THE VIABILITY THEOREM FOR STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT

The aim of this paper is to combine two ways for representing uncertainty through stochastic differential inclusions: a "stochastic uncertainty", driven by a Wiener process, and a "contingent uncertainty", driven by a set-valued map, as well as to consider stochastic control problems with continuous dynamic and state dependent controls.

This paper is also devoted to viability of a closed subset under stochastic differential inclusions, characterized in terms of stochastic tangent sets to closed subsets.

1 Introduction

The first aim of this paper is to combine two ways for representing uncertainty through stochastic differential inclusions: a "stochastic uncertainty", driven by a Wiener process, and a "contingent uncertainty", driven by a set-valued map [2] as well as to consider stochastic control problems with continuous dynamic and state dependent controls.

The second objective is to extend to stochastic differential inclusions the Viability Theorem for nonstochastic differential inclusions (see [7, Aubin] for an exhaustive presentation, historical comments and a bibliography). In [3, Aubin & Da Prato], the Nagumo Theorem was extended to stochastic differential equations with Lipschitz or monotone dynamics. Here, we extend this theorem not only for continuous single-

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[2] which can even be a "fuzzy set-valued map" as in [5, Aubin & Dordan].

valued drift and diffusion coefficients, but also to the case of what can be called stochastic differential inclusions.

Let us set $X := \mathbf{R}^n$, $Y := \mathbf{R}^m$, and let us consider a closed subset K of X and a stochastic differential inclusion

$$d\xi \;\in\; F(\xi(t))dt + g(\xi(t))dW(t),$$

where F is a set–valued maps vith values in X, and g is a map in $\mathcal{L}(Y, X)$ respectively, and W is an Y–valued Wiener process. It describes the contribution of two kinds of randomness: a contingent one, represented by a set–valued drift, and a stochastic one described by a diffusion.

More generally we shall consider the case of control problems of the form

$$i)\;\; d\xi \;=\; f(\xi(t), u(t))dt + g(\xi(t), u(t))dW(t)$$

$$ii)\;\; \text{for almost all } (\omega, t) \in \Omega \times [0, 1], \;\; u_\omega(t) \;\in\; U(x_\omega(t)),$$

where the constraints in the controls depend upon the states.

They fit to the following framework. If H is a set–valued map from X to $X \times \mathcal{L}(Y, X)$, we introduce stochastic differential inclusions of the form

$$d\xi \in H(\xi, dt \oplus dW(t)).$$

We look for "weak solutions" $\xi(\cdot)$ in the sense of [14, Stroock & Varadhan] (also known under the name of "martingale solutions" in the sense of [11, Da Prato & Zabczyk]): there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbf{P}})$ with a filtration $\{\widehat{\mathcal{F}}_t\}$ and a Wiener process $\widehat{W}$ such that

$$\widehat{\xi}(t) \;=\; \widehat{\xi}(0) + \int_0^t \widehat{u}(s)ds + \int_0^t \widehat{v}(s))d\widehat{W}(s)$$

where for almost all $(\omega, t) \in \widehat{\Omega} \times [0, \infty[$,

$$(\widehat{u}_\omega(t), \widehat{v}_\omega(t)) \in H(\widehat{\xi}_\omega(t)).$$

We also want to prove the (stochastic) viability property of K with respect to H: from any initial random variable x in K, starts at least one solution $\xi(\cdot)$ to the stochastic differential inclusion which is viable in K, in the sense that

$$\forall\, t \in [0, \infty[, \;\; \text{for almost all } \omega \in \Omega, \;\; \xi_\omega(t) \;\in\; K$$

whenever the stochastic dynamics H satisfy adequate tangential conditions. For that purpose, we use the concept of stochastic contingent set $\mathcal{T}_K(x)$ introduced in [2, Aubin & Da Prato].

Then we shall prove in essence that subset K enjoys the viability property with respect to H, whenever

$$H(x) \cap \mathcal{T}_K(x) \neq \emptyset,$$

for any $x \in K$.

For instance, this condition means that for every $\mathcal{F}_0$-random variable x viable in K

- when K is a vector subspace,

$$\forall x \in H, \ H(x) \cap (K \times K) \neq \emptyset$$

- There exists $(u, v) \in H(x)$ such that

$$vx \ = \ 0 \ \& \ \ \langle x, u \rangle + \frac{1}{2} \, \|v\|^2 \ = \ 0$$

  when K is the unit sphere.

- There exists $(u, v) \in H(x)$ such that

$$vx \ = \ 0 \ \& \ \ \langle x, u \rangle + \frac{1}{2} \, \|v\|^2 \ \leq \ 0$$

  when K is the unit ball.

