Regulation of Dynamic Networks

Jean-Pierre Aubin

Abstract

Socio-economic networks, neural networks and genetic networks

- **describe collective phenomena**
- **through constraints relating actions of several actors,**
- **coalitions of these actors**
- **and multilinear connectionist operators acting on the set of actions of each coalition.**

We provide a class of control systems governing the evolution of actions, coalitions and multilinear connectionist operators under which **the architecture of the network remains viable.**

The controls are the **“viability multipliers”** of the “resource space” in which the constraints are defined.

They are involved as **“tensor products”** of the actions of the coalitions and the viability multiplier, allowing to encapsulate in this dynamical and multilinear framework the concept of **Hebbian learning rules** in neural networks in the form of “multi-Hebbian” dynamics in the evolution of connectionist operators.

They are also involved in the evolution of coalitions through the “cost” of the constraints under the viability multiplier regarded as a price.

Outline

We shall present examples in order of increasing complexity:

1. **Linear Case**
2. **Example: Multi-stage Production Processes**
3. **Bilinear Case**
4. **Multilinear Case**
5. **Multilinear Case with Evolving Coalitions**
6. **The Regulation Map**
7. **Reestablishing Viability by Viability Multipliers**

The Linear Case : Dynamic Production Economy

We regard the finite dimensional vector spaces X and Y as the input (resource) and the output space respectively and the space $\mathcal{L}(X,Y)$ of linear operators A as input-output (production, connectionist) operators

Let $M \subset Y$ be the "demand set" that the producer must fulfill.

The problem takes the form:

$$\forall\, t \geq 0,\;\; A(t)x(t) \;\in\; M$$

where both the resource x and the connectionist operator A evolve. We start with a dynamic system of the form

$$\begin{cases} i) & x'(t) \;=\; c(x(t)) \\ ii) & A'(t) \;=\; \alpha(A(t)) \end{cases}$$

under which the above constraints are not necessarily viable.

Viability Multipliers

We can reestablish viability by involving **viability multipliers** $q \in Y^*$ **ranging over the dual** $Y^* := Y$ **of the resource space** Y **(identified with** Y **for simplicity)**
.

We denote by $A^* \in \mathcal{L}(Y^*, X^*)$ **the transpose of** A**:**

$$\forall\, q \in Y^*,\ \forall\, x \in X,\ \ \langle A^*q, x\rangle\ :=\ \langle q, Ax\rangle$$

We denote by $x \otimes q \in \mathcal{L}(X^\star, Y^*)$ **denotes the** tensor product **defined by**

$$x \otimes q\ :\ p \in X^* := X \mapsto (x \otimes q)(p)\ :=\ \langle p, x\rangle q$$

the matrix of which is made of entries

$$(x \otimes q)_i^j = x_i q^j$$

The **contingent cone** $T_M(x)$ **to** $M \subset Y$ **at** $y \in M$ is the set of directions $v \in Y$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $y + h_n v_n \in M$ for every n. The normal cone is

$$N_M(y) := \{q \in Y^* \mid \forall\, v \in T_M(y),\ \langle q, v\rangle \leq 0\}$$

For simplicity, we shall assume that M is convex.

The Viable Control System

The viability of the constraints

$$\forall\, t \geq 0,\ \ A(t)x(t)\ \in\ M$$

can be reestablished when the initial system is replaced the above system by the control system

$$\begin{cases} (i) & x'(t)\ =\ c(x(t)) - A^*(t)q(t) \\ (ii) & A'(t)\ =\ \alpha(A(t)) - x(t) \otimes q(t) \end{cases}$$

which involves viability multipliers $q(t) \in Y^\star$ which satisfy the regulation law

$$\forall\, t \geq 0,\ \ q(t)\ \in\ R_M(A(t), x(t))$$

where the regulation map R_M is defined by

$$R_M(A,x)\ =\ (AA^\star + \|x\|^2 \mathsf{I})^{-1}(Ac(x) + \alpha(A)(x) - T_M(A(x)))$$

One can even require that on top of it, the viability multiplier satisfies

$$q(t)\ \in\ N_M(A(t)x(t)) \cap R_M(A(t), x(t)))$$

Dynamical Correction Index and Heavy Evolutions

The norm $\|q(t)\|$ of the viability multiplier $q(t)$ measures the intensity of the viability discrepancy of the dynamic since

$$\begin{cases} (i) & \|c(x(t)) - x'(t)\| \;\leq\; \|A^*(t)\| \, \|q(t)\| \\ (ii) & \|\alpha(A(t)) - A'(t)\| \;=\; \|x(t)\| \, \|q(t)\| \end{cases}$$

The viability multipliers of minimal norm in the regulation map provide the smallest velocities of the corrections, and thus, heavy evolutions.

