Connectionist Complexity and its Evolution

Jean-Pierre Aubin*

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Among all the lessons I learned from Jacques-Louis Lions, both of human, moral and scientific nature, trying to look for motivations for mathematics and find applications of mathematics in other fields of knowledge was the most important one, and doubtless, the most difficult to learn, the most arduous to implement. For this, and the fact that I am owing him everything, professionally and intellectually, this contribution is far from witnessing the wholehearted and devoted recognition he deserves.

Abstract

Whenever a constrained set of states is not viable under a system of "autonomous" differential equations, controls or regulation parameters are usually introduced (in several ways) for restoring the viability of a "global system". We propose also to serve the same purpose of rendering a system viable by connecting the dynamics of elementary autonomous dynamical systems and/or the constraints (whenever "cycles" enter constraints, as the biological cycles, where each component of the system feeds back on the preceding component). This paper is devoted to both methods, to the relations between them, in order to substantiate on mathematical grounds issues such as biodiversity, co-evolution or co-viability in biological evolution as well as dynamical economic decentralization or multi-stage production in economic theory. This view point leads us to the definition of *static and dynamical "connectionist complexity" indices* and to the search of evolution minimizing at each instant those indices.

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1 Background Motivations

... la complexité, c'est ce qui reste lorsque l'on a tout simplifié (nous connaissons tous des orfèvres de la complexification de situations simples). Plus précisément : la complexité d'un système correspond au "minimum" des données, des connaissances, des connexions logiques, des structures et des moyens de communication qui permettent en principe de définir ce système, de le comprendre et de le gérer.
Jacques-Louis Lions, Discours à l'Académie des Sciences du premier décembre 1997 en présence du Président de la République

Connecting the dynamics of elementary autonomous dynamical systems and/or the constraints in order to sustain the viability of a "global system" could be a general feature of what we propose to call "connectionist complexity" — to make more precise a meaning of such a polysemous concept as "complexity".

1. Is not complexity meaning in the day-to-day language not only the number of variables of a system, but also and above all the labyrinth of connections between the components of a living organism or an organization or a "system" ?
2. Is not the purpose of complexity to sustain the "stability" — another polysemous word — or more precisely the viability constraints set by the environment ?
3. Is not the increase of complexity parallel to the growth of the web of constraints whenever the system cannot comply to them in an autonomous or decentralized way ?
4. Is not the organization of organisms as a hierarchical structure of relatively "autonomous" organs due to biological "cycles" involved in the viability constraints or multi-stage production processes ?
5. Is not the organization in organisms and organs rooted in the need to offer them slowly evolving partial environments to specialize them in specific activities ?

The Claude Bernard's "constance du milieu intérieur", the "homeostasis' of Walter Cannon, viability constraints to which dynamical systems must comply, may be at the origin of connectionist complexity. Life as adaptation to this thin annulus wrapping the surface of the earth — according to Gaia hypothesis — has shown an everlasting trend toward a highly connected network of labyrinthine — connectionist

— complexity. In this sense, complexity arose with the apparition of life and seems necessary to the pursuit of its evolution.

This growth of "structural" complexity is the legacy that Jean-Baptiste de Monet, chevalier de Lamarck, offered to us, the backbone of his theory of evolution which was forgotten ever since, overshadowed as it was by other aspects of the evolution of species, such as the Darwinian natural selection or genetics.

In this framework of adaptation to viability constraints, the evolution of the state no longer derives from intrinsic dynamical laws valid in the absence of constraints, but requires some "organization" that evolves together with the state of the system in order to adapt to the viability constraints.

The evolution law of both the state and the connection matrix results from the confrontation of the intrinsic dynamics to the viability constraints.

This collective behavior is also one of the characteristics of the concept of complexity as it is often understood.

We regard here *connectionism* — a less normative and more neutral term than *cooperation* whenever the system, the organ, the organism or the organization arise in biology or economics or social sciences — as an answer to adapt to more and more viability constraints, which implies the emergence of links between the components of a dynamical system and their evolution.

A system is *disconnected (or autonomous, free, autarchic, decentralized, etc.)* if the connection matrix is the identity (or unit) matrix. The distance between the *connection matrix* W — which is the matrix linking each component of the dynamical system to the others — and the identity matrix could capture and measure the concept of an *index of connectionist complexity.* The larger such a connectionist complexity index, the more complex — or labyrinthine or intricate — the connectionist feature of the system. Or one can propose instead to measure a *dynamical index of connectionist complexity* through the norm of the velocity $W'(t)$ of a time dependent connection matrix $W(t)$ starting from the identity matrix as we shall do in the case of "heavy evolution" of global systems. One can also measure other features of connectionist complexity through the sparsity of the *connection matrix*, i.e., the number — or the position — of entries which are equal to zero or "small". The sparser such a connection matrix, the less complex the system.

Compounded with these issues of connectionist complexity — that is naturally involved in neural networks[1] — is the question of whether specific links relating one component to another can play the role of regulation parameters and whether and when they govern the viable evolutions of the state of the system. In order to distinguish the regulation parameters — on which no identified agent is acting — from the control parameters of control theory guided by agents for pursuing a definite

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[1] See for instance [7, Aubin] for the links between the role of synaptic matrices in neural networks and connectionist matrices for describing connectionist complexity.

teleological objective, as the optimization of an intertemporal or final criterion, we call in short *regulons* these regulation parameters.

Living and social systems are “complex”, although there is no consensus on the definition of complexity. However, reading the literature on complexity, and quoting George Cowan, the founder of the Santa Fe Institute, “in the universe, everything is connected with every thing” seems to be the consensual agreement of the members of this Institute.

This attempt to sustain the viability of the system by connecting the dynamics or the constraints of its agents may be a general feature of “complex systems”.

However, Seth Loyd had found 31 different definitions of complexity at the beginning of the 90’s. This number increased a lot since. Complexity is indeed a polysemous word, that tries to embrace too many distinct phenomenon of interest in biological and social sciences.

Physicists and computer scientists have attempted to *measure* it in various ways, through the concept of Clausius’s entropy, Claude Shannon’s information, Gilbert Chauvet’s nonsymmetric information, the degree of regularity instead of randomness, “hierarchical complexity” in the display of level of interactions, Andrei Kolmogorov, Gregory Chaitin & Ray Solomonoff “algorithmic information contents” (see [14, Chaitin] for instance) and other temporal or spatial computational complexity indices measuring the computer time or the amount of computer memory needed to describe a system, “grammatical complexity” measuring the language to describe it, etc. Some economists link complexity issues with chaos theory as in [20, 21, Day] for instance. Other investigators link complexity issues with catastrophe theory, or fractals. See among many references [40, Peliti & Vulpiani].

Physicists — and among them, specialists of “spin glasses” such as Giorgio Parisi — propose the number of equilibria of a dynamical system as a characteristic of complexity. Or, even more to the point, “quasi equilibria”, that are “small” areas of the state space in which the evolution remains a “long time”, before “jumping quickly” to another quasi equilibrium (see [37, 38, 39, Parisi]).

However, these attempts did not answer directly the question that some biologists or economists asked: *Complexity and hierarchical organization, yes, but for what purpose*?

1. Since at least the works [25, Elton] of Charles Elton and [30, Hutchinson] of George Hutchinson at the end of the fifties, the conventional wisdom of biologists proposes in some loose ways that complexity, regarded as the number of variables of the systems and their links — is justified for maintaining “stability” — a fuzzy word, meaning confinement, or rather, viability as it was proposed to single out this meaning from the numerous intendments of “stability” in mathematics. Biodiversity is presently and actually championed on the basis of this objective. In a series of papers summarized in [35, May],

Robert May and his collaborators disclaimed this proposition by showing that the higher the dimension, the less stable were dynamical models of Lotka-Volterra types. Therefore, either the biologists' assumption was false, or such mathematical connotation of complexity — the dimension of the state space — or the chosen mathematical model is inadequate. This is a suggestion made by John-Maynard Smith in [43, Smith], when he concluded that stability of ecosystems is due to some specific interactions. *This present study captures and supports this viewpoint by explaining the necessity of these links described by a connection matrix and deriving their evolution from the requirement to obey "collective" viability constraints* (a mathematical implementation of the biological connotations of stabilization and homeostasis).

