi(x(t), W(t), p(t)) \\ iii) & \|p'(t)\| & \leq & \psi(x(t), W(t), p(t)) \end{array}\right. \tag{11}$$

subject to the viability constraints

$$\mathcal{K} \ := \ \{(x, W, p) \mid h'(x)Wg(x) - h'(x)p \ \in \ T_M^{\square}(h(x))\}$$

We set $v = Wg(x) - p$

$$G_M(x, W, p) \ :=$$
$$DT_M^{\square}(h(x), h'(x)v)(h'(x)v) - h'(x)Wg'(x)v - h''(x)(v, v)$$

We set

$$r(x,W,p) \ := \ h'(x)^{+}\left(\pi^{h'(x)}_{DT^{\square}_{M}(h(x),h'(x)v)(h'(x)v)} - 1\right)$$
$$(h'(x)Wg'(x)v + h''(x)(v,v))$$

where $v := Wg(x) - p$**, which is a map independent from** α **and** β**. Then the differential equation governing the evolution of the heavy evolution of the connecting matrix is given by the system**

$$\left\{\begin{array}{llll} i) & x' & = & Wg(x) - p \\ ii) & W' & = & \dfrac{\beta g(x)}{\alpha + \beta\|g(x)\|^2} \otimes r(x,W,p) \\ iii) & p' & = & \dfrac{-\alpha}{\alpha + \beta\|g(x)\|^2} r(x,W,p) \end{array}\right.$$

Then from any initial $(x_0, W_0, p_0) \in \mathcal{K}$ **starts one solution** $(x(\cdot), W(\cdot), p(\cdot))$ **to the above system of differential equations sustaining the viability of the system in the sense that the viability constraints**

$$\forall\, t \geq 0, \ \ h'(x(t))(W(t)g(x(t)) - p(t)) \ \in \ T^{\square}_{M}(h(x(t)))$$

are always met.

We next briefly investigate the intricate case when the connection matrices are involved in the constraints in a hierarchical process of the form

$$W_{m-1} \cdots W_j \ldots W_0 x - x_m \ \in \ M$$

This describes both chemical or biological "cycles" where the m organisms are related by constraints $x_j = W_j x_{j-1}$ and $x_m \in x_{m-1} - M$

Instead of imposing a given sequence of connection matrices and studying the evolution of the commodities under dynamical systems of the form

$$x_j'(t) \ = \ g_j(x_j(t))$$

we want to derive also the evolution of connection matrices $W_j(t)$ in such a way that the above viability constraint is satisfied at each instant. This involved situation is best understood if it is illustrated by some examples.

One aspect of the biological and ecological problems of "co-evolution" and "co-viability" can be translated (in its simplest form) in the following way. The states $x_0(\cdot)$ and $x_1(\cdot)$ of two organisms evolve in state spaces X_0 and X_1. For that purpose, the organism # 1 consumes $\langle C_1, x_1 \rangle$ units of (scalar) "metabolites" (where $C_1 \in X_1^\star$) which are provided by the activity $\langle A_0, x_0 \rangle$ of organism # 0 proportionally to its state x_0 (where $A_0 \in X_0^\star$). So, for surviving, organism # 1 needs the activity of w units of organisms # 0 for satisfying the "co-viability" constraint

$$w_0 \langle A_0, x_0 \rangle - \langle C_1, x_1 \rangle \ \leq \ 0$$

When the "metabolites" range over a vector space Y_1 instead of R, then the constraint becomes

$$W_0 A_0 x_0 - C_1 x_1 \ \leq \ 0$$

where A_0 and C_1 are given linear operators and where the connection matrix W_0 enters the co-viability constraints. Hence, we have to devise co-evolutions $x_0(\cdot)$ and $x_1(\cdot)$ of both the organisms as well as the evolution $W_0(\cdot)$ of the connection matrix such that the co-viability constraint is satisfied at each instant. $\square$