Hebbian Rules

Differential equation

$$A'(t) \ = \ \alpha(A(t)) - x(t) \otimes q(t)$$

involves the Hebbian rule **proposed by Hebb in his** The organization of behavior **in 1949 as the basic learning process of synaptic weight:**

Taking $\alpha(A) = 0$, the evolution of the synaptic matrix $A := (a_i^j)$ **obeys the differential equation**

$$\frac{d}{dt} a_i^j(t) \ = \ -x_i(t) q^j(t)$$

The velocity of the synaptic weight is the product of the presynaptic activity and the postsynaptic activity**.**

Observe that in this case

$$\|A'(t)\| \ = \ \|x(t)\| \|q(t)\|$$

The inertia of the connection matrix, which can be regarded as a index of dynamic connectionist complexity, is proportional to the norm of the viability multiplier.

Introducing Another Factor

We introduce now a scalar $\chi(t) \in [0,1]$ aimed at "intensify" the resource $x(t)$ regarded as a potential action **evolving under**

$$\chi'(t) \;=\; \kappa(\chi(t))$$

The constraint becomes

$$\forall\, t \geq 0,\;\; A(t)\chi(t)x(t) \;\in\; M$$

They are viable under

$$\begin{cases} (i) & x'(t) \;=\; c(x(t)) - \chi(t)A^{*}(t)q(t) \\ (ii) & \chi'(t) \;=\; \kappa(\chi(t)) - \langle q(t), A(t)x(t)\rangle \\ (iii) & A'(t) \;=\; \alpha(A(t)) - \chi(t)x(t) \otimes q(t) \end{cases}$$

which involves viability multipliers $q(t) \in Y^{\star}$ which satisfy the regulation law

$$\forall\, t \geq 0,\;\; q(t) \;\in\; R_M(A(t), x(t), \chi(t))$$

where the regulation map R_M is defined by

$$R_M(A,x,\chi) \; = \; (\chi A A^\star + \chi \|x\|^2 \mathsf{I} + Ax \otimes Ax)^{-1}(Ac(x) + \alpha(A)(x) + \kappa(\chi) Ax - T_M(A(x)))$$

One can even require that on top of it, the viability multiplier satisfies

$$q(t) \; \in \; N_M(\chi(t)A(t)x(t)) \cap R_M(A(t), x(t), \chi(t))$$

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Example: Multi-stage Production Processes

The constraints are of the form

$$A_{\mathbb{H}}^{\mathbb{H}-1}\cdots A_h^{h-1}\ldots A_2^1 x_1 \ \in \ M_{\mathbb{H}}$$

This describes for instance a production process associating with the resource x_1 the intermediate outputs $x_2 := A_2^1 x_1$, which itself produces an output $x_3 := A_2^1 x_2$, and so on, until the final output $x_{\mathbb{H}} := A_{\mathbb{H}}^{\mathbb{H}-1}\cdots A_h^{h-1}\ldots A_2^1 x_1$ which must belong to the production set $M_{\mathbb{H}}$.

The evolution without constraints of the commodities and the operators is governed by dynamical systems of the form

$$\left\{\begin{array}{ll} (i) & x_h'(t) \ = \ c_h(x_h(t)) \\ & \\ (ii) & \dfrac{d}{dt}A_{h+1}^h(t) \ = \ \alpha_{h+1}^h(A_h(t)) \end{array}\right.$$

Hierarchical System

The constraints

$$\forall\, t \geq 0,\ \ A_{\mathbb{H}}^{\mathbb{H}-1}(t)\cdots A_{h}^{h-1}(t)\ldots A_{2}^{1}(t)x_1(t)\ \in\ M_{\mathbb{H}}$$

are viable under the system

$$\left\{\begin{array}{lll}
x_1'(t) \ = \ c_1(x_1(t)) + A_2^1(t)^{\star}(t)p^1(t) & & (h=1)\\
x_h'(t) \ = \ c_h(x_h(t)) - p^{h-1}(t) + A_{h+1}^{h}(t)^{\star}p^h(t) & & (h=1,\ldots,\mathbb{H}-1)\\
x_{\mathbb{H}}'(t) \ = \ c_{\mathbb{H}}(x_{\mathbb{H}}(t)) - p^{\mathbb{H}-1}(t) & & (h=\mathbb{H})\\
\dfrac{d}{dt}A_{h+1}^{h}(t) \ = \ \alpha_{h+1}^{h}(A_h(t)) + x_h(t)\otimes p^h(t) & & (h=1,\ldots,\mathbb{H}-1)
\end{array}\right.$$

involving viability multipliers $p^h(t)$ (intermediate "shadow price"). The input-output matrices $A_{h+1}^{h}(t)$ obey dynamics involving the tensor product of $x_h(t)$ and $p^h(t)$.

Structural Topologies

We assume that $X := \prod_{h=1}^{\mathbb{H}}$ and $Y := \prod_{k=1}^{\mathbb{K}}$ and that $A := (A_h^k)$ where $A_h^k \in \mathcal{L}(X_k, Y_h)$. We introduce a set-valued map $J : \{1, \ldots, \mathbb{H}\} \rightsquigarrow \{1, \ldots, \mathbb{K}\}$.