Here the norm $\| \cdot \|$ is the Hilbert–Schmidt norm, that is

$$\|v\|^2 \ = \ \mathrm{Tr} \ [vv^*] \, .$$

We mention that an elementary calculus of stochastic tangent sets to direct images, inverse images and intersections of closed subsets can be found in [2, Aubin & Da Prato].

Let us conclude this introduction by mentioning that the “dual” concept of invariance, requiring that all solutions remain viable in K follows from [2, Aubin & Da Prato] and [10, Da Prato& Frankowska]: under adequate lipschitzianity assumptions the invariance follows from the comdition

$$H(x) \subset \mathcal{T}_K(x)$$

2 Stochastic Tangent Sets

Let us consider a σ–complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, an increasing family of σ–sub–algebras $\mathcal{F}_t \subset \mathcal{F}$ and two finite dimensional vector-spaces $X := \mathbf{R}^n, \ Y := \mathbf{R}^m$.

The constraints are defined by a closed subset $K \subset X$. We denote by $\mathcal{K}$ the subset

$$\mathcal{K} \ := \ \{u \in L^2(\Omega, \mathcal{F}, P) \mid \text{for almost all } \omega \in \Omega, \ \ u_\omega \in K\}$$

Definition 2.1 (Stochastic Contingent Set) *Let us consider a $\mathcal{F}_\tau$-measurable random variable $x \in \mathcal{K}$. We define the* stochastic contingent set $\mathcal{T}_K(x)$ *to* K *at* x *as the set of pairs $(\gamma, v) \in \mathbf{R}^n \times \mathcal{L}(Y, X)$ satisfying the following property: For any $\alpha, \rho > 0$, there exist $h \in]0, \alpha[$ and $\mathcal{F}_{\tau+t}$-measurable random variables a^h and b^h with values in $\mathbf{R}^n$, such that*

$$\begin{array}{ll} i) & \mathbf{E}(\|a^h\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^h\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^h) \ = \ 0 \\ iv) & b^h \ \text{is independent of} \ \mathcal{F}_\tau \end{array}$$

and satisfying

$$x + h\gamma + v(W(\tau + h) - W(\tau)) + ha^h + \sqrt{h}b^h \in \mathcal{K}.$$

Definition 2.2 *We shall say that $H : X \mapsto X \times \mathcal{L}(Y, X)$ is a Marchaud set–valued map if its graph is closed, its images are convex and its growth is linear in the sense that*

$$\forall\, x \in X,\ \forall (u, v) \in H(x),\ \max\left(\|u\|^2,\ \|v\|^2\right) \leq c\left(\|x\|^2 + 1\right) \tag{2.1}$$

We consider the stochastic differential inclusion

$$d\xi \ \in \ H(\xi, dt \oplus dW(t)) \tag{2.2}$$

where H is a Marchaud set–valued map. We say that a stochastic process $\xi(t)$ is a weak or martingale solution to the stochastic differential inclusion (2.2) starting at ξ_0 if there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbf{P}})$ with a filtration $\{\widehat{\mathcal{F}}_t\}$ and a $\{\widehat{\mathcal{F}}_t\}$-Wiener process $\widehat{W}$ such that

$$\begin{array}{ll} i) & \widehat{\xi}(t) \ = \ \widehat{\xi}(0) + \displaystyle\int_0^t \widehat{u}(s)ds + \int_0^t \widehat{v}(s))d\widehat{W}(s) \\ ii) & \text{for almost all } (\omega, t) \in \Omega \times [0, \infty[, \ \ (\widehat{u}_\omega(t), \widehat{v}_\omega(t)) \in H(\widehat{\xi}_\omega(t)) \end{array} \tag{2.3}$$

where $\widehat{\xi}_0$ has the same law than ξ_0.

Definition 2.3 *We shall say that a stochastic process $x(\cdot)$ is* viable in K *if and only if*

$$\forall\, t \geq 0,\ \ x(t) \ \in \ \mathcal{K} \tag{2.4}$$

i.e., if and only if

$$\forall\, t \geq 0,\ \ \text{for almost all } \omega \in \Omega,\ \ \xi_\omega(t) \ \in \ K$$

We shall say that K enjoys the (stochastic) viability property *with respect to H — or, in short, that K is viable under H — if for any $\mathcal{F}_0$-measurable random variable ξ_0 in K starts one $\mathcal{F}_0$-measurable martingale solution $t \mapsto \xi(t)$ to the stochastic differential inclusion (2.2) starting at x which is* viable in K.

3 Stochastic Viability

Theorem 3.1 (Stochastic Viability) *Let K be a closed subset of X. We assume that H is a Marchaud set–valued map. If*

$$\forall x \in K, \quad H(x) \cap \mathcal{T}_K(x) \neq \emptyset,$$

then, from any initial $\mathcal{F}_0$-random variable $\xi_0 \in \mathcal{K}$ starts a martingale solution to the stochastic differential inclusion (2.2) which is viable in $\mathcal{K}$.