Let us consider the case when the set of allocations

$$K := \left\{ x \in \prod_{i=1}^{n} L_i \mid \sum_{i=1}^{n} x_i \in M \right\}$$

of scarce resources $y \in M$ is empty. We regard here $M \subset Y := \mathbf{R}^n$ as the production set of one producer, which cannot meet the demand using only one unit of labor, say. The question arises whether one can revive the viability of the allocation mechanism by increasing the number of labor units : We introduce for that purpose a scalar $w \in [1, +\infty[$ measuring the number of labor units and consider the new set of allocations defined by

$$K := \left\{ (x, y, w) \in \prod_{i=1}^{n} L_i \times M \times [1, +\infty[\mid \sum_{i=1}^{n} x_i \leq wy \right\} \tag{12}$$

Assume that the autonomous dynamics of the consumers i are defined by $g_i : Y \mapsto Y$, that the autonomous dynamics of the producer is described by $g_0 : Y \mapsto Y$ and that the autonomous dynamics governing the number of labor units is given by $e : \mathbf{R} \mapsto \mathbf{R}$ (a reasonable choice is to take $e = 0$). We derive that *the constraints (12) involving labor units are viable under "dynamical economies" of the form*

$$\begin{cases} x_i'(t) \ = \ g_i(x_i(t)) - p(t) \ \ (i = 1, \ldots, n) \\ y'(t) \ = \ g_0(y(t)) + w(t)p(t) \\ w'(t) \ := \ e(w(t)) + \langle p(t), y(t) \rangle \end{cases}$$

where $p(t) \in Y^\star$ is a price regulating the evolution by slowing down the consumptions of the consumer, increasing the production of the producer and governing the evolution of the labor units in terms of the income $\langle p(t), y(t) \rangle$ they receive from the production at each instant. $\square$

We regard the connection matrices as technological matrices in production economics. The constraints are of the form

$$W_{m-1} \cdots W_j \ldots W_0 x \in M$$

This describes for instance a production process associating with the commodity x the intermediate input $x_1 := W_0 x$, which itself requires an input $x_2 := W_1 x_1$, and so on.

Instead of imposing a given sequence of technological matrices and studying the evolution of the commodities under dynamical systems of the form

$$x_j'(t) = g_j(x_j(t))$$

we want to derive also the evolution of technological matrices $W_j(t)$ in such a way that the viability constraint

$$\forall\, t \geq 0, \;\; W_{m-1}(t) \cdots W_j(t) \ldots W_0(t) x(t) \in M \subset Y$$

is satisfied at each instant.

Using *viability multipliers*, **one can prove that dynamical systems of the form**

$$\begin{cases} x_0'(t) \; = \; g_0(x_0(t)) - W_0^\star(t)p_1(t) & (j=0) \\ x_j'(t) \; = \; g_j(x_j(t)) + p_j(t) - W_j^\star(t)p_{j+1}(t) & (j=1,\ldots,m-1) \\ x_m'(t) \; = \; g_m(x_m(t)) + p_m(t) & (j=m) \\ W_j'(t) \; = \; e(W_j(t)) - x_j(t) \otimes p_{j+1}(t) & (j=0,\ldots,m-1) \end{cases}$$

govern viable solutions provided that the evolution of technological matrices $W_j(t)$ **obeys dynamics involving the tensor product of** $x_j(t)$ **and the "intermediate price"** $p_{j+1}(t)$**.** $\square$

Let X_j, Y_j , Z_j ($j = 0, \dots m-1$) and Y_j ($j = i, \dots m$) denote finite dimensional spaces.

Consider linear operators $A_j \in \mathcal{L}(X_j, Y_j)$, $B_j \in \mathcal{L}(Z_j, X_{j+1})$ and $C_j \in \mathcal{L}(X_j, Y_j)$ and "technological matrices" $W_j \in \mathcal{L}(Y_j, Z_j)$.

The hierarchical process takes the form

$$\forall\, j = 1, \dots m, \;\; B_{j-1}W_{j-1}A_{j-1}x_{j-1} + C_j x_j \;\in\; M_j$$

where $M_j \subset Y_j$ are closed convex subsets.

Here $x_0 \in X_0$ is regarded as an output, $x_m \in X_m$ as an input, whereas $x_j \in X_j$ is regarded either as intermediate outputs or inputs.