The constraints are defined by

$$\forall\ h = 1, \ldots, \mathbb{H},\ \sum_{k \in J(h)} A_h^k(t) x_k(t) \ \in\ M_h\ \subset\ Y_h$$

We consider a system of differential equations

$$\left\{ \begin{array}{ll} (i) & x_h'(t) \ = \ c_h(x_h(t)),\ \ h = 1, \ldots, \mathbb{H} \\ \\ (ii) & \dfrac{d}{dt} A_h^k(t) \ = \ \alpha_h^k(A_h^k(t)) \end{array} \right.$$

The Corrected System

Then the constraints

$$\forall\, h = 1, \ldots, \mathbb{H}, \ldots \sum_{k \in J(h)} A_h^k(t) x_k(t) \;\in\; M_h \;\subset\; Y_h$$

are viable under the corrected system

$$\begin{cases} (i) \quad x_h'(t) \;=\; c_h(x_h(t)) - \sum_{k \in J^{-1}(h)} A_k^h(t)^\star p^k, \quad h = 1, \ldots, \mathbb{H},\; k = 1, \ldots, \mathbb{K} \\ \\ (ii) \quad \dfrac{d}{dt} A_h^k(t) \;=\; \alpha_h^k(A_h^k(t)) - x_k(t) \otimes p^h(t), \; (h,k) \in \mathrm{Graph}(J) \end{cases}$$

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Bilinear Case

When $X := X_1 \times X_2$**, constraints take the form:** $\forall\, t \geq 0$

$$A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) \in \; M$$

where $A_i \in \mathcal{L}(X_i, Y)$ **(**$i = 1, 2$**) and** $A_{\{1,2\}} \in \mathcal{L}_2(X_1 \times X_2, Y)$ **is bilinear.**

We set

$$A_{\{1,2\}}(x_i)x_j := A_{\{1,2\}}(x_i, x_j),\; i = 1, 2,\; j = 2, 1$$

We start with the system

$$\begin{cases} (i) & x_i'(t) \;=\; c_i(x(t)),\; i = 1, 2 \\ (ii) & A_i'(t) \;=\; \alpha_i(A_i(t)),\; i = 1, 2 \\ (iii) & A_{\{1,2\}}'(t) \;=\; \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) \end{cases}$$

Connectionist Tensor

The **tensor product** $x_1 \otimes x_2 \otimes q$ **is a bilinear operator from** $X_1^* \times X_2^*$ **to** Y^* **associating with any pair** $(p_1, p_2) \in X_1^* \times X_2^*$ **the element**

$$(x_1 \otimes x_2 \otimes q)(p_1, p_2) \;:=\; \langle p_1, x_1\rangle\langle p_2, x_2\rangle q$$

The components of this bilinear form — the "tensors" — can be written

$$(x_1 \otimes x_2 \otimes q)^j_{i_1,i_2} \;=\; x_{1_{i_1}} x_{2_{i_2}} q^j$$

as the products of the components of the three factors of this tensor product.

The Associated Control System

We correct the above system by the control system

$$\left\{\begin{array}{llll} (i) & x_1'(t) & = & c_1(x(t)) - A_1(t)^\star q(t) - A_{\{1,2\}}(t)(x_2(t))^\star q(t) \\ (ii) & x_2'(t) & = & c_2(x(t)) - A_2(t)^\star q(t) - A_{\{1,2\}}(t)(x_1(t))^\star q(t) \\ (iii) & A_1'(t) & = & \alpha_1(A_1(t)) - x_1(t) \otimes q(t) \\ (iv) & A_2'(t) & = & \alpha_2(A_2(t)) - x_2(t) \otimes q(t) \\ (vi) & A_{\{1,2\}}'(t) & = & \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) - x_1(t) \otimes x_2(t) \otimes q(t) \end{array}\right.$$

so that the constraints

$$A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t) \in \ M$$

are viable under the corrected system.

We may require that the viability multiplier $q(t)$ **belongs to normal cone to** M**:**

$$q(t) \ \in \ N_M(A_{\{1,2\}}(t)(x_1(t), x_2(t)) + A_1(t)x_1(t) + A_2(t)x_2(t))$$

Comments

The structure of this control system involves

1. **the transposes** $A_i^*(t)q(t)$ **and**
$$A_{\{1,2\}}(t)(x_j(t))^*(t)q(t)$$
$(i = 1, 2)$ **in the evolution of the variables** $x_i(t)$**,**
2. **the tensor products** $x_i(t) \otimes q(t)$ **(Hebbian rules) in the evolution of the linear operators** $A_i(t)$**,**
3. **the tensor product** $x_1(t) \otimes x_2(t) \otimes q(t)$ **in the evolution of the bilinear form** $A_{\{1,2\}}$**.**

Taking $\alpha_{1,2}(A) = 0$**, the evolution of** $A_{\{1,2\}} := (a^j_{i_1,i_2})$ **obeys**

$$\frac{d}{dt} a^j_{i_1,i_2}(t) \;= -\, x_{1_{i_1}}(t) x_{2_{i_2}}(t) q^j(t)$$

stating that the velocity of the synaptic tensor is the product of the presynaptic activities of the neurons arriving at the synapse (i_1, i_2, j) **and the postsynaptic activity.**