2. Ever since Adam Smith's invisible hand, what we call nowadays decentralization is justified by the need of agents to behave in a decentralized way for complying to scarcity constraints, using for this purpose "messages" such as prices or "rationing" mechanisms which involve shortages (and lines, queues, unemployment), or "frustration" of consumers, or "monetary" mechanisms, or others. "Price" p constitute the main examples of messages, actually, the messages with the smallest dimension (see for instance an introduction to this issue in [42, Saari]): Knowing at each instant the price p, consumers are only supposed to know how to choose their consumption bundle in a decentralized way, i.e., *without*

    - knowing the behavior of their fellow consumers
    - knowing the set of scarce resources

    We shall show that this type of regulation — which is also present in physical or mechanical models — provides the same evolution of the system than some associated connection matrices. Furthermore, *decentralization through prices with minimal norm is equivalent to minimal complexity* in some specific models, providing another mathematical justification of price decentralization.

Since the dawn of our present representation of science, more or less severe meanings of "reductionism" forge the prevailing consensus, despite assaults coming from some militant systems theorists pleading for at least half a century for a "global or holist treatment". These views should not be conflicting, since global systems could be represented as "connected" simpler systems. Hence "going from complex to simple" is mandatory before attempting to reconcile and build from them consistent and coherent "global systems". *A method for constructing such global systems from individual subsystems and global constraints is also a purpose of this paper*.

Indeed, the problem is that living systems being open, there is no way to describe them entirely and consequently, to demonstrate the truth of any proposition. We are left with metaphors, and among them, mathematical ones, to validate a consensual definition of one meaning — actually, one submeaning, so to speak — of complexity.

Therefore, a mathematical metaphor begins by dividing a system into at least two classes, the system under study and its environment. This division is always arbitrary, but often justified by the specialists of each nonmathematical field of knowledge. We could call "phenotype" the part of the system interacting with its environment. This is consistent with the concept of Dawkins' "extended phenotype" advocating that everything is connected with everything (see [19, Dawkins]).

Once conceptually isolated from its environment, a living system fuels itself in the last analysis on external energy through the consumptions of wastes of the other components of the open system set apart in the description of the environment. This can be translated by saying that it has to obey viability constraints bearing on its phenotype. Each component of the system which can evolve independently in the absence of constraints, must interact each other in order to maintain the viability of the system imposed by its environment: We propose mathematical metaphors designing evolution laws governing the evolution of connection matrices linking the autonomous dynamics of each component in order to sustain the viability constraints prescribed by the environment of the isolated system.

Our point is the following: Starting with a disconnected system — for which the nondiagonal elements of the connection matrix are all equal to zero — the viability can be maintained whenever each component of the system is linked with other components through an evolving connection matrix, *dictating cooperation among the components.* The larger the number of independent constraints, the more numerous the links between any component with the others. In the mathematical version of the metaphor, a class of evolution laws governing the evolution of these connection matrices can be designed from the disconnected dynamical system and the set of constraints. Among them, we advocate the "heavy evolution law", in which the connection matrix evolves as slowly as possible for governing viable evolutions of the state, and in particular, *remains constant as long as the viability of the system is not at stakes* — what we call the *inertia principle.* It has the property of "locking in" any connection matrix which can regulate for ever a nonempty subset of states, called the *viability niche of this connection matrix.*

Connecting the dynamics of a system to obey viability constraints is not by far the only way which can be designed. One can also envisage to *connect the constraints.* This appears whenever "cycles" enter constraints, as the chemical and biological cycles, where each component of the system feeds back on the preceding component. See the Eigen hypercycle theory in [**?**, Eigen] and [24, Eigen & Schuster] among many other references on this topic. Instead of connecting the dynamics as it is done in the framework of replicator systems (see [29, Hofbauer & Sigmund] for instance), one can connect the constraints in a chain of the form

$$W_m \ldots W_1 W_0 x_0 \in M$$

where each connection matrix W_j maps level j to the next level $j+1$. These types of constraints appear also in co-evolution and co-viability issues, in multi-layer neural networks, where the connection matrices are regarded as synaptic matrices (see [7, Aubin] for instance) or in multi-stage production processes, where connection matrices are interpreted as technological matrices mapping the output of an intermediate commodity to the input needed to produce it (see [8, Aubin] for instance). But, instead of assuming that the connection matrices W_j are given, we derive their evolution laws in order to sustain the viability of a regulated global system.

Hence, constraints of the above type induce a hierarchical organization and the appearance of connection matrices linking one level to the other. The choice of these connection matrices is not given in advance, but the solution to the adaptation of the system to viability constraints.

2 Survey of the Main Results

2.1 Regulation by Regulees and Connection Matrices

Let $X := \mathbf{R}^n$, $Y := \mathbf{R}^p$ be finite dimensional vector spaces, $g : X \mapsto X$ describe the continuous dynamics of the system, $h : X \mapsto Y$ represent the continuously differentiable resource map and $M \subset Y$ the closed subset of available resources.

As the simplest example, we assume that disconnected or autonomous systems are given in the form of a system of differential equations

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; g_i(x_i(t))$$

We also postulate that the state of the system must comply to *collective viability constraints* of the form

$$\forall\, t \geq 0, \;\; h(x_1(t), \ldots, x_n(t)) \; \in \; M$$

Since the nature of the constraints is collective and the dynamics are autonomous, so to speak, the *viability property* requiring that from any initial state satisfying the constraints starts at least one solution obeying them forever is not generally met.

Hence, we have to couple or connect the dynamics of the system

1. either by connecting the components of the original dynamics through connecting matrices $W := (w_i^j)_{i,j} \in \mathcal{L}(X, X)$:

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; \sum_{j=1}^{n} w_i^j(t) g_j(x_j(t))$$

2. or by subtracting regulons (or fluxes, prices, controls, etc.) to each component of the original dynamical system:

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; g_i(x_i(t)) - p_i(t)$$

3. or by a combination of both regulation procedures subsumed in the system

$$\forall\, i = 1, \ldots, n,\ \ x_i'(t) \ = \ \sum_{j=1}^{n} w_i^j(t) g_j(x_j(t)) - p_i(t)$$

   regulated by connection matrix-regulon pairs (W, p).

We investigate these two ways and encompass them in the third one.

We shall first observe that one can associate with regulons connection matrices providing the same evolutions, revealing the actual links relating the dynamics when regulons are used as a kind of decentralizing messages. In economics, this provides another explanation of the decentralizing role of prices.

2.2 Minimal Static Connectionist Complexity

We next define the *index of (connectionist) complexity* of a connection matrix W as the distance between this *connection matrix* W and the identity matrix which describes the autonomous situation. We then prove that in this context, the connection matrix which minimizes this connectionist complexity index is associated with the regulation mechanism driving slow viable evolutions, i.e., that the least complex connection matrix governs slow evolutions. This justifies further the decentralizing role of prices.

On the mathematical side, the natural framework to study these connection matrices is tensor algebra of matrices because the connection matrix which minimizes the connectionist complexity index happens to involve the tensor product of the dynamics and the regulons. Recall that whenever $p \in X^\star$ and $y \in Y$, we denote by $p \otimes y \in \mathcal{L}(X, Y) = X^\star \otimes Y$ the rank one linear operator defined by $x \mapsto (p \otimes y)(x) := \langle p, x \rangle y$, the matrix of which is

$$\left(p^i y_j\right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,p}}$$

the kernel of which is equal to the one of p and the image of which is the line spanned by y (see [7, Aubin] for an explanation of the relevance of the tensor structure and a short introduction to the parts of tensor algebra needed for our purpose). In terms of sparsity of the matrix, we observe that whenever an entry $p^i y_j$ is equal to 0, then the line or the column passing through this entry is made of zeros.

It happens that *the viable connection matrices* $W^\bullet(x)$ *minimizing the complexity index are associated with the viable regulons* $\varpi^\circ(x)$ *with minimal norm through the formula*

$$W^\bullet(x) := \mathbf{1} - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi^\circ(x)$$

If one adopts this static measure of connectionist complexity and the relevance of such a mathematical metaphor, we derive that *the least complex connectionist matrix enjoy the specific sparsity structure of tensor products.*

Actually, we shall show that in the framework of the mixed regulated system (3), the viability theorem provides a *regulation* law of the form

$$\forall\, t \geq 0, \;\; (W(t), p(t)) \;\in\; R_M(x(t))$$

providing the evolution of the connection matrix-regulon pairs $(W(t), p(t))$ regulating viable evolutions where R_M is a set-valued feedback map built from the confrontation between the dynamics and the constraints. The solutions governed by the connection matrix-regulon pairs $(W(t), p(t))$ minimizing at each instant t the measure

$$\alpha \|\mathbf{1} - W(t)\|^2 + \beta \|p(t)\|^2$$

in $R_M(x(t))$ do not depend upon the choice of the nonnegative weights α and β.