It is useful to translate this theorem in terms of stochatic control problems with state-dependent constraints bearing on the controls where the set $H(x) := \{(f(x,u), g(x,u))\}_{u \in U(x)}$ is parametrized by controls:

Theorem 3.2 (Viable Stochastic Control) *Let K be a closed subset of X. We assume that the set-valued map $U : X \mapsto Z$ is Marchaud and that $f : \mathrm{Graph}(U) \mapsto X$ is continuous and affine with respect to u.*

We introduce the feedbak map $x \mapsto \mathcal{R}_K(x)$ associating with every $\mathcal{F}_t$-random variable x in $\mathcal{K}$ the set of $\mathcal{F}_t$-random variable $u \in U(x)$ such that

$$(f(x,u), g(x,u)) \ \in \ \mathcal{T}_{\mathcal{K}}(x)$$

If for every $\mathcal{F}_t$–measurable random variable x in $\mathcal{K}$, the set $\mathcal{R}_K(x)$ is not empty, then, from any initial $\mathcal{F}_0$–measurable random variable $\xi_0 \in \mathcal{K}$ starts a martingale solution to the control problem

$$\begin{aligned} d\xi \ &= \ f(\xi(t), u(t))dt + g(\xi(t), u(t))dW(t) \\ &\text{for almost all } (\omega, t) \in \Omega \times [0, \infty[, \ \ u_\omega(t) \ \in \ U(x_\omega(t)), \end{aligned} \tag{3.1}$$

which is viable in $\mathcal{K}$. The evolution of viable martingale solutions is regulated by

$$\textit{for almost all } (\omega, t) \ \in \Omega \times [0, \infty[, \ \ u_\omega(t) \in \mathcal{R}_K(\xi_\omega(t))$$

Proof of Theorem 3.1 — It is enough to prove the existence of a solution to (2.2) on an arbitrary closed line interval, $[0,1]$ for instance. We set

$$\|u(\cdot)\|_\infty \ := \ \sup_{t \in [0,1]} \|u(t)\| \ \& \ \|\xi(\cdot)\|_\alpha := \sup_{t \neq s} \frac{\|\xi(t) - \xi(s)\|}{|t-s|^\alpha}$$

We shall first provide a priori estimates, second construct approximate solutions to the stochastic differential inclusion and third, prove that they converge to a solution to our problem.

3.1 A priori estimates

We shall prove that the linear growth assumption implies that :

$\forall\, T>0,\ \exists \gamma>0$ such that

$$\max\left\{\sup_{t\in[0,T]}\mathbf{E}(\|\xi(t)\|^2),\ \sup_{t\in[0,T]}\mathbf{E}(\|u(t)\|^2),\ \sup_{t\in[0,T]}\mathbf{E}(\|v(t)\|^2)\right\}\le\gamma$$

For simplicity, set $T=1$. Hence

$$\mathbf{E}(\|\xi(t)\|^2)\le 2\mathbf{E}(\|\xi_0\|^2)+2\mathbf{E}\left(\left\|\int_0^t u(s)ds\right\|^2\right)$$

$$+2\mathbf{E}\left(\left\|\int_0^t v(s)dW(s)\right\|^2\right)+4\mathbf{E}\left(\left\langle\int_0^t u(s)ds,\int_0^t v(s)dW(s)\right\rangle\right)$$

$$\le 2(\mathbf{E}(\|\xi_0\|^2+t\int_0^t\mathbf{E}(\|u(s)\|^2)ds+\int_0^t\mathbf{E}(\|v(s)\|^2)ds)$$

because

$$\mathbf{E}\left(\left\|\int_0^t\varphi(s)ds\right\|^2\right)\le t\int_0^t\mathbf{E}(\|\varphi(s)\|^2)ds, \tag{3.2}$$

and

$$\mathbf{E}\left(\left\|\int_0^t\varphi(s)dW(s)\right\|^2\right)=\int_0^t\mathbf{E}(\|\varphi(s)\|^2)ds \tag{3.3}$$

Using linear growth assumption (2.1), and setting $\varphi(s):=\mathbf{E}(\|\xi(s)\|^2)$, we deduce that

$$\varphi(t)\le(c+1)+2\varphi(0)+2(c+1)\int_0^t\varphi(s)ds.$$

Then the claim follows from the Gronwall lemma. ■

3.2 Construction of Approximate Solutions

We begin by constructing approximate viable solutions to the stochastic differential inclusion.

Let B denote the unit ball of $X\times\mathcal{L}(X,Y)$.