We introduce coefficients $\chi_i(t) \in [0,1]$ **(**$i = 1,2$**) evolving under**

$$\chi_i'(t) \;=\; \kappa_i(\chi_i(t)), \;\; i = 1,2$$

The constraint becomes: $\forall\, t \geq 0$,

$$\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t),x_2(t))$$

$$+\chi_1(t)A_1(t)x_1(t) + \chi_2(t)A_2(t)x_2(t) \;\in\; M$$

They are viable under the control system

$$
\left\{
\begin{array}{llcl}
(i) & x_1'(t) & = & c_1(x(t)) - \chi_1(t)A_1(t)^*q(t) - \chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_2(t))^*q(t)\\
(ii) & x_2'(t) & = & c_2(x(t)) - \chi_2(t)A_2(t)^*q(t) - \chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t))^*q(t)\\
(iii) & \chi_1'(t) & = & \kappa_1(\chi_1(t)) - \langle q(t), A_1(t)x_1(t)\rangle - \chi_2(t)\langle q(t), A_{\{1,2\}}(t)(x_1(t), x_2(t))\rangle\\
(iv) & \chi_2'(t) & = & \kappa_2(\chi_2(t)) - \langle q(t), A_2(t)x_2(t)\rangle - \chi_1(t)\langle q(t), A_{\{1,2\}}(t)(x_1(t), x_2(t))\rangle\\
(v) & A_1'(t) & = & \alpha_1(A_1(t)) - \chi_1(t)x_1(t)\otimes q(t)\\
(vi) & A_2'(t) & = & \alpha_2(A_2(t)) - \chi_2(t)x_2(t)\otimes q(t)\\
(vii) & A_{\{1,2\}}'(t) & = & \alpha_{\{1,2\}}(A_{\{1,2\}}(t)) - \chi_1(t)\chi_2(t)x_1(t)\otimes x_2(t)\otimes q(t)
\end{array}
\right.
$$

We may require that the viability multiplier $q(t)$ ranges over the normal cone

$$q(t) \;\in\; N_M(\chi_1(t)\chi_2(t)A_{\{1,2\}}(t)(x_1(t), x_2(t))\chi_1(t)A_1(t)x_1(t) + \chi_2(t)A_2(t)x_2(t))$$

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Introducing Networks

The coordination problem in a dynamic environment **involves the evolution**

1. **of actions** $x(t) := (x_1(t), \dots, x_n(t)) \in \prod_{i=1}^n X_i$**,**
2. **of connectionist operators**

$$A_{S(t)}(t) : \prod_{i=1}^{n} X_i \mapsto Y$$

3. **acting on** coalitions $S(t) \subset N := \{1, \dots, n\}$ **of the** n **actors**

and requires that

$$\forall\, t \geq 0, \;\; g\left(\{A_S(t)(x(t))\}_{S \subset N}\right) \;\in\; M$$

where $g : \prod_{S \subset N} Y_S \mapsto Y$ **is a** propagation rule**.**

An Economic Interpretation

Actors here are economic agents (producers) $i = 1, \ldots, n$ ranging over the set $N := \{1, \ldots, n\}$. Each coalition $S \subset N$ of economic agents is regarded as a production unit (a firm) using resources of their agents to produce (or not produce) commodities. Each agent $i \in N$ provides a resource vector (capital, competencies, etc.) $x_i \in X$ in a resource space $X_i := \mathbb{R}^{m_i}$ used in production processes involving coalitions $S \subset N$ of economic agents (regarded as firms employing economic agents)

We describe the production process of a firm $S \subset N$ by an operator $A_S : \prod_{i=1}^{n} X_i \mapsto Y$ operating on the resources $x := (x_1, \ldots, x_n)$ provided by the economic agents to the production units described by coalitions of economic agents for producing a commodity $A_S(x)$.

The supply constraints are described by a subset $M \subset Y$ of the commodity space, representing the set of commodities that must be produced by the firms: Condition

$$\sum_{S \subset N} A_S(t)(x(t)) \in M$$

express that at each instant, the total production must belong to M.

It defines the architecture of the corresponding network requiring that at each instant, agents supply adequate resources to the firms in order that the production objectives are fulfilled.

Question Raised

Assume that we may know the intrinsic (independently of the constraints) laws of evolution of the variables x_i, of the connectionist operator $A_S(t)$ and of the coalitions $S(t)$.

There is no reason why collective constraints defining the above architecture are viable **under these dynamics, i.e, satisfied at each instant.**

We use the concept of "viability multipliers" $q(t) \in Y^*$ **that can be used as** "controls" or "regulons" **involved for modifying the initial dynamics.**

We design classes of dynamics "correcting" the initial (intrinsic) ones in such a way that the viability of the constraints is guaranteed. This may allow us to explain the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves.