2.3 Minimal Dynamic Connectionist Complexity

By differentiating this regulation law thanks to the differential calculus of set-valued maps, we derive under adequate assumptions a *dynamical regulation (or adjustment) law* of the form

$$\forall\, t \geq 0, \;\; (W'(t), p'(t)) \;\in\; G_M(x(t), W(t), p(t))$$

which, together with (3), governs the evolution of the triple $(x(t), W(t), p(t))$.

Heavy evolutions are governed by the particular dynamical feedback obtained when the velocities $(W'(t), p'(t))$ of the connection matrix-regulon pairs $(W(t), p(t))$ minimize at each instant t the measure $\alpha\|W'(t)\|^2 + \beta\|p'(t)\|^2$ in $G_M(x(t), W(t), p(t))$. Taking $\alpha = 1$ and $\beta = 0$ furnishes the case of a *dynamical complexity index* measuring the norm of the velocity of the connection matrix. *Heavy connected viable solutions*, heavy in the sense of *“heavy trends”*, are the ones for which *the connection matrix evolves with minimal velocity*. We thus prove that in this case, the connection matrix is governed by a differential equation of the form

$$W'(t) \;=\; a(x(t)) \otimes b(x(t))$$

where the maps $x \mapsto a(x)$ involving the dynamics and $x \mapsto b(x)$ involving “viability errors” can be explicitly computed. Such differential equations are typical *Hebbian learning processes* appearing in the neural network literature, where the velocity of the entries (synaptic weights) are of the form

$$w_j^{i'}(t) \;=\; a^i(x(t)) b_j(x(t))$$

and thus, proportional to the product of activities of the ith and jth components (neurons in the case of neural networks).

Heavy viable solutions obey the *inertia principle*: “*keep the connection matrix constant as long as they provide viable evolutions*”. The *viability niche* of a given matrix W is the viability kernel of the constrained set $K := h^{-1}(M)$ under the system $x'(t) = Wg(x(t))$ where W is kept constant. We say that a connection matrix W is a *punctuated equilibrium* if its viability niche is not empty. If at some time t_f, $W(t_f)$ is a "punctuated equilibrium", then the heavy connected viable solution enters its viability niche and remains in this viability niche forever[2] whereas the connection matrix remains equal to this punctuated equilibrium. In other words, *punctuated equilibria are "locked-in" by heavy solutions.*

2.4 Hierarchical Complexity

We next briefly investigate the intricate case when the connection matrices are involved in the constraints in a hierarchical process of the form

$$W_{m-1} \cdots W_j \ldots W_0 x - x_m \ \in \ M$$

This describes both chemical or biological "cycles" where the m organisms are related by constraints $x_j = W_j x_{j-1}$ and $x_m \in x_{m-1} - M$

Instead of imposing a given sequence of connection matrices and studying the evolution of the commodities under dynamical systems of the form

$$x_j'(t) \ = \ g_j(x_j(t))$$

we want to derive also the evolution of connection matrices $W_j(t)$ in such a way that the above viability constraint is satisfied at each instant. This involved situation is best understood if it is illustrated by some examples.

Example # 1: Co-viability and Co-evolution

One aspect of the biological and ecological problems of "co-evolution" and "co-viability" can be translated (in its simplest form) in the following way. The states $x_0(\cdot)$ and $x_1(\cdot)$ of two organisms evolve in state spaces X_0 and X_1. For that purpose, the organism # 1 consumes $\langle C_1, x_1\rangle$ units of (scalar) "metabolites" (where $C_1 \in X_1^\star$) which are provided by the activity $\langle A_0, x_0\rangle$ of organism # 0 proportionally to its state x_0 (where $A_0 \in X_0^\star$). So, for surviving, organism # 1 needs the activity of w units of organisms # 0 for satisfying the "co-viability" constraint

$$w_0\langle A_0, x_0\rangle - \langle C_1, x_1\rangle \ \leq \ 0$$

When the "metabolites" range over a vector space Y_1 instead of $\mathbf{R}$, then the constraint becomes

$$W_0 A_0 x_0 - C_1 x_1 \ \leq \ 0$$

where A_0 and C_1 are given linear operators and where the connection matrix W_0 enters the co-viability constraints. Hence, we have to devise co-evolutions $x_0(\cdot)$ and

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[2] as long as the constrained set of does not change for external reasons which are not taken into account here (see [9, Aubin] for an investigation of constrained set evolving under morphological equations).

$x_1(\cdot)$ of both the organisms as well as the evolution $W_0(\cdot)$ of the connection matrix such that the co-viability constraint is satisfied at each instant. □

Example # 2: Increasing Production Capacities in Economics Let us consider the case when the set of allocations

$$K := \left\{ x \in \prod_{i=1}^{n} L_i \mid \sum_{i=1}^{n} x_i \in M \right\}$$

of scarce resources $y \in M$ is empty. We regard here $M \subset Y := \mathbf{R}^n$ as the production set of one producer, which cannot meet the demand using only one unit of labor, say. The question arises whether one can revive the viability of the allocation mechanism by increasing the number of labor units : We introduce for that purpose a scalar $w \in [1, +\infty[$ measuring the number of labor units and consider the new set of allocations defined by

$$(1) \qquad K := \left\{ (x, y, w) \in \prod_{i=1}^{n} L_i \times M \times [1, +\infty[\mid \sum_{i=1}^{n} x_i \leq wy \right\}$$

Assume that the autonomous dynamics of the consumers i are defined by $g_i : Y \mapsto Y$, that the autonomous dynamics of the producer is described by $g_0 : Y \mapsto Y$ and that the autonomous dynamics governing the number of labor units is given by $e : \mathbf{R} \mapsto \mathbf{R}$ (a reasonable choice is to take $e = 0$). We derive that *the constraints (1) involving labor units are viable under "dynamical economies" of the form*

$$\left\{ \begin{array}{lll} x_i'(t) & = & g_i(x_i(t)) - p(t) \ \ (i = 1, \ldots, n) \\ y'(t) & = & g_0(y(t)) + w(t)p(t) \\ w'(t) & := & e(w(t)) + \langle p(t), y(t) \rangle \end{array} \right.$$

where $p(t) \in Y^\star$ is a price regulating the evolution by slowing down the consumptions of the consumer, increasing the production of the producer and governing the evolution of the labor units in terms of the income $\langle p(t), y(t) \rangle$ they receive from the production at each instant. □

Example: Multi-stage Production Processes Both co-viable processes in biology and ecology and production processes in economics require constraints involving connection matrices, regarded as technological matrices in production economics. The constraints are of the form

$$W_{m-1} \cdots W_j \ldots W_0 x \in M$$

This describes for instance a production process associating with the commodity x the intermediate input $x_1 := W_0 x$, which itself requires an input $x_2 := W_1 x_1$, and so on.

Instead of imposing a given sequence of technological matrices and studying the evolution of the commodities under dynamical systems of the form

$$x_j'(t) \;=\; g_j(x_j(t))$$

we want to derive also the evolution of technological matrices $W_j(t)$ in such a way that the viability constraint

$$\forall\, t \geq 0,\;\; W_{m-1}(t)\cdots W_j(t)\ldots W_0(t)x(t) \;\in\; M \;\subset\; Y$$

is satisfied at each instant. Using *viability multipliers* (see chapter 5 of [8, Aubin]), one can prove that dynamical systems of the form

$$\left\{\begin{array}{lll} x_0'(t) \;=\; g_0(x_0(t)) - W_0^\star(t)p_1(t) & & (j=0) \\ \\ x_j'(t) \;=\; g_j(x_j(t)) + p_j(t) - W_j^\star(t)p_{j+1}(t) & & (j=1,\ldots,m-1) \\ \\ x_m'(t) \;=\; g_m(x_m(t)) + p_m(t) & & (j=m) \\ \\ W_j'(t) \;=\; e(W_j(t)) - x_j(t)\otimes p_{j+1}(t) & & (j=0,\ldots,m-1) \end{array}\right.$$

govern viable solutions provided that the evolution of technological matrices $W_j(t)$ obeys dynamics involving the tensor product of $x_j(t)$ and the “intermediate price” $p_{j+1}(t)$. $\square$

Actually, we shall study the general case when constraints at each level of a hierarchical organization are of the form

$$\forall\, j=1,\ldots m,\;\; B_{j-1}W_{j-1}A_{j-1}x_{j-1} + C_jx_j \;\in\; M_j \subset Y_j$$

in the linear case and

$$\forall\, j=1,\ldots m,\;\; h_j(W_{j-1}l_{j-1}(x_{j-1}), x_j) \in M_j \subset Y_j$$

in the nonlinear case.