Lemma 3.3 *Let K be a closed subset of X. Assume that H is a bounded set–valued map. Then there exists a constant $\gamma>0$ such that, for any $\varepsilon>0$, the set $\mathcal{S}_\varepsilon(\xi_0)$ of adapted stochastic processes*

$$\xi(t)\ =\ \xi(0)+\int_0^t u(s)ds+\int_0^t v(s)dW(s)$$

on $[0,1]$ *satisfying* $\xi(0)=\xi_0$,

$$\textit{for almost all } (\omega,t)\in\Omega\times[0,1],\ \ (u_\omega(t),v_\omega(t))\ \in\gamma B$$

and

$$\begin{array}{ll} i) & \mathbf{E}\left(\sup_{t\in[0,1]} d^2(\xi(t),\mathcal{K})\right)\leq 2\varepsilon^2 \\ ii) & \mathbf{E}\left(\text{ess sup}_{t\in[0,1]} d^2\left((\xi(t),u(t),v(t)),\text{Graph}(H)\right)\right)\ \leq\ 2\varepsilon^2 \\ iii) & \max\left(\mathbf{E}\left(\|\xi(\cdot)\|_\alpha\right),\ \ \mathbf{E}(\|u(\cdot)\|_\infty),\ \ \mathbf{E}(\|v(\cdot)\|_\infty)\right)\ \leq\ \gamma, \end{array}\tag{3.4}$$

is not empty.

Proof — Let us fix $\varepsilon>0$. We denote by $\mathcal{A}_\varepsilon(\xi_0)$ the set of pairs $(T_\xi,\xi(\cdot))$ where $T_\xi\in[0,1]$ and $\xi(\cdot)$ is a stochastic process satisfying $\xi(0)=\xi_0$ and

$$\begin{array}{ll} i) & \forall\, t\in[0,T_\xi],\ \mathbf{E}d^2(\xi(T_\xi),\mathcal{K})\ \leq\ \varepsilon^2 T_\xi \\ ii) & \mathbf{E}\left(\sup_{t\in[0,T_\xi]} d^2(\xi(t),\mathcal{K})\right)\leq 2\varepsilon^2 \\ iii) & \mathbf{E}\left(\text{ess sup}_{t\in[0,T_x]} d^2\left((\xi(t),u(t),v(t)),\text{Graph}(H)\right)\right)\ \leq\ 2\varepsilon^2 \\ iv) & \max\left(\mathbf{E}\left(\|\xi(\cdot)\|_\alpha\right),\ \ \mathbf{E}(\|u(\cdot)\|_\infty),\ \mathbf{E}(\|v(\cdot)\|_\infty)\right)\ \leq\ \gamma \end{array}\tag{3.5}$$

The set $\mathcal{A}_\varepsilon(\xi)$ is not empty: take $T_\xi=0$ and $\xi(0)\equiv\xi_0$. It is an inductive set for the order relation

$$(T_{\xi_1},\xi_1(\cdot))\ \preceq\ (T_{\xi_2},\xi_2(\cdot))$$

if and only if

$$T_{\xi_1}\ \leq\ T_{\xi_2}\ \ \&\ \ \xi_2(\cdot)|_{[0,T_{\xi_1}]}\ =\ \xi_1(\cdot)$$

Zorn's Lemma implies that there exists a maximal element $(T_\xi,\xi(\cdot))\in\mathcal{A}_\varepsilon(\xi_0)$. The Lemma follows from the claim that for such a maximal element, we have $T_\xi=1$. If not, we shall extend $\xi(\cdot)$ by a stochastic process $\widehat{\xi}(\cdot)$ on an interval $[T_\xi,S_\xi]$ where $S_\xi>T_\xi$, contradicting the maximal character of $(T_\xi,\xi(\cdot))$.

Since $\xi_\omega(T_\xi)$ is $\mathcal{F}_{T_\xi}$ measurable, the projection map $\Pi_K(\xi_\omega(T_\xi))$ is also $\mathcal{F}_{T_\xi}$-measurable (see [6, Theorem 8.2.13, p. 317]). Then there exists a $\mathcal{F}_{T_\xi}$-measurable selection $y_\omega\in\Pi_K(\xi_\omega(T_\xi))$, which we call a projection of the random variable $\xi(T_\xi)$ onto K.

For simplicity, *we set* $x=\xi(T_\xi)$ *and thus choose a projection* $y\in\Pi_K(x)$. We take

$$\rho:=\frac{\varepsilon\sqrt{1-T_\xi}}{2}>0$$

and we set

$$c^2 \ := \ \max_{(u,v)\in\gamma B}(\mathbf{E}(\|u\|^2), 4\mathbf{E}(\|v\|^2)) \ < \ +\infty \tag{3.6}$$

We then introduce

$$\beta \ := \ \min\left(\varepsilon, \frac{\varepsilon^2(1-T_x)}{2c^2}\right) \ > \ 0$$

which is positive whenever $T_\xi < 1$.