Fuzzy Coalitions

We shall

1. **restrict the connectionist operators to be** multiaffine, **(which involve** tensor products),
2. **next, allow coalitions S to become** fuzzy coalitions **so that they can evolve continuously.**

Fuzzy coalitions $\chi = (\chi_1, \ldots, \chi_n)$ **are defined by memberships $\chi_i \in [0, 1]$ between 0 and 1, instead of being equal to either 0 or 1 as in the case of usual coalitions.**

Player i participates fully in χ if $\chi_i = 1$, does not participate at all if $\chi_i = 0$ and participates in a fuzzy way if $\chi_i \in]0, 1[$. We associate with a fuzzy coalition χ the set $P(\chi) := \{i \in N \mid \chi_i \neq 0\} \subset N$ of agents i participating to the fuzzy coalition χ.

Using fuzzy coalitions allows us to define their velocities and study their evolution.

Fuzzy Coalitions

We embed the family of subsets of a (discrete) set N of n players to the space R^n through the map χ associating with any coalition $S \in \mathcal{P}(N)$ its characteristic function $\chi_S \in \{0,1\}^n \subset \mathsf{R}^n$, since R^n can be regarded as the set of functions from N to R.

By definition, the family of fuzzy sets is the convex hull $[0,1]^n$ of the power set $\{0,1\}^n$ in R^n. Therefore, we can write any fuzzy set in the form

$$\chi = \sum_{S \in \mathcal{P}(N)} m_S \chi_S \text{ where } m_S \geq 0 \;\&\; \sum_{S \in \mathcal{P}(N)} m_S = 1$$

The memberships are then equal to

$$\forall\, i \in N, \;\; \chi_i \; = \sum_{S \ni i} m_S$$

Memberships of Coalitions

We also introduce the membership

$$\gamma_S(\chi) := \prod_{j \in S} \chi_j$$

of a coalition S in the fuzzy coalition χ as the product of the memberships of agents i of the coalition S.

It vanishes whenever one the membership of one agent does and boils down to individual memberships for one agent coalitions. When two coalitions are disjoint ($S \cap T = \emptyset$), then $\gamma_{S \cup T}(\chi) = \gamma_S(\chi)\gamma_T(\chi)$.

In particular, for any agent $i \in S$, $\gamma_S(\chi) = \chi_i \gamma_{S \setminus i}(\chi)$.

L. A. Zadeh

introduced in 1965 the concept of fuzzy set.

Since then, it has been wildly successful, even in many areas outside mathematics!. Lately, we found in "La lutte finale", Michel Lafon (1994), p.69 by A. Bercoff the following quotation of the late François Mitterand, president of the French Republic (1981-1995):

"Aujourd'hui, nous nageons dans la poésie pure des sous ensembles flous" ...

(Today, we swim in the pure poetry of fuzzy subsets)

Characteristic Functions of Coalitions

We associate with any coalition $S \subset N$ **its characteristic function** $\chi_S : N \mapsto R$ **associating with any** $i \in N$

$$\chi_S(i) \ := \begin{cases} 1 & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S \end{cases}$$

and we set

$$\forall \ = 1, \ldots, n, \ \ (\chi_S \circ x)_i \ := \begin{cases} x_i & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S \end{cases}$$

Multilinear Operators

We associate with any coalition $S \subset N$ the space $\mathcal{L}_S(X^S, Y)$ of S-linear operators A_S. We extend such a S-linear operator A_S to a n-linear operator $A_S \in \mathcal{L}_n\left(\prod_{i=1}^n X_i, Y\right)$ defined by: $\forall\, x \in \prod_{i=1}^n X_i$,

$$A_S(x) \;=\; A_S(x_1, \ldots, x_n) \;:=\; A_S(\chi_S \circ x)$$

A multiaffine **operator $A \in \mathcal{A}_n\left(\prod_{i=1}^n X_i, Y\right)$ is a sum of S-linear operators $A_S \in \mathcal{L}_S(X^S, Y)$ when S ranges over the family of coalitions:**

$$A(x_1, \ldots, x_n) \;:=\; \sum_{S \subset N} A_S(\chi_S \circ x) \;=\; \sum_{S \subset N} A_S(x)$$

For any $i \in S$, we denote by $(x_{-i}, u_i) \in X^N$ the sequence $y \in X^N$ where $y_j := x_j$ when $j \neq i$ and $y_i = u_i$ when $j = i$.

We denote by $A_S\,(x_{-i}) \in \mathcal{L}(X_i, Y)$ the linear operator defined by $u_i \mapsto A_S(x_{-i})u_i := A_S(x_{-i}, u_i)$ and its transpose $A_S\,(x_{-i})^* \in \mathcal{L}(Y^*, X_i^*$ defined by

$$\forall\, q \in Y^*,\ \forall\, u_i \in X_i,$$

$$\langle A_S\left(x_{-i}\right)^* q, u_i\rangle \;=\; \langle q, A_S\left(x_{-i}\right) u_i\rangle$$