3 Connecting the Dynamics

3.1 Regulees and Connection Matrices

Here, both finite dimensional vector spaces $X := \mathbf{R}^n$ and $Y := \mathbf{R}^p$ are identified with their duals. We supply X with the Euclidian norm (for simplicity). Let $h : X \mapsto Y$ define the constraint map, $M \subset Y$ the constrained set, $g : X \mapsto X$ the dynamics of the disconnected system $x' = g(x)$. When the necessary and sufficient viability condition

$$\forall\, x \in K,\;\; h'(x)g(x) \;\in\; T_M(x)$$

for complying to the viability constraints

$$\forall\, t \geq 0, \;\; h(x(t)) \;\in\; M$$

does not hold true, we have either to

1. reinforce the constraints by replacing the original constrained set $K := h^{-1}(M)$ by its *viability kernel*, which is the largest closed subset of K viable under g (possibly empty),
2. or modify the dynamics by introducing regulation controls (called regulons) and finding their regulation law (if any).

We refer to [6, 8, Aubin] for more informations on the first procedure and the properties of the viability kernel of $K := h^{-1}(M)$. For the second kind of procedures, we investigate successively

1. the regulation obtained by subtracting prices to the original dynamical behavior
$$x'(t) \;=\; g(x(t)) - p(t)$$
as planning procedures devised in [22, Drèze & De la Vallée-Poussin] and [35, Malinvaud] and as differential variational inequalities studied in [27, 28, Henry] and [16, 17, 18, Cornet] in the 70's (which are also studied in chapters 6 and 7 of [6, Aubin]),
2. the regulation by connecting the agents of the original dynamics through connecting matrices $W := (w_i^j) \in \mathcal{L}(X, X)$:
$$x'(t) \;=\; W(t)g(x(t))$$
3. a combination of both regulation procedures
$$x'(t) \;=\; W(t)g(x(t)) - p(t)$$

Naturally, we call the original dynamical system $x' = g(x)$ *a fully disconnected or decentralized system because the connection matrix is the identity matrix* or the regulon is equal to 0.

We first observe the following link between regulons and connection matrices:

Proposition 3.1 *Let* $\Gamma : X \mapsto X$ *be any continuous map such that*

$$\inf_{x \in K} \langle \Gamma(x), g(x) \rangle > 0$$

Then solutions to the differential equations $x'(t) = g(x(t)) - p(t)$ *and* $x'(t) = W(t)g(x(t))$ *do coincide whenever the connection matrix* $W(t)$ *is associated with the regulon* $p(t)$ *by the formula*

$$W(t) := \mathbf{1} - \frac{\Gamma(x(t))}{\langle \Gamma(x(t)), g(x(t)) \rangle} \otimes p(t) \tag{2}$$

the entries of which are equal to

$$w_j^i(t) \;=\; \delta_j^i - \frac{\Gamma(x(t))^i}{\langle \Gamma(x(t)), g(x(t)) \rangle} p_j(t)$$

Proof — Indeed, we observe that

$$W(t)g(x(t)) \;=\; g(x(t)) - \frac{\langle \Gamma(x(t)), g(x(t)) \rangle}{\langle \Gamma(x(t)), g(x(t)) \rangle} p(t) \;=\; g(x(t)) - p(t)$$

so that the two differential equations $x'(t) = W(t)g(x(t))$ and $x'(t) = g(x(t)) - p(t)$ are the same. □

3.2 Differentiating constraints

Consider the initial — disconnected — dynamical system

$$x'(t) \;=\; g(x(t)) \tag{3}$$

subject to collective viability constraints of the form

$$\forall\, t \geq 0, \;\; h(x(t)) \;\in\; M \tag{4}$$

Nagumo's Invariance Theorem provides a necessary and sufficient condition for the subset

$$K \;:=\; h^{-1}(M) \;=\; \{x \in X \quad \text{such that} \;\; h(x) \in M\}$$

to be viable in the sense that from any initial state $x_0 \in K$ starts a solution $x(\cdot)$ to the differential equation $x' = g(x)$ viable in K in the sense that

$$\forall\, t \geq 0, \;\; h(x(t)) \;\in\; M$$

For that purpose, we shall need to "differentiate" these viability constraints by implementing the concept of tangency to any subset. The adequate choice to obtain viability theorems is the concept of *contingent cone*[3] introduced for the first time by Georges Bouligand and Francesco Severi in the thirties (and which happens to be the cornerstone to set-valued analysis).

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[3] For a presentation of the ménagerie of tangent cones, we refer to chapter 4 of [13, Aubin & Frankowska] and [41, Rockafellar & Wets].

Definition 3.2 *When K is a subset of X and x belongs to K, the contingent cone $T_K(x)$ to K at x is the closed cone of elements v satisfying*

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} \; = \; 0$$

We observe that

$$\text{if} \quad x \in \operatorname{Int}(K), \;\; \text{then} \;\; T_K(x) \; = \; X$$

and that

$$\text{if} \quad K := \{\bar{x}\}, \;\; \text{then} \;\; T_{\{\bar{x}\}}(\bar{x}) \; = \; \{0\}$$

We recall that a differentiable function viable in K satisfies

$$\forall\, t \geq 0, \;\; x'(t) \; \in \; T_K(x(t))$$

Let us mention that the contingent cone coincides with the tangent space $T_K(x)$ of differential geometry when K is a "smooth manifold". Also, when K is convex, one can prove that the contingent cone coincides with the *tangent cone* $T_K(x)$ to K at $x \in K$ of convex analysis, which is the closed cone spanned by $K - x$:

$$T_K(x) \; = \; \overline{\bigcup_{h>0} \frac{K-x}{h}}$$

In this case, the tangent cone is convex.

One can prove that the contingent cone $T_K(x)$ is convex whenever K "is *sleek* at x", which means that the set-valued map $T_K(\cdot)$ is lower semicontinuous at x. Convex subsets are sleek at every elements.

There is a calculus of contingent cones which allows us to "compute them", and thus, to "apply" the viability theorems[4]. We recall in particular that if $K := h^{-1}(M)$ where $h : X \mapsto Y$ is a continuously differentiable map such that $h'(x)$ is surjective[5] and M is closed and convex (or, more generally, sleek), then

$$T_K(x) \; = \; h'(x)^{-1} T_M(h(x))$$

The Viability Theorem holds true for the class of *Marchaud systems*

$$\text{for almost all } \; t, \;\; x'(t) = f(x(t), p(t)) \text{ where } p(t) \in P(x(t)) \tag{5}$$

which satisfy the following conditions:

$$\left\{ \begin{array}{ll} i) & \text{Graph}(P) \;\text{ is closed} \\ ii) & f \;\text{ is continuous} \\ iii) & \text{the velocity subsets } \; F(x) \; := \; f(x, P(x)) \;\text{ are convex} \\ iv) & f \;\text{ and }\; P \;\text{ have linear growth} \end{array} \right. \tag{6}$$

---

[4] as well as the "equilibrium theorems under constraints" and "optimization theorems under constraints".

[5] A weaker requirement is the "transversality assumption"

$$\operatorname{Im}(h'(x)) + T_M(h(x)) \; = \; Y$$

It states that *a closed subset K is viable under (5) if and only if the regulation map R_K defined by*

$$\forall\, x \in K,\;\; R_K(x) \;:=\; \{p \in P(x) \mid f(x,p) \in T_K(x)\}$$

has nonempty values. Furthermore, any "open loop" *control* $p(\cdot)$ *regulating a viable solution* $x(\cdot)$ *in the sense that obeys the* regulation law

$$\text{for almost all }\; t,\;\; p(t) \in R_K(x(t)) \tag{7}$$

Otherwise, if K is not a viability domain of the system (5), there exists a largest closed viability domain of (U,f) contained in K (possibly empty), denoted $\mathrm{Viab}(K)$, *called the viability kernel of K, and equal to the set of states $x_0 \in K$ from which starts a solution of the system (5) viable in K.*

For differential equations $x' = g(x)$, it was proved in 1942 by Nagumo, stating that K is viable under $x' = g(x)$ if and only if for any $x \in K$, the dynamics and the constraints are linked by the following relation :

$$g(x) \;\in\; T_K(x)$$

When $K := h^{-1}(M)$ where M is sleek and h is continuously differentiable, then this condition becomes

$$\forall\, x \in K,\;\; h'(x)g(x) \;\in\; T_M(h(x))$$

When the necessary and sufficient condition $h'(x)g(x) \;\in\; T_M(h(x))$ is not met by every $x \in K$, we denote by $(\mathbf{1} - \pi_{T_M(h(x))})h'(x)g(x)$ the *viability error* (where $\pi_{T_M(h(x))}$ is the projector of best approximation onto the contingent cone $T_M(h(x))$).