We know that there exists $(u,v) \in H(y)$ such that (u,v) belongs to the stochastic contingent set $\mathcal{T}_\mathcal{K}(y)$: There exist $h \in]0,\beta]$ and $\mathcal{F}_{T_x+h}$-random variables a^h and b^h such that

$$\begin{array}{ll} i) & \mathbf{E}(\|a^h\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^h\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^h) \ = \ 0 \\ iv) & b^h \ \text{ is independent of } \ \mathcal{F}_t \end{array} \tag{3.7}$$

and satisfying

$$y + v(W(T_x+h) - W(T_x)) + hu + ha^h + \sqrt{h}b^h \ \in \ \mathcal{K} \tag{3.8}$$

We then set $S_x := T_x + h > T_x$ and we *define the stochastic process* $\widehat{\xi}(t)$ *on the interval* $[T_x, S_x]$ by

$$\widehat{\xi}(t) \ := \ x + (t-T_x)u + v(W(t) - W(T_x)) \tag{3.9}$$

Therefore, setting $h := t - T_x$,

$$d_\mathcal{K}^2(\widehat{\xi}(S_x)) - d_\mathcal{K}^2(\widehat{\xi}(T_x)) \ \leq \left\|x - y - ha^h - \sqrt{h}b^h\right\|^2 - \|x-y\|^2 =$$

$$\|ha^h + \sqrt{h}b^h\|^2 - 2\left\langle x-y, ha^h\right\rangle - 2\left\langle x-y, \sqrt{h}b^h\right\rangle$$

We take the expectation in both sides of this inequality and estimate each term of the right hand-side. First, we use the estimate

$$\mathbf{E}(\|ha^h + \sqrt{h}b^h\|^2) \ \leq \ 2h(h\mathbf{E}(\|a^h\|^2) + \mathbf{E}(\|b^h\|^2))$$

thanks to estimates (3.2), and (3.3).

Next,

$$\mathbf{E}\left(\left\langle x-y, a^h\right\rangle\right) \ \leq \ \mathbf{E}\left(\|x-y\|^2\right)^{\frac{1}{2}}\left(\mathbf{E}\left(\left\|a^h\right\|^2\right)\right)^{\frac{1}{2}}$$

and we observe that

$$\mathbf{E}\left\langle x-y, \frac{1}{\sqrt{h}}b^h\right\rangle \ = \ 0$$

since b^h is independent of $x-y$ and $\mathbf{E}(b^h)=0$. We obtain, by the very choice of ρ,

$$\mathbf{E}(d^2(\widehat{\xi}(S_x),\mathcal{K})) \;=\; \mathbf{E}(d^2(\widehat{\xi}(T_x+h),\mathcal{K}))$$

$$\leq\; \mathbf{E}d^2(\widehat{\xi}(T_x),\mathcal{K}) + 2h\mathbf{E}\left(\|x-y\|^2\right)^{\frac{1}{2}}\left(\mathbf{E}\left(\left\|a^h\right\|^2\right)\right)^{\frac{1}{2}}$$

$$+2h(h\mathbf{E}(\|a^h\|^2)+\mathbf{E}(\|b^h\|^2))$$

$$\leq\; \mathbf{E}d^2(\widehat{\xi}(T_x),\mathcal{K}) + h\left(\mathbf{E}\left(\|x-y\|^2\right)+3\mathbf{E}(\|a^h\|^2)+\mathbf{E}(\|b^h\|^2))\right)$$

$$\leq\; \varepsilon^2 T_x + h(\varepsilon^2 T_x + 4\rho^2) \;\leq\; \varepsilon^2 T_x + h\varepsilon^2 \;=\; \varepsilon^2 S_x$$

by (3.5)i).

Hence $\widehat{\xi}(\cdot)$ satisfies (3.5)i) for S_x.

We observe also that for any $t\in[T_x,S_x]$,

$$d^2_{\mathcal{K}}(\widehat{\xi}(t)) \;\leq\; \|\widehat{\xi}(t)-y\|^2$$

and that

$$\|\widehat{\xi}(t)-y\|^2 \;=\; \|x-y+(t-T_x)u+v(W(t)-W(T_x))\|^2$$

$$=\; d^2_{\mathcal{K}}(x) + 2\langle x-y,(t-T_x)u+v(W(t)-W(T_x))\rangle$$

$$+\|(t-T_x)u+v(W(t)-W(T_x))\|^2$$

$$\leq\; d^2_{\mathcal{K}}(x)+\|x-y\|+2\left\|(t-T_x)u+v(W(t)-W(T_x))\right\|^2$$

$$\leq\; 2d^2_{\mathcal{K}}(x)+4(t-T_x)^2\left\|u\right\|^2+4\left\|v(W(t)-W(T_x))\right\|^2$$