Tensor Products

We associate with $q \in Y^*$ and elements $x_i \in X_i$ the multilinear operator

$$x_1 \otimes \cdots \otimes x_n \otimes q \ \in \ \mathcal{L}_n\left(\prod_{i=1}^{n} X_i^*, Y^*\right)$$

associating with any $p := (p_1, \ldots, p_n) \in \prod_{i=1}^n X_i^*$ the element :

$$(x_1 \otimes \cdots \otimes x_n \otimes q)(p) := \left(\prod_{i=1}^{n} \langle p_i, x_i \rangle\right) q \in Y^*$$

This multilinear operator $x_1 \otimes \cdots \otimes x_n \otimes q$ is called the tensor product **of the x_i's and q.**

The Corrected System

Using viability multipliers, **we can modify**

$$\begin{cases} i) & x_i'(t) = c_i(x(t)),\ i = 1, \dots, n \\ ii) & A_S'(t) = \alpha_S(A(t)),\ S \subset N \end{cases}$$

for the constraints

$$\forall\, t \geq 0,\ \sum_{S \subset N} A_S(t)(x(t)) \ \in\ M$$

to be viable under the control system

$$\begin{cases} (i) & x_i'(t) = c_i(x_i(t)) - \sum_{S \ni i} A_S(t)(x_{-i}(t))^* q(t),\ i = 1, \dots, n \\ \\ (ii) & \forall\, S \subset N,\ A_S'(t) = \alpha_S(A(t)) - \left(\bigotimes_{j \in S} x_j(t) \right) \otimes q(t) \end{cases}$$

Multi-Hebbian Rule

When we regard the multilinear operator A_S **as a tensor product of components**

$$A^j_{S_{\Pi_{i\in S} i_k}},\ j=1,\ldots,p,\ i_k=1,\ldots,n_i,\ i\in S$$

differential equation ii) can be written in the form: $\forall\ i\in S,\ j=1,\ldots,p,\ k=1,\ldots,n_i$**,**

$$\begin{array}{rcl}\frac{d}{dt}A^j_{S_{\Pi_{i\in S} i_k}} & = & \alpha_{S_{\Pi_{i\in S} i_k}}(A(t)) \\ & & -\left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)\end{array}$$

The correction term of the component $A^j_{S_{\Pi_{i\in S} i_k}}$ **of the** S**-linear operator is the product of the components** $x_{i_k}(t)$ **actions** x_i **in the coalition** S **and of the component** q^j **of the viability multiplier.**

This can be regarded as a multi-Hebbian rule in neural network learning algorithms.

Outline

Evolving Fuzzy Coalitions

For encapsulating the idea that at each instant, only a number of fuzzy coalitions χ are active, we introduce the following evolving architecture: $\forall\, t \geq 0$,

$$\sum_{S \subset N} A_S(t)(\chi(t) \circ x(t))$$

$$\sum_{S \subset N} \left(\prod_{j \in S} \chi_j(t) \right) A_S(t)(x(t)) \;\in\; M$$

Assume that the intrinsic evolution of fuzzy coalitions is governed by

$$\chi_i'(t) \;=\; \kappa_i(\chi(t)),\; i = 1, \ldots, n$$

The Corrected Control System

Assume that the functions c_i, κ_i and α_S are continuous and that $M \subset Y$ are closed.

Recall that

$$\gamma_S(\chi) \;:=\; \prod_{j \in S} \chi_j$$

is the membership **of a coalition S in the fuzzy coalition χ.**

Then the constraints

$$\sum_{S \subset N} (\gamma_S(\chi(t)))\, A_S(t)(x(t)) \;\in\; M$$

are viable under the control system

$$\left\{\begin{array}{llll}
(i) & x_i'(t) & = & c_i(x_i(t)) - \displaystyle\sum_{S\ni i} (\gamma_S(\chi(t)))\, A_S(t)(x_{-i}(t))^* q(t),\ i = 1, \ldots, n \\
(ii) & \chi_i'(t) & = & \kappa_i(\chi(t)) - \displaystyle\sum_{S\ni i} \left(\gamma_{S\setminus i}(\chi(t))\right) \langle q(t), A_S(t)\,(x(t))\rangle\,,\ i = 1, \ldots, n \\
(iii) & A_S'(t) & = & \alpha_S(A(t)) - \gamma_S(\chi(t)) \left(\displaystyle\bigotimes_{j\in S} x_j(t)\right) \otimes q(t),\ S \subset N
\end{array}\right.$$

Comments

For each agent i, the velocities $x'_i(t)$ of the state and the velocities $\chi'_i(t)$ of its membership in the fuzzy coalition $\chi(t)$ are corrected by subtracting

1. **the sum over all coalitions S to which he belongs of the $A_S(t)(x_{-i}(t))^* q(t)$ weighted by the membership $\gamma_S(\chi(t))$:**

$$x'_i(t) \;=\; c_i(x_i(t)) - \sum_{S \ni i} \gamma_S(\chi(t)) A_S(t)(x_{-i}(t))^* q(t)$$

2. **the sum over all coalitions S to which he belongs of the costs $\langle q(t), A_S(t)\,(x(t))\rangle$ of the constraints associated with connectionist tensor A_S of the coalition S weighted by the membership $\gamma_{S\setminus i}(\chi(t))$:**

$$\chi'_i(t) \;=\; \kappa_i(\chi(t)) - \sum_{S \ni i} \gamma_{S\setminus i}(\chi(t))\,\langle q(t), A_S(t)\,(x(t))\rangle$$

Herd Behavior

This type of dynamics describes a mimetic or herd behavior or panurgean effect (from a famous story by François Rabelais (1483-1553), where Panurge sent overboard the head sheep, followed by the whole herd).