We then shall regulate it either by introducing connection matrices W or controls p and prove that, under some adequate assumptions, these two modes of regulation are related and that specific models are equivalent in the sense that they provide the same viable evolutions of the state.

3.3 Minimal Connectionist Complexity

Actually, if the regulons $p(t)$ regulate viable evolutions, so do the connection matrices $W(t)$ associated to them by formula (2). Indeed, the Viability Theorem (see Theorem 6.1.4 of [6, Aubin] for instance) implies that if we introduce the *regulation map* Π_M defined by

$$\Pi_M(x) \;:=\; \{p \mid h'(x)(g(x) - p) \in T_M(h(x))\}$$

the regulons $p(t)$ regulating viable solutions are given by the *regulation law*

$$p(t) \;\in\; \Pi_M(x(t))$$

In the same way, if we introduce the new regulation map

$$R_M(x) \;:=\; \{W \in \mathcal{L}(X,X) \mid h'(x)Wg(x) \;\in\; T_M(h(x))\}$$

the connection matrices W regulating viable solutions to $x' = Wg(x)$ are given by the *regulation law*

$$W(t) \;\in\; R_M(x(t))$$

We observe that whenever the regulons $p(\cdot)$ obey the regulation law, so do the associated connection matrix because the equation

$$h'(x(t))W(x(t))g(x(t)) \; = \; h'(x(t))(g(x(t)) - p(t)) \; \in \; T_M(h(x))$$

implies that the condition $W(t) \in R_M(x(t))$ holds true whenever $p(t) \in \Pi_M(x(t))$.

If $\varpi^\circ(x) \in \Pi_M(x)$ denotes the regulon in $\Pi_M(x)$ with minimal norm (which exists whenever M is sleek and h is differentiable, because in this case $\Pi_M(x)$ is closed and convex), we recall that solutions to differential equation

$$(8) \qquad x'(t) \; = \; g(x(t)) - \varpi^\circ(x(t))$$

are called *slow viable solutions* to control system $x' = g(x) - p$. They do exist whenever the regulation map Π_M is lower semicontinuous with closed convex images (see [26, Falcone & Saint-Pierre] or Theorem 6.6.3 of [6, Aubin] for instance) despite the fact that ϖ° is not necessarily continuous.

The problem of minimal complexity is then to find connection matrices $W^\bullet(x) \in R_M(x)$ *as close as possible to the identity matrix* $\mathbf{1}$:

Definition 3.3 *If* $W \in \mathcal{L}(X,X)$ *is a connection matrix regulating the connected system* $x' = Wg(x)$*, we regard the norm*[6] $\|W - \mathbf{1}\|_{\mathcal{L}(X,X)}$ *as the complexity index of the connected system. We denote by* $W^\bullet(x) \in R_M(x)$ *the connection matrix closest to the identity matrix* $\mathbf{1}$ *among the matrices of* $R_M(x)$*.*

The solutions to differential equation

$$(9) \qquad x'(t) \; = \; W^\bullet(x(t))g(x(t))$$

are called the simplest — least complex — viable solutions to the connected system $x' = Wg(x)$*.*

It happens that *such connection matrices minimizing the complexity index are associated with the regulons* $\varpi^\circ(x)$ *through the formula*

$$W^\bullet(x) := \mathbf{1} - \frac{g(x)}{\|g(x)\|^2} \otimes \varpi^\circ(x)$$

To make this statement precise, we recall that when $B \in \mathcal{L}(X,Y)$ is surjective, its orthogonal right inverse is equal to

$$B^+ \; = \; B^\star(BB^\star)^{-1}$$

---

[6] We choose the Hilbert-Schmidt norm associated with the Euclidian norm supplied on X, This can be generalized to the case of any Hilbertian norm, or even, to the case when $\mathcal{L}(X,X)$ is supplied with the usual norm of linear operators.

that we can supply Y with the final norm μ^B defined by $\mu^B(z) := \|B^+ z\|$. Denote by π_K^B the projector of best approximation onto the closed subset K for this final norm. One can prove that the unique solution $\bar{x}$ to the minimization problem

$$\inf_{Bx \in K+v} \|x-u\|$$

is equal to

$$\bar{x} \;=\; u - B^+(\mathbf{1} - \pi_K^B)(Bu - v)$$

(See Proposition 2.1.4 of [7, Aubin] for instance).

We finally recall that when M is sleek (and in particular, convex), then its tangent cones $T_M(y)$ are convex (see Theorem 4.1.10 of [13, Aubin & Frankowska]). In this case, since the polar cone to the contingent cone $T_M(y)$ is the normal cone $N_M(y)$, we also know that the solution to the minimization problem

$$\inf_{Bx \in T_M(y)+v} \|x-u\|$$

is equal to

$$\bar{x} \;=\; u - B^+(\mathbf{1} - \pi_{T_M(y)}^B)(Bu - v) \;=\; u - B^\star \pi_{N_M(y)}^{B^\star}(BB^\star)^{-1}(Bu - v)$$

where $\pi_{N_M(y)}^{B^\star}$ denotes the projector onto the normal cone $N_M(y)$ when the dual $Y^\star$ is supplied with the "dual final norm" $\|B^\star q\|$ (see Theorem 10.1.4 of [8, Aubin] for instance).

We then can prove the existence theorem of slow solutions:

Theorem 3.4 *Let us assume that the map g is continuous and bounded and that the resource map h is continuously differentiable and satisfies the uniform surjectivity condition:*

$$(10) \qquad \forall\, x \in K,\;\; h'(x) \text{ is surjective } \& \; \sup_{x \in K} \|h'(x)^+\| \;<\; +\infty$$

and that M is closed convex (or more generally, sleek). Then the slow solution of the control system

$$(11) \qquad x'(t) \;=\; g(x(t)) - p(t)$$

subjected to the viability constraints

$$\forall\, t \geq 0,\;\; h(x(t)) \;\in\; M$$

is the solution to the differential equation

$$x'(t) \;=\; g(x(t)) - \varpi^\circ(x(t))$$

where

$$(12) \qquad \left\{ \begin{array}{lcl} \varpi^\circ(x) & = & h'(x)^+ \left(\mathbf{1} - \pi_{T_M(h(x))}^{h'(x)}\right) h'(x) g(x) \\ \\ & = & h'(x)^\star \pi_{N_M(h(x))}^{h'(x)^\star} (h'(x)h'(x)^\star)^{-1} h'(x) g(x) \end{array} \right.$$

(See for instance Proposition 4.3.3 of [8, Aubin]).

The first formula of (12) expresses that the feedback regulon $\varpi^\circ(x)$ is "proportional" — through the linear operator $h'(x)^+$ — to the smallest *viability error*

$$\left(\mathbf{1} - \pi^{h'(x)}_{T_M(h(x))}\right) h'(x)g(x)$$

made when the tangential characterization $h'(x)g(x) \in T_M(h(x))$ of the viability of the autonomous system is violated.

The second formula of (12) shows that

$$q(x) \ := \ \pi^{h'(x)^\star}_{N_M(h(x))}(h'(x)h'(x)^\star)^{-1}h'(x)g(x)$$

is a *viability multiplier* associated with the viability constraints $K := h^{-1}(M)$ because $\varpi^\circ(x) = h'(x)^\star q(x)$ (see Proposition 5.2.1 of [8, Aubin]).

In the same way, the existence theorem of sow solutions to control problems and the projection theorem on spaces of operators imply

Theorem 3.5 *Let us assume that the map g is continuous and bounded and that the resource map h is continuously differentiable and satisfies the uniform surjectivity condition:*

$$(13) \qquad \forall\, x \in K, \ \ h'(x) \text{ is surjective } \& \ \sup_{x\in K} \frac{\|h'(x)^+\|}{\|g(x)\|} \ < \ +\infty$$

Let $M \subset Y$ be a closed convex (or more generally, sleek) subset. Then the scarcity set $K := h^{-1}(M)$ is viable under the connected system $x' = Wg(x)$ if and only if for every $x \in K$, the image $R_M(x)$ of the regulation map is not empty.

The solution with minimal complexity subjected to the viability constraints

$$\forall\, t \geq 0, \ \ h(x(t)) \ \in \ M$$

is governed by the differential equation

$$x'(t) \ = \ W^\bullet(x(t))g(x(t))$$

where

$$(14) \quad \left\{ \begin{array}{lcl} W^\bullet(x) & = & \mathbf{1} - \dfrac{g(x)}{\|g(x)\|^2} \otimes h'(x)^+ \left(\mathbf{1} - \Pi^{h'(x)}_{T_M(h(x))}\right) h'(x)g(x) \\ \\ & = & \mathbf{1} - \dfrac{g(x)}{\|g(x)\|^2} \otimes h'(x)^\star \pi^{h'(x)^\star}_{N_M(h(x))}(h'(x)h'(x)^\star)^{-1}h'(x)g(x) \end{array} \right.$$

Furthermore, the slow viable solutions to control system (11) regulated by regulons and the viable solutions to the connected system $x' = Wg(x)$ under minimal complexity coincide.