Therefore

$$\sup_{t\in[T_x,S_x]}\|\widehat{\xi}(t)-y\|^2$$

$$\leq\; 2d^2_{\mathcal{K}}(x)+4(S_x-T_x)^2\left\|u\right\|^2+4\sup_{t\in[T_x,S_x]}\left\|v(W(t)-W(T_x))\right\|^2$$

By taking the expectations and by using inequality

$$\mathbf{E}\left(\sup_{t\in[a,b]}\left\|\int_a^b h(s)dW(s)\right\|^2\right) \;\leq\; 4\int_a^b\|h(s)\|^2ds$$

we obtain

$$\mathbf{E}\left(\sup_{t\in[T_x,S_x]}\|\widehat{\xi}(t)-y\|^2\right) \leq\; 2\varepsilon^2 T_x + 4(S_x-T_x)c^2 \;\leq\; 2\varepsilon^2 T_x + 4\beta c^2$$

since $\max(\mathbf{E}(\|u\|^2), 4\mathbf{E}(\|v\|^2)) = c^2$ and since $S_x - T_x \leq \beta$. By the choice of $\beta \leq \dfrac{\varepsilon^2(1-T_x)}{2c^2}$, we infer that

$$\mathbf{E}\left(\sup_{t\in[T_x,S_x]} \|\widehat{\xi}(t) - y\|^2\right) \leq 2\varepsilon^2 T_x + 4\beta c^2 \leq 2\varepsilon^2 T_x + 2\varepsilon^2(1-T_x) = 2\varepsilon^2 \quad (3.10)$$

Hence $\widehat{\xi}(\cdot)$ satisfies (3.5)ii) for S_x.

We then observe that

$$\forall\, t \in [S_x, T_x],\ \ (\widehat{\xi}(t), u, v) = (y, u, v)+(\widehat{\xi}(t)-y, 0, 0) \in \mathrm{Graph}(H)+ \sup_{t\in[T_x,S_x]} \|\widehat{\xi}(t)-y\|B$$

Therefore,

$$\mathbf{E}\left(\text{ess sup}(\sup_{t\in[T_x,S_x]} d^2\left((\widehat{\xi}(t), u, v), \mathrm{Graph}(H)\right)\right) \leq \mathbf{E}\left(\sup_{t\in[T_x,S_x]} \|\widehat{\xi}(t) - y\|^2\right) \leq 2\varepsilon^2$$

We introduce now the concatenation $\widetilde{\xi}(\cdot)$ of $\xi(\cdot)$ defined on the interval $[0, T_x]$ and $\widehat{\xi}(\cdot)$ on $[T_x, S_x]$ defined by

$$\widetilde{\xi} := \begin{cases} \xi(t) & \text{if} \quad t \in [0, T_x] \\ \widehat{\xi}(t) & \text{if} \quad t \in [T_x, S_x] \end{cases}$$

One observe easily that

$$\|\widetilde{\xi}\|_\alpha \leq 2^{\alpha-1} \max(\|\xi\|_\alpha, \|\widehat{\xi}\|_\alpha) \quad (3.11)$$

We have to estimate $\mathbf{E}(\|\widehat{u}\|_\infty)$ and $\mathbf{E}(\|\widehat{\xi}\|_\alpha)$. The first estimate follows from

$$\mathbf{E}\left(\|u\|_\infty\right) \leq \sup_{(u,v)\in\gamma B} \mathbf{E}(\|u\|)$$

Writing

$$\widehat{\xi}(t) := \xi_1(t) + \xi_2(t)$$

where

$$\xi_1(t) := x + (t - T_x)u \;\&\; \xi_2(t) := \int_{T_x}^t v dW(s)$$

we infer that

$$\mathbf{E}\left(\|\xi_1\|_\alpha^2\right) \leq \sup_{(u,v)\in\gamma B)} \mathbf{E}(\|u\|) \;\&\; \mathbf{E}\left(\|\xi_2\|_\alpha^2\right) \leq c_{\alpha,p} \sup_{(u,v)\in\gamma B} \mathbf{E}(\|v\|^p)$$

thanks to inequality

$$\mathbf{E}\left(\left\|\int_a^b a(s)dW(s)\right\|_\alpha^2\right) \leq c_{\alpha,p} \int_a^b \|a(s)\|^p ds,$$

(see [11, Da Prato & Zabczyk]).)

3.3 Convergence of Martingale Solutions

The existence of martingale solutions follows from Lemma 3.3 (with $\varepsilon := \frac{1}{n}$) and the following Martingale Convergence Theorem.