The (algebraic) increase of agent i's membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the agents of the coalition other than him.

The correction of the velocities of the connectionist tensors A_S is a weighted "multi-Hebbian" rule: for each component $A^j_{S_{\Pi_{i\in S} i_k}}$ of A_S, the correction term is the product of the membership $\gamma_S(\chi(t))$ of the coalition S, of the components $x_{i_k}(t)$ and of the component $q^j(t)$ of the regulon:

$$\frac{d}{dt}A^j_{S_{\Pi_{i\in S} i_k}} = \alpha_{S_{\Pi_{i\in S} i_k}}(A(t)) - \gamma_S(\chi(t))\left(\prod_{i\in S} x_{i_k}(t)\right) q^j(t)$$

Viability and Lagrange Multipliers

If we minimize $\sum_{i=1}^{n} \mathbf{u}_i(x_i) + \sum_{i=1}^{n} \mathbf{v}_i(\chi_i) + \sum_{S \subset N} \mathbf{w}_S(A_S)$ **under constraints**

$$\sum_{S \subset N} \gamma_S(\chi) A_S(x) \ \in \ M$$

then optimality conditions imply the existence of Lagrange multipliers p **such that:**

$$\begin{cases} \nabla \mathbf{u}_i(x_i) \ = \ \sum_{S \ni i} \gamma_S(\chi) A_S(x_{-i})^* p, \ i = 1, \ldots, n \\ \\ \nabla \mathbf{v}_i(\chi_i) \ = \ \sum_{S \ni i} \gamma_{S \setminus i}(\chi) \left\langle p, A_S\left(x\right)\right\rangle, \ i = 1, \ldots, n \\ \\ \nabla \mathbf{w}_S(A_S) \ = \ \gamma_S(\chi(t)) \left(\bigotimes_{j \in S} x_j \right) \otimes p, \ S \subset N \end{cases}$$

Outline

1. **Linear Case**
2. **Example: Multi-stage Production Processes**
3. **Bilinear Case**
4. **Multilinear Case**
5. **Multilinear Case with Evolving Coalitions**
6. **The Regulation Map**
7. **Reestablishing Viability by Viability Multipliers**

The Regulation Map

$$\begin{cases} \text{Set } h(x,\chi,A) \ := \ \sum_{S\subset N} A_S(\chi\circ x) \\ \text{and } H(x,\chi,A) \ := \ \displaystyle\sum_{S\subset N}\left(\prod_{j\in S}\chi_j^2\|x_j\|^2\right)\mathbf{I} \\ +\displaystyle\sum_{R,S\subset N}\sum_{i\in R\cap S}\left(\gamma_R(\chi)\gamma_S(\chi)A_R(x_{-i})A_S(x_{-i})^* + \gamma_{R\setminus i}(\chi)\gamma_{S\setminus i}(\chi)A_R(x)\otimes A_S(x)\right) \end{cases}$$

Then the regulation map R_M is defined by

$$\begin{cases} R_M(x,\chi,A) \ := \ H(x,\chi,A)^{-1} \\ \left(\displaystyle\sum_{S\subset N}\left(\alpha_S(A)(x) + \sum_{i\in S}\left(\gamma_S(\chi)A_S(x_{-i},c_i(x)) + \gamma_{S\setminus i}(\chi)\kappa_i(\chi)A_S(x)\right)\right) - T_M(h(x,\chi,A))\right) \end{cases}$$

and the viability multipliers $q(t)$obey the regulation law of the form

$$\forall\, t\geq 0,\ \ q(t) \ \in \ R_M(x(t),\chi(t),A(t))$$

(an "adjustment law", in the vocabulary of economists). We can even choose

$$\forall\, t \geq 0, \quad q(t) \;\in\; R_M(x(t), \chi(t), A(t)) \cap N_M\left(\sum_{S \subset N}\left(\prod_{j \in S} \chi_j(t)\right) A_S(t)(x(t))\right)$$

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Reestablishing Viability

Consider the case when a subset K is not viable under a differential equation $x' \in F(x(t))$. The Basic Viability Theorem tells us that when F is a Marchaud map, the necessary and sufficient tangential condition

$$F(x) \cap \overline{co}(T_K(x)) \neq \emptyset$$

is not satisfied for at least one state $x \in K$.

The question arises whether we can correct the dynamics F in such a way that K becomes viable under the new one.

Viability Multipliers

The first idea that jumps to the mind is to introduce viability multipliers $p \in F(x) - \overline{co}(T_K(x))$ measuring the "viability discrepancy" between the velocity $F(x)$ and the (closed convex hull) of the cone tangent directions to K at x.