Proof — We can apply the same proof as the one of Proposition 3.4 where the connected system is a control system $f(x, W) := Wg(x)$ and where the role of the regulon p is played by the connection matrix W.

Then, using the definition of tensor products $(g(x) \otimes h'(x))W := h'(x)Wg(x)$ of linear operators (see [7, Aubin]), the regulation map can be written in the form:

$$R_M(x) \;:=\; \{W \in \mathcal{L}(X, X) \mid (g(x) \otimes h'(x))W \;\in\; T_M(h(x))\}$$

so that viable solutions are regulated by the regulation law

$$(g(x(t)) \otimes h'(x(t)))W(t) \;\in\; T_M(h(x))$$

Since the map $g(x) \otimes h'(Wx)$ is surjective from the space $\mathcal{L}(X, X)$ of connection matrices to the resource space Y, we observe that

$$\|(g(x) \otimes h'(x))^+\| \;=\; \frac{\|h'(x)^+\|}{\|g(x)\|}$$

so that the assumptions of Proposition 1.5.1 of [13, Aubin & Frankowska] are satisfied thanks to (13). Hence the regulation map R_M is lower semicontinuous.

Therefore, the viable connection matrices closest to the identity matrix $\mathbf{1}$ are the solutions $W^\bullet$ minimizing the distance $\|W - \mathbf{1}\|$ among the connection matrices satisfying the above constraints.

We infer from Theorem 2.5.6 of [7, Aubin] that since the map $g(x) \otimes h'(Wx)$ is surjective from the space $\mathcal{L}(X, X)$ of connection matrices to the resource space Y, the solution is given by formula (14).

Therefore, we derive from Theorem 6.6.3 of [6, Aubin] the existence of a viable solutions to the differential inclusion $x' = W^\bullet(x)g(x)$. □

3.4 Mixed Regulation by Regulees and Connection Matrices

Let us consider the mixed regulation mode using both connection matrices W and regulons p:

$$x'(t) \;=\; Wg(x(t)) - p(t)$$

subjected to the same viability constraints

$$\forall\, t \geq 0, \;\; h(x(t)) \;\in\; M$$

Then the regulation map takes the form

$$R_M(x) \;:=\; \{(W, p) \in \mathcal{L}(X, X) \times X \mid h'(x)Wg(x) - h'(x)p \;\in\; T_M(x)\}$$

We supply the space $\mathcal{L}(X, X) \times X$ with the norm $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}(W, p)^2 \;:=\; \alpha\|W\|^2 + \beta\|p\|^2$$

Then the slow solutions are regulated by the pair $(W_{\alpha,\beta}(x), \varpi_{\alpha,\beta}(x))$ in the regulation map which minimizes the distance $\nu_{\alpha,\beta}(W-\mathbf{1},p)$ in $R_M(x)$. By Theorem 10.3.7 of [8, Aubin], the solution is given by

$$\left\{\begin{array}{lcl} W_{\alpha,\beta}(x) & = & \mathbf{1} - \dfrac{\beta g(x)}{\alpha+\beta\|g(x)\|^2} \otimes h'(x)^+ \left(\mathbf{1} - \pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x) \\ \\ \varpi_{\alpha,\beta}(x) & = & \dfrac{\alpha}{\alpha+\beta\|g(x)\|^2} h'(x)^+ \left(\mathbf{1} - \pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x) \end{array}\right.$$

We then observe that, setting $\varpi^\circ(x) := h'(x)^+ \left(\mathbf{1} - \Pi_{T_M(h(x))}^{h'(x)}\right) h'(x)g(x)$,

$$\left\{\begin{array}{l} W_{\alpha,\beta}(x)g(x) - \varpi_{\alpha,\beta}(x) \\ = \; g(x) - \dfrac{\beta\|g(x)\|^2}{\alpha+\beta\|g(x)\|^2}\varpi^\circ(x) - \dfrac{\alpha}{\alpha+\beta\|g(x)\|^2}\varpi^\circ(x) \\ = \; g(x) - \varpi^\circ(x) \end{array}\right.$$

Therefore, for every pair (α,β), the slow viable solutions, which are solutions to the differential equations

$$x'(t) \;=\; W_{\alpha,\beta}(x(t))g(x(t)) - \varpi_{\alpha,\beta}(x(t))$$

are the slow solutions to the initial control system, i.e., solutions to

$$x'(t) \;=\; g(x(t)) - \varpi^\circ(x(t)) \;\; \square$$

3.5 Heavy Connected Evolution

Instead of minimizing at each instant t the complexity index $\|W(t)-\mathbf{1}\|_{\mathcal{L}(X,X)}$, we advocate to consider the problem of minimizing at each instant the *norm* $\|W'(t)\|_{\mathcal{L}(X,X)}$ *of the velocity of the connection matrix*, regarded as a *dynamical complexity index*. In a "regulation perspective" — in which no agent controlling the system is identified — when autonomous systems has to be connected to remain viable, a system is "dynamically simpler" whenever its connection matrix evolves slower (whereas in control theory slow viable evolutions constitute a more reasonable choice).

Heavy connected viable evolutions are the ones associated with connection matrices evolving while minimizing at each instant the norm $\|W'(t)\|_{\mathcal{L}(X,X)}$ *of the velocity of the viable connection matrix.*

We need more details to make precise the set of velocities W' over which this minimization process is effected.

Not only shall we consider such heavy connected evolution, but also the *heavy viable evolutions* of the system regulated by regulons, introduced in [12, Aubin & Frankowska] (see also Chapter 7 of [6, Aubin]), obtained by minimizing at each instant the norm of the velocity of the regulon (in control theory, this is the problem of minimal chattering).

In order to treat both problems in the same time, we shall consider for slow evolutions, the mixed regulation of the initial problem $x' = g(x)$ by both regulons and connection matrices

$$x'(t) \ = \ Wg(x(t)) - p(t)$$

subjected to the same viability constraints

$$\forall\, t \geq 0, \ \ h(x(t)) \ \in \ M$$

and where we choose velocities of the connection matrices and of the regulons with minimal norm when we supply the space $\mathcal{L}(X,X) \times X$ with the norm $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}(W,p)^2 \ := \ \alpha\|W\|^2 + \beta\|p\|^2$$

The case of heavy connected viable evolutions is obtained by taking $\alpha = 1, \ \beta = 0$ and the case of heavy viable evolution by taking $\alpha = 0, \ \beta = 1$.

The regulons and the connection matrices regulating viable solutions obey the regulation law

$$(W(t), p(t)) \ \in \ R_M(x(t))$$

where

$$(15) \quad R_M(x) \ := \ \{(W,p) \in \mathcal{L}(X,X) \times X^\star \mid h'(x)(Wg(x) - p) \ \in \ T_M(h(x))\}$$

The difficulty lies in the fact that the graph of the contingent cone map $y \rightsquigarrow T_M(y)$ is not necessarily closed, which happens whenever inequality constraints are involved, for instance. In order to overcome this obstacle, we introduce a closed graph selection $T_M^\Box(\cdot)$ of $T_M(\cdot)$ whenever the graph of the contingent cone is not closed. There are several ways to built closed graph selections of the contingent cone map, such as the global contingent set (see Definition 4.4.1 of [6, Aubin]) or such as the Maderner selection when M is defined by inequality constraints (see [33, Maderner] or Theorem 5.1.11 of [6, Aubin]). We associate with such a closed graph selection $T_M^\Box(\cdot)$ the set subset $\mathcal{K}^\Box \subset X \times \mathcal{L}(X,X) \times Y$ defined by

$$\mathcal{K}^\Box \ := \ \{(x,W,p) \mid h'(x)Wg(x) - h'(x)p \ \in \ T_M^\Box(h(x))\}$$

which is a closed subset contained in the graph of the regulation map R_M. We now regard $\mathcal{K}^\Box$ as a constrained set of the "metasystem" obtained by setting bonds on the velocity of the connection matrices and the regulons:

$$(16) \qquad \begin{cases} i) & x'(t) \ = \ W(t)g(x(t)) - p(t) \\ ii) & \|W'(t)\| \ \leq \ \varphi(x(t), W(t), p(t)) \\ iii) & \|p'(t)\| \ \leq \ \psi(x(t), W(t), p(t)) \end{cases}$$

If $F : X \rightsquigarrow Y$ is a set-valued map, we denote by $DF(x,y) : X \rightsquigarrow Y$ the *contingent derivative* of F at $(x,y) \in \mathrm{Graph}(F)$ defined by

$$\mathrm{Graph}(DF(x,y)) \; := \; T_{\mathrm{Graph}(F)}(x,y)$$

introduced in [4, Aubin] (see also [13, Aubin & Frankowska], [41, Rockafellar & Wets] and [9, Aubin] for mutations of set-valued maps).