Theorem 3.4 (Martingale Convergence Theorem) *Let K be a closed subset of X. We assume that H is a Marchaud set–valued map.*

Consider a sequence of $\mathcal{F}_t$-adapted stochastic processes

$$\xi_n(t) \ = \ \xi_n(0) + \int_0^t u_n(s)ds + \int_0^t v_n(s)dW(s)$$

on $[0,1]$ satisfying

$$\textit{for almost all}\, (\omega,t) \in \Omega \times [0,1], \ \ (u_{n_\omega}(t), v_{n_\omega}(t)) \ \in \ \gamma B$$

and

$$\begin{array}{ll} i) & \mathbf{E}\left(\sup_{t\in[0,1]} d^2(\xi_n(t),\mathcal{K})\right) \ \to \ 0 \\ ii) & \mathbf{E}\left(\text{ess sup}_{t\in[0,1]} d^2\left((\xi_n(t),u_n(t),v_n(t)),\text{Graph}(H)\right)\right) \ \to \ 0 \\ iii) & \max\left(\mathbf{E}\left(\|\xi_n(\cdot)\|_\alpha\right), \ \ \mathbf{E}(\|u_n(\cdot)\|_\infty), \ \mathbf{E}(\|v_n(\cdot)\|_\infty)\right) \ \le \ \gamma. \end{array} \tag{3.12}$$

Then there exist a probability space $(\widehat{\Omega},\widehat{\mathcal{A}},\widehat{\mathbf{P}})$ with a filtration $\{\widehat{\mathcal{F}}_t\}$ and a $\{\widehat{\mathcal{F}}_t\}$-Wiener process $\widehat{W}$, other stochastic processes $(\widehat{\xi}_n,\widehat{u}_n,\widehat{v}_n)$ and $(\widehat{\xi},\widehat{u},\widehat{v})$ such that their laws are equal:

$$\mathcal{L}(\widehat{\xi}_n,\widehat{u}_n,\widehat{v}_n) \ = \ \mathcal{L}(\xi_n,u_n,v_n)$$

and such that a subsequence (again denoted by) $\widehat{\xi}_n$ converges almost surely to $\widehat{\xi}$ in $\mathcal{C}([0,1];X)$, $(\widehat{u}_n,\widehat{v}_n)$ converges almost surely to $(\widehat{u},\widehat{v})$ in $L^1_\sigma([0,1],X\times\mathcal{L}(Y,X))$ which satisfy

$$\begin{array}{ll} i) & \widehat{\xi}(t) \ = \ \widehat{\xi}(0) + \int_0^t \widehat{u}(s)ds + \int_0^t \widehat{v}(s)d\widehat{W}(s) \\ ii) & \textit{for almost all} \ \ (\omega,t) \in \Omega\times[0,1], \ \ (\widehat{u}_\omega(t),\widehat{v}_\omega(t)) \ \in \ H(\widehat{\xi}_\omega(t)) \\ iii) & \textit{for almost all}\, \omega\in\widehat{\Omega}, \ \ \forall\, t\in[0,1], \ \ \xi_\omega(t)\in K \end{array} \tag{3.13}$$

Proof — Let us denote by $\mathcal{B}_\infty$ the unit ball of $L^\infty([0,1],X\times\mathcal{L}(Y,X))$. Then the subset $L^1_\sigma([0,1],\gamma\mathcal{B}_\infty)$ of functions of $L^1([0,1],X\times\mathcal{L}(Y,X))$ with values in the relatively compact subset $\gamma\mathcal{B}_\infty$ of $X\times\mathcal{L}(Y,X)$ is a Polish space (i.e., a separable complete metric space) when L^1 is supplied with the weakened topology. We thus consider the random variable (ξ_n, u_n, v_n) taking its values in the Polish space

$$\Phi := \mathcal{C}([0,1], X) \times L^1_\sigma([0,1], \gamma\mathcal{B}_\infty)$$

Their probability laws $\mathcal{L}(\xi_n, u_n, v_n)$ are defined by

$$\mathcal{L}(\xi_n, u_n, v_n)(A \times B \times C) \ := \ \mathbf{P}((\xi_n(\cdot), u_n(\cdot), v_n(\cdot)) \in A \times B \times C)$$

so that, for any function $\varphi \in \mathcal{C}(\Phi, \mathbf{R})$, we have by definition

$$\begin{aligned} & \int_\Omega \varphi(\xi_n(\cdot), u_n(\cdot), v_n(\cdot))\mathbf{P}(d\omega) \\ = & \int_\Phi \varphi(y(\cdot), u(\cdot) v(\cdot))\mathcal{L}(\xi_n, u_n, v_n)(dy, du, dv) \end{aligned} \tag{3.14}$$

Since the expectations of the norms $\mathbf{E}(\|u_n(\cdot)\|_\infty) \leq \gamma$, $\mathbf{E}(\|v_n(\cdot)\|_\infty) \leq \gamma$ and $\mathbf{E}(\|\xi_n(\cdot)\|_\alpha) \leq \gamma$ are bounded, we know by Lemma 3.5 below that their probability laws $\mathcal{L}(\xi_n, u_n, v_n)$ defined by (3.14) are tight. Therefore, they converge "weakly" to some measure $\mathcal{L}_\star$.