They are used to correct the differential inclusion $x'(t) \in F(x(t))$ under which K is not viable by replacing it by the control system

$$\begin{cases} (i) & x'(t) \ \in \ F(x(t)) - p(t) \\ (ii) & p(t) \ \in \ P(x(t)) \end{cases} \tag{1}$$

regulated by "viability multipliers" $p(t) \in P(x(t))$ measuring the "viability discrepancy", where $P : X \rightsquigarrow X$ is a set-valued map satisfying the property:

$$\forall \, x \in K, := \ P(x) \cap (F(x) - \overline{co}(T_K(x))) \ \neq \ \emptyset$$

The Viability Theorem implies that when F and P are Marchaud, K is viable under this corrected control system (1).

Examples

The question arises how to find examples of such set-valued maps P.

The first class of example is to take balls $P(x) := c(x)B$ centered at 0 of radius $c(x)$. The function $c(\cdot)$ must be upper semicontinuous for the set-valued map $c(\cdot)B$ to be continuous, larger than or equal to $d(F(x),\overline{co}(T_K(x)))$ in order that

$$\forall\, x \in K,\ \ c(x)B \cap (F(x) - \overline{co}(T_K(x)))\ \neq\ \emptyset$$

and smaller than $d(0,F(x))$ to avoid finding equilibria.

Minimizing Viability Discrepancy

In particular, the question arises whether we can “minimize the viability discrepancy” by singling out the viability multiplier $p^\circ(x)$ of minimal norm:

$$\|p^\circ(x)\| \;=\; \inf_{v\in F(x),\; u\in\overline{co}(T_K(x))} \|v-u\|$$

which exist whenever $F(x)$ is convex and compact thanks to the Projection Theorem : there exist $v^\circ(x) \in F(x)$ and $u^\circ(x) \in T_K(x)$ such that $p^\circ(x) = v^\circ(x) - u^\circ(x)$, where

$$u^\circ(x) \;:=\; \Pi_{\overline{co}(T_K(x))}(v^\circ(x))$$

is **the projection of the velocity $v^\circ(x)$ onto the (closed convex hull of) the tangent cone to K at x.**

The solutions to the control system

$$x'(t) \;\in\; F(x(t)) - p^\circ(x(t)) \tag{2}$$

are called **slow evolutions** of the corrected system (1).

Variational Inequalities

Even though control system (2) was built to meet the tangential conditions of the Viability Theorem, we observe that it loses the minimal continuity requirements sufficient (but not necessary) for guaranteeing the existence of a solution. However, when K is closed and convex and when F is Marchaud and lower semicontinuous, one can prove that *slow solutions do exist.*

This is the case in particular when $F(x) := \{f(x)\}$ is a continuous single-valued map. We observe that control system (2) can be written in the form of “variational inequalities”:

$$\forall\, y \in K, \;\; \langle x'(t) - f(x(t)), x(t) - y \rangle \;\leq\; 0 \tag{3}$$

Explicit Constraints

If the constrained set K is of the form

$$K \;:=\; \{x \in X \quad \textbf{such that} \;\; h(x) \in M\}$$

where $h : X \mapsto Y := \mathbb{R}^m$ and $M \subset Y$, the tangent cone $T_K(x)$ can be couched under adequate assumptions in terms of the tangent cone $T_M(h(x))$:

$$T_K(x) \;=\; h'(x)^{-1} T_M(h(x))$$

In this case, we can measure the viability discrepancy by viability multipliers

$$q \in (h'(x)h'(x)^\star)^{-1}(h'(x)F(x) - T_M(h(x)))$$

instead of viability multipliers $p \in F(x) - h'(x)^{-1}T_M(h(x))$ and correct the differential inclusion $x'(t) \in F(x(t))$ by replacing it by the control system

$$x'(t) \;\in\; F(x(t)) - h'(x(t))^\star q(t)$$

regulated by viability multipliers **$q(t) \in Y^\star$, which play a role analogue to Lagrange multipliers in optimization under constraints.**

Lagrange and Viability Multipliers

When $X := \mathbb{R}^n$, $Y := \mathbb{R}^m$ and $h(x) := (h_1(x), \ldots, h_m(x))$, this control system can be written more explicitly under the form

$$x_j'(t) = f_j(x(t)) + \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} q_k(t), \quad j = 1, \ldots, m$$

We recall that the minimization of a function $x \mapsto J(x)$ over K is equivalent to the minimization without constraints of the function

$$x \mapsto J(x) + \sum_{k=1}^{m} \frac{\partial h_k(x)}{\partial x_j} q_k$$

for an appropriate **“Lagrange multiplier”** $q \in Y^\star$.

These Lagrange and viability multipliers associated to constraints of the form $h(x) \in M$ share the same interpretations. In economics, for instance, Lagrange are often regarded as (shadow) prices in optimization or equilibrium models. Therefore, viability multipliers enjoy the same status as prices, slowing down consumption to meet scarcity constraints, boosting production to guarantee given outputs, etc.