From now on, we set $v \; := \; Wg(x) - p$ and

$$H_M^{\square}(x,W,p) \; := \; DT_M^{\square}(h(x), h'(x)v)(h'(x)v) - h'(x)Wg'(x)v - h''(x)(v,v)$$

When $g(x) \neq 0$ and $h'(x)$ is surjective, when g is continuously differentiable and h twice continuously differentiable, we infer that (v,V,q) belongs to the contingent cone $T_{\mathcal{K}^{\square}}(x,W,p)$ if and only if

$$\left\{ \begin{array}{l} h'(x)Vg(x) - h'(x)q + h'(x)Wg'(x)v + h''(x)(v, Wg(x) - p) \\ \in \; DT_M^{\square}(h(x), h'(x)(Wg(x) - p))(h'(x)v) \end{array} \right.$$

Therefore (v,V,q) belongs to $T_{\mathcal{K}^{\square}}(x,W,p)$ if and only if

$$h'(x)Vg(x) - h'(x)q \; \in \; H_M^{\square}(x,W,p)$$

We now introduce

$$\left\{ \begin{array}{l} G_M^{\square}(x,W,p) \; := \; \{(V,q) \in \varphi(x,W,p)B \times \psi(x,W,p)B \\ \text{such that } h'(x)Vg(x) - h'(x)q \; \in \; H_M^{\square}(x,W,p)\} \end{array} \right.$$

Hence, we deduce from the Viability Theorem the following

Proposition 3.6 *Let us assume that φ and ψ are continuous with linear growth, that g is continuously differentiable, that h is twice continuously differentiable and that for every $(x,W,p) \in \mathcal{K}^{\square}$, the subset of viable connection matrix-regulon pair $G_M^{\square}(x,W,p)$ is not empty.*

Then from any initial $(x_0, W_0, p_0) \in \mathcal{K}^{\square}$ starts one solution $(x.,W.,p.)$ to the metasystem (16) satisfying the differential inclusion

$$\text{for almost all } t \geq 0, \;\; (W'(t), p'(t)) \; \in \; G_M^{\square}(x(t), W(t), p(t))$$

sustaining the viability of the system in the sense that the viability constraints

$$\forall \, t \geq 0, \;\; h'(x(t))(W(t)g(x(t)) - p(t)) \; \in \; T_M^{\square}(h(x(t)))$$

are always met.

In particular, for any selection $r(x,W,p)$ of $h'(x)^{-1}(H_M^{\square}(x,W,p))$ and any non-negative real-valued function $\lambda(\cdot)$ such that

$$\lambda(x) \; \leq \; \min\left(\frac{\varphi(x,W,p)}{\|g(x)\| \|r(x,W,p)\|}, \frac{\psi(x,W,p) + \|r(x,W,p)\|}{\|g(x)\|^2 \|r(x,W,p)\|} \right)$$

the solutions to the system of differential equations

$$\begin{cases} i) & x' & = & Wg(x) - p \\ ii) & W' & = & \lambda(x)g(x) \otimes r(x,W,p) \\ iii) & p' & = & (\lambda(x)\|g(x)\|^2 - 1)r(x,W,p) \end{cases}$$

are viable.

Proof — Indeed, we observe that

$$W'g(x) - p' \;=\; \lambda(x)\|g(x)\|^2 r(x,W,p) - (\lambda(x)\|g(x)\|^2 - 1)r(x,W,p) \;=\; r(x,W,p)$$

so that

$$h'(x)(W'g(x) - p') \;=\; h'(x)r(x,W,p) \;\in\; H_M^\square(x,W,p) \;\; \square$$

From this, we derive the differential equations governing the evolution of heavy solutions, the ones which choose velocities of the connection matrices and of the regulons with minimal norm when we supply the space $\mathcal{L}(X,X) \times X$ with the norm $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}(W,p)^2 \;:=\; \alpha\|W\|^2 + \beta\|p\|^2$$

Theorem 3.7 *We posit the assumptions of Proposition 3.6 and we assume that*

$$\forall\, x \in K,\;\; h'(x) \text{ is surjective } \& \sup_{x \in K} \frac{\|h'(x)^+\|}{\|g(x)\|} \;<\; +\infty \tag{17}$$

and that $(x,W) \rightsquigarrow H_M^\square(x,W,p)$ *is lower semicontinuous. We set*

$$r^\circ(x,W,p) \;:=\; h'(x)^+\left(\pi^{h'(x)}_{DT^\square_M(h(x),h'(x)v)(h'(x)v)} - \mathbf{1}\right)\left(h'(x)Wg'(x)v + h''(x)(v,v)\right)$$

where $v := Wg(x) - p$*, which is a map independent from* α *and* β*. Then the differential equation governing the evolution of the heavy evolution of the connecting matrix is given by the system*

$$\begin{cases} i) & x' & = & Wg(x) - p \\ ii) & W' & = & \dfrac{\beta g(x)}{\alpha + \beta\|g(x)\|^2} \otimes r^\circ(x,W,p) \\ iii) & p' & = & \dfrac{-\alpha}{\alpha + \beta\|g(x)\|^2} r^\circ(x,W,p) \end{cases}$$

Proof — Proposition 1.5.1 of [13, Aubin & Frankowska] implies that the regulation map R_M defined by (15) is lower semicontinuous because the set-valued map $H_M^\square(\cdot)$ is lower semicontinuous thanks to the surjectivity assumption (17).

Heavy solutions are obtained by choosing the minimal velocity $(W'(t), p'(t))$ in the set $H_M^\square(x(t), W(t), p(t))$ for the norm $\nu_{\alpha,\beta}$. Setting $x := x(t)$, $W := W(t)$

and $p(t) = p$, then $(V, q) := (W'_{\alpha,\beta}(t), p'_{\alpha,\beta}(t))$ is the pair (V, q) solution to the minimization problem:

$$\inf_{h'(x)Vg(x)-h'(x)q \,\in\, H^{\square}_{M}(x,W,p)} \nu_{\alpha,\beta}(V, q)$$

We infer from Theorem 10.3.7 of [8, Aubin] that since the map $g(x) \otimes h'(x)$ is surjective from the space $\mathcal{L}(X, X)$ of connection matrices to the resource space Y, the solution is given by

$$\left\{\begin{array}{lll} W' & = & \dfrac{\beta g(x)}{\alpha + \beta\|g(x)\|^2} \otimes h'(x)^{+}(\pi^{h'(x)}_{DT^{\square}_{M}(h(x),h'(x)v)(h'(x)v)} - \mathbf{1})q(x, W, p) \\ \text{where} & & \\ q(x, W, p) & := & h'(x)Wg'(x)v + h''(x)(v, v) \\ & & \\ p' & = & \dfrac{-\alpha}{\alpha + \beta\|g(x)\|^2} h'(x)^{+}(\pi^{h'(x)}_{DT^{\square}_{M}(h(x),h'(x)v)(h'(x)v)} - \mathbf{1})q(x, W, p) \end{array}\right.$$

The velocities $W'(t)$ and $p'(t)$ depend upon α and β. However, we observe that

$$W'(t)g(x(t)) - p'(t) \;=\; r^{\circ}(x(t), W(t), p(t))$$

is independent of the parameters α and β.

4 Connecting the Constraints

Let X_j, Y_j , Z_j $(j = 0, \ldots m-1)$ and Y_j $(j = i, \ldots m)$ denote finite dimensional spaces.

Consider linear operators $A_j \in \mathcal{L}(X_j, Y_j)$, $B_j \in \mathcal{L}(Z_j, X_{j+1})$ and $C_j \in \mathcal{L}(X_j, Y_j)$ and "technological matrices" $W_j \in \mathcal{L}(Y_j, Z_j)$.

The hierarchical process takes the form

$$\forall\, j = 1, \ldots m, \;\; B_{j-1}W_{j-1}A_{j-1}x_{j-1} + C_j x_j \;\in\; M_j$$

where $M_j \subset Y_j$ are closed convex subsets.

Here $x_0 \in X_0$ is regarded as an output, $x_m \in X_m$ as an input, whereas $x_j \in X_j$ is regarded either as intermediate outputs or inputs.