Observe that taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m$, the above constraints read:

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0W_0A_0x_0 - x_m \ \in \ M_m$$

Taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m-1$ and $C_m = 0$, we find constraints of the form

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0W_0A_0x_0 \ \in \ M_m$$

Therefore, in a hierarchical process, the state variables are the sequences

$$((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1})$$

made of vectors and of matrices. The above constraints are defined by inverse images under a multilinear map.

$$\left\{\begin{array}{ccc} & A_{j-1} & \\ x_{j-1} \in X_{j-1} & \longmapsto & Y_{j-1} \\ & W_{j-1} & \downarrow \\ & & Z_{j-1} \\ & B_{j-1} & \downarrow \\ x_j \in X_j & \longmapsto & Y_j \supset M \\ & C_j & \end{array}\right.$$

However, the linearity of the operators $A_j,\ B_j$ & C_j does not really bring any simplification in the viability analysis. This is the reason why we study this example in the more general nonlinear form described in the following way:

1. $x_j \in L_j \subset X_j$, $(j = 1, \ldots m)$,

2. continuously differentiable maps $l_j : X_j \mapsto Y_j$ **and** $h_j : Z_{j-1} \times X_j \mapsto Y_j$**,**

3. "technological matrices" $W_j \in \mathcal{W}_j \subset \mathcal{L}(Y_j, Z_j)$

4. "scarcity constraint" $h_j(W_{j-1}l_{j-1}(x_{j-1}), x_j) \in M_j \subset Y_j$

The constrained set is defined by

$$\begin{array}{l} K \ := \ \{((x_j)_{j=0,\ldots,m}, \ (W_j)_{j=0,\ldots,m-1}) \in \prod_{j=0}^{m} L_j \times \prod_{j=0}^{m-1} \mathcal{W}_j \\ \textbf{\textit{such that}}\, h_j(W_{j-1}l_{j-1}(x_{j-1}), x_j) \ \in \ M_j\} \end{array} \tag{13}$$

We set

$$A_j \ := \ l'_j(x_j), \ \ B_{j-1} \ := \ h'_{j_{z_{j-1}}}(W_{j-1}l_{j-1}(x_{j-1}), x_j)$$

and

$$C_j \ := \ h'_{j_{x_j}}(W_{j-1}l_{j-1}(x_{j-1}), x_j)$$

Hence, the directional derivative is equal to

$$\begin{cases} h'_j(W_{j-1}l_{j-1}(x_{j-1}), x_j)(u_{j-1}, u_j, V_{j-1}) \\ = \ C_j u_j + B_{j-1}W_{j-1}A_{j-1}u_{j-1} + B_{j-1}V_{j-1}l_{j-1}(x_{j-1}) \end{cases}$$

and can be written in the form

$$B_{j-1}W_{-1}A_{j-1}u_{j-1} + C_j u_j + (l_{j-1}(x_{j-1}) \otimes B_{j-1})V_{j-1}$$

The transpose of this operator maps $(q_j) \in \prod_{j=1}^{m} Y_j^\star$ to

$$(X_0 \times \mathcal{L}(Y_0^\star, Z_0^\star)) \times \left(\prod_{j=1}^{m-1} (X_j^\star \times \mathcal{L}(Y_j^\star, Z_j^\star)\right) \times X_m^\star$$

in the following way:

$$\begin{cases} (A_0^\star W_0^\star B_0^\star q_1 \ , \ l_0(x_0) \otimes B_0^\star q_1) & (j=0) \\ (A_j^\star W_j^\star B_j^\star q_{j+1} + C_j^\star q_j \ , \ l_j(x_j) \otimes B_j^\star q_{j+1}) & (j=1,\ldots,m-1) \\ C_m^\star q_m & (j=m) \end{cases}$$

We posit a sufficient condition guaranteeing the transversality conditions:

$$\forall \; ((x_j)_{j=0,\ldots,m}, \ (W_j)_{j=0,\ldots,m-1}), \ \ h'_{j_{x_j}}(W_{j-1} l(x_{j-1}), x_j) \text{ are surjective} \tag{14}$$

Assume now that the dynamical behaviors of the autonomous systems are described by

$$\begin{cases} x_j'(t) \ = \ g_j(x_j(t))(t) \quad (j=0,\ldots m) \\ \\ W_j'(t) \ = \ e(W_j(t)) \quad (j=0,\ldots m-1) \end{cases}$$

where $y_j(t) \in Y_j$ and $q_j(t) \in Z_j^\star$.