Example Observe that taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m$, the above constraints read:

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0 W_0 A_0 x_0 - x_m \;\in\; M_m$$

Taking $M_j := \{0\}$ for $j = 1, \ldots, m-1$ and $C_j := -\mathbf{1}$ for $j = 1, \ldots, m-1$ and $C_m = 0$, we find constraints of the form

$$B_{m-1}W_{m-1}A_{m-1} \cdots B_{j-1}W_{j-1}A_{j-1} \cdots B_0W_0A_0x_0 \;\in\; M_m$$

Therefore, in a hierarchical process, the state variables are the sequences

$$((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1})$$

made of vectors and of matrices. The above constraints are defined by inverse images under a multilinear map.

$$\begin{tikzcd}
x_{j-1} \in X_{j-1} \arrow[r, mapsto, "A_{j-1}"] & Y_{j-1} \arrow[d, "W_{j-1}"'] \\
& Z_{j-1} \arrow[d, "B_{j-1}"'] \\
x_j \in X_j \arrow[r, mapsto, "C_j"'] & Y_j \supset M
\end{tikzcd}$$

However, the linearity of the operators A_j, B_j & C_j does not really bring any simplification in the viability analysis. This is the reason why we study this example in the more general nonlinear form described in the following way:

1. $x_j \in L_j \subset X_j$, $(j = 1, \ldots m)$,
2. continuously differentiable maps $l_j : X_j \mapsto Y_j$ and $h_j : Z_{j-1} \times X_j \mapsto Y_j$,
3. "technological matrices" $W_j \in \mathcal{W}_j \subset \mathcal{L}(Y_j, Z_j)$
4. "scarcity constraint" $h_j(W_{j-1}l_{j-1}(x_{j-1}), x_j) \in M_j \subset Y_j$

The constrained set is defined by

$$\begin{array}{l} K \;:=\; \{((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1}) \in \prod_{j=0}^{m} L_j \times \prod_{j=0}^{m-1} \mathcal{W}_j \\ \textit{such that}\, h_j(W_{j-1}l_{j-1}(x_{j-1}), x_j) \;\in\; M_j\} \end{array} \tag{18}$$

We set

$$A_j \;:=\; l_j'(x_j),\quad B_{j-1} \;:=\; h'_{j_{z_{j-1}}}(W_{j-1}l_{j-1}(x_{j-1}), x_j)$$

and

$$C_j \;:=\; h'_{j_{x_j}}(W_{j-1}l_{j-1}(x_{j-1}), x_j)$$

Hence, the directional derivative is equal to

$$\left\{ \begin{array}{l} h_j'(W_{j-1}l_{j-1}(x_{j-1}), x_j)(u_{j-1}, u_j, V_{j-1}) \\ =\; C_ju_j + B_{j-1}W_{j-1}A_{j-1}u_{j-1} + B_{j-1}V_{j-1}l_{j-1}(x_{j-1}) \end{array} \right.$$

and can be written in the form

$$B_{j-1}W_{-1}A_{j-1}u_{j-1}+C_ju_j+(l_{j-1}(x_{j-1})\otimes B_{j-1})V_{j-1}$$

The transpose of this operator maps $(q_j)\in\prod_{j=1}^m Y_j^\star$ to

$$(X_0\times\mathcal{L}(Y_0^\star,Z_0^\star))\times\left(\prod_{j=1}^{m-1}(X_j^\star\times\mathcal{L}(Y_j^\star,Z_j^\star)\right)\times X_m^\star$$

in the following way:

$$\left\{\begin{array}{cl}(A_0^\star W_0^\star B_0^\star q_1\ ,\ l_0(x_0)\otimes B_0^\star q_1) & (j=0)\\ (A_j^\star W_j^\star B_j^\star q_{j+1}+C_j^\star q_j\ ,\ l_j(x_j)\otimes B_j^\star q_{j+1}) & (j=1,\ldots,m-1)\\ C_m^\star q_m & (j=m)\end{array}\right.$$

We posit a sufficient condition guaranteeing the transversality conditions:

(19) $\quad\forall\,((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1}),\ \ h'_{j_{x_j}}(W_{j-1}l(x_{j-1}),x_j)$ are surjective

Assume now that the dynamical behaviors of the autonomous systems are described by

$$\left\{\begin{array}{rcll}x_j'(t)&=&g_j(x_j(t))(t) & (j=0,\ldots m)\\ \\ W_j'(t)&=&e(W_j(t)) & (j=0,\ldots m-1)\end{array}\right.$$

where $y_j(t)\in Y_j$ and $q_j(t)\in Z_j^\star$.

We deduce from Proposition 5.2.2 of [8, Aubin] on viability multipliers that *the constraints (18) involving connection matrices are viable under control systems of the form*

$$\left\{\begin{array}{rcl}x_0'(t)&=&g_0(x_0(t))-A_0^\star W_0^\star(t)B_0^\star p_1(t)\\ \\ x_j'(t)&=&g_j(x_j(t))-C_j^\star p_j(t)-A_j^\star W_j^\star(t)B_j^\star p_{j+1}(t)\ \ (j=1,\ldots,m-1)\\ \\ x_m'(t)&=&g_m(x_m(t))-C_m^\star p_m(t)\\ \\ W_j'(t)&:=&e(W_j(t))-l_j(x_j(t))\otimes B_j^\star p_{j+1}(t)\ \ (j=0,\ldots,m-1)\end{array}\right.$$

The regulation map can be written in the form in the following way: $(p_j)_{j=1,\ldots,m}$ belongs to $\Pi_K((x_j)_{j=0,\ldots,m},\ (W_j)_{j=0,\ldots,m-1})$ if and only if p is a solution to the system of equations

$$\left\{\begin{array}{l}
\left(C_1C_1^\star + B_0W_0A_0A_0^\star W_0^\star B_0^\star\right) p + \langle p, B_0l_0(x_0)\rangle B_0l_0(x_0) + C_1A_1^\star W_1^\star B_1^\star p_2 \ = \ z_1 \\
\\
\Big(C_jC_j^\star + B_{j-1}W_{j-1}A_{j-1}A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\Big)\, p_j \\
+\langle p_j, B_{j-1}l_{j-1}(x_{j-1})\rangle B_{j-1}l_{j-1}(x_{j-1}) + C_jA_j^\star W_j^\star B_j^\star p_{j+1} \\
+B_{j-1}W_{j-1}A_{j-1}C_{j-1}^\star p_{j-1} \ = \ z_j \ \ (j = 2, \ldots, m-1) \\
\\
\left(C_mC_m^\star + B_{m-1}W_{m-1}A_{m-1}A_{m-1}^\star W_{m-1}^\star B_{m-1}^\star\right) p_m \\
+\langle p_m, B_{m-1}l_{m-1}(x_{m-1})\rangle B_{m-1}l_{m-1}(x_{m-1}) + B_{m-1}W_{m-1}A_{m-1}C_{m-1}^\star p_{m-1} \ = \ z_m
\end{array}\right.$$

where where r_j belongs to $T_{M_j}(h_j(W_{j-1}l(x_{j-1}), x_j))$ and where

$$z_j \ := \ C_jg(x_j) + B_{j-1}W_{j-1}A_{j-1}g_{j-1}(x_{j-1}) + B_{j-1}e_{j-1}(W_{j-1})l(x_{j-1}) - r_j$$

In other words, this solution is obtained by inverting a tri-diagonal block matrix:

$$\left(\begin{array}{ccccccccccccc}
\mathcal{J}_1 & \mathcal{M}_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\mathcal{M}_1^\star & \mathcal{J}_2 & \mathcal{M}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \mathcal{M}_{j-1}^\star & \mathcal{J}_j & \mathcal{M}_j & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\mathcal{M}_{m-2}^\star & \mathcal{J}_{m-1} & \mathcal{M}_{m-1} \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mathcal{M}_{m-1}^\star & \mathcal{J}_m
\end{array}\right)$$

where we set

$$\left\{\begin{array}{l}
\mathcal{J}_jp_j \\
:= \ \Big(C_jC_j^\star + B_{j-1}W_{j-1}A_{j-1}A_{j-1}^\star W_{j-1}^\star B_{j-1}^\star\Big)\, p_j + \langle p_j, B_{j-1}l_{j-1}(x_{j-1})\rangle B_{j-1}l_{j-1}(x_{j-1})
\end{array}\right.$$

and

$$\mathcal{M}_j \ := \ C_jA_j^\star W_j^\star B_j^\star$$

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Jean-Pierre Aubin

Centre de Recherche Viabilité, Jeux, Contrôle
Université de Paris-Dauphine
F-75775 Paris cx (16)
aubin@dmi.ens.fr