We deduce that *the constraints (13) involving connection matrices are viable under control systems of the form*

$$\left\{\begin{array}{lcl} x_0'(t) & = & g_0(x_0(t)) - A_0^\star W_0^\star(t) B_0^\star p_1(t) \\ \\ x_j'(t) & = & g_j(x_j(t)) - C_j^\star p_j(t) - A_j^\star W_j^\star(t) B_j^\star p_{j+1}(t) \ \ (j = 1, \ldots, m-1) \\ \\ x_m'(t) & = & g_m(x_m(t)) - C_m^\star p_m(t) \\ \\ W_j'(t) & := & e(W_j(t)) - l_j(x_j(t)) \otimes B_j^\star p_{j+1}(t) \ \ (j = 0, \ldots, m-1) \end{array}\right.$$

The regulation map can be written in the form in the following way: $(p_j)_{j=1,\ldots,m}$ **belongs to** $\Pi_K((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1})$ **if and only if** p **is a solution to the system of equations**

$$\left\{\begin{array}{l} (C_1 C_1^\star + B_0 W_0 A_0 A_0^\star W_0^\star B_0^\star)\, p + \langle p, B_0 l_0(x_0)\rangle B_0 l_0(x_0) + C_1 A_1^\star W_1^\star B_1^\star p_2 \ = \ z_1 \\ \\ \left(C_j C_j^\star + B_{j-1} W_{j-1} A_{j-1} A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\right) p_j \\ + \langle p_j, B_{j-1} l_{j-1}(x_{j-1})\rangle B_{j-1} l_{j-1}(x_{j-1}) + C_j A_j^\star W_j^\star B_j^\star p_{j+1} \\ + B_{j-1} W_{j-1} A_{j-1} C_{j-1}^\star p_{j-1} \ = \ z_j \ \ (j = 2, \ldots, m-1) \\ \\ (C_m C_m^\star + B_{m-1} W_{m-1} A_{m-1} A_{m-1}^\star W_{m-1}^\star B_{m-1}^\star)\, p_m \\ + \langle p_m, B_{m-1} l_{m-1}(x_{m-1})\rangle B_{m-1} l_{m-1}(x_{m-1}) + B_{m-1} W_{m-1} A_{m-1} C_{m-1}^\star p_{m-1} \ = \ z_m \end{array}\right.$$

where where r_j belongs to $T_{M_j}(h_j(W_{j-1}l(x_{j-1}), x_j))$ and where

$$z_j \ := \ C_j g(x_j) + B_{j-1}W_{j-1}A_{j-1}g_{j-1}(x_{j-1}) + B_{j-1}e_{j-1}(W_{j-1})l(x_{j-1}) - r_j$$

In other words, this solution is obtained by inverting a tri-diagonal block matrix:

$$\begin{pmatrix} \mathcal{J}_1 & \mathcal{M}_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \mathcal{M}_1^\star & \mathcal{J}_2 & \mathcal{M}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \mathcal{M}_{j-1}^\star & \mathcal{J}_j & \mathcal{M}_j & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\mathcal{M}_{m-2}^\star & \mathcal{J}_{m-1} & \mathcal{M}_{m-1} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mathcal{M}_{m-1}^\star & \mathcal{J}_m \end{pmatrix}$$

where we set

$$\begin{cases} \mathcal{J}_j p_j \\ := \ \left(C_j C_j^\star + B_{j-1}W_{j-1}A_{j-1}A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\right) p_j + \langle p_j, B_{j-1}l_{j-1}(x_{j-1})\rangle B_{j-1}l_{j-1}(x_{j-1}) \end{cases}$$

and

$$\mathcal{M}_j \ := \ C_j A_j^\star W_j^\star B_j^\star$$