By Skorohod's Theorem (see for instance Theorem 2.4 of [11, Da Prato & Zabczyk])), there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbf{P}})$ with a filtration $\{\widetilde{\mathcal{F}}_t\}$, stochastic processes $\widetilde{\xi}_n$ and $\widetilde{\xi}$ and stochastic processes $(\widetilde{u}_n, \widetilde{v}_n)$ and $(\widetilde{u}, \widetilde{v})$ such that

$$\mathcal{L}(\widetilde{\xi}_n, \widetilde{u}_n, \widetilde{v}_n) \ = \ \mathcal{L}(\xi_n, u_n, v_n) \ \& \ \mathcal{L}(\widetilde{\xi}, \widetilde{u}, \widetilde{v}) \ = \ \mathcal{L}_\star$$

and converging to $(\widetilde{\xi}, \widetilde{u}, \widetilde{v})$ in the sense that

i) almost surely, $\widetilde{\xi}_n$ converges to $\widetilde{\xi}$ in $\mathcal{C}([0,1]; X)$
ii) almost surely, $(\widetilde{u}_n, \widetilde{v}_n)$ converges weakly to $(\widetilde{u}, \widetilde{v})$ in $\mathcal{L}^1_\sigma([0,1]; \gamma\mathcal{B}_\infty)$

Therefore

$$\begin{aligned} & \int_\Omega \varphi(\xi(\cdot), u(\cdot), v(\cdot))\mathbf{P}(d\omega) \\ = & \int_\Phi \varphi(y(\cdot), u(\cdot), v(\cdot))\mathcal{L}_\star(dy, du, dv) \end{aligned}$$

Set

$$M_n(t) \ := \ \xi_n(t) - \xi(0) - \int_0^t u_n(s)ds \ \& \ \widetilde{M}_n(t) \ := \ \widetilde{\xi}_n(t) - \widetilde{\xi}(0) - \int_0^t \widetilde{u}_n(s)ds$$

Then M_n is a martingale with respect to the filtration $\sigma(\xi_n, u_n)$ whose quadratic variation is

$$\langle\langle M_n(t)\rangle\rangle \ = \ \int_0^t \|v_n(s)\|^2 ds$$

because, for any bounded and continuous function φ,

$$\mathbf{E}\left([M_n(t) - M_n(s)]\varphi(\xi_n(s), u_n(s), v_n(s))\right) \ = \ \mathbf{E}\left(\left[\int_s^t v_n(s)dW(s)\right]\varphi(\xi_n(s), u_n(s), v_n(s))\right) \ = \ 0$$

Therefore, $\widetilde{M}_n$ is a martingale because for any bounded and continuous function φ,

$$\mathbf{E}\left([\widetilde{M}_n(t) - \widetilde{M}_n(s)]\varphi(\widetilde{\xi}_n(s), \widetilde{u}_n(s), \widetilde{v}_n(s))\right) \;=\; \mathbf{E}\left([M_n(t) - M_n(s)]\varphi(\xi_n(s), u_n(s), v_n(s))\right) \;=\; 0 \tag{3.15}$$

Its quadratic variation is equal to

$$\langle\langle \widetilde{M}_n(t)\rangle\rangle \;=\; \int_0^t \|\widetilde{v}_n(s)\|^2 ds.$$

Since

$$\sup_{n\geq 0} \mathbf{E}(\|M_n(\cdot)\|^2_{\mathcal{C}([0,1];X)}) \;<\; +\infty$$

we deduce[3] that the martingales converge to $\widetilde{M}$ defined by

$$\widetilde{M}(t) \;:=\; \widetilde{\xi}(t) - \widetilde{\xi}(0) - \int_0^t \widetilde{u}(s)ds$$

which is a martingale because, passing to the limit in (3.15), we obtain

$$\mathbf{E}\left([\widetilde{M}(t) - \widetilde{M}(s)]\varphi(\widetilde{\xi}(s), \widetilde{u}(s), \widetilde{v}(s))\right) \;=\; 0$$

for any bounded and continuous function φ.

Therefore, by the Martingale Representation Theorem (see for instance Theorem 8.2 of [11, Da Prato & Zabczyk])), there exist another probability space $(\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{P})$ with a filtration $\{\widehat{\mathcal{F}}_t\}$, a Wiener process $\widehat{W}$, other stochastic processes $(\widehat{\xi}_n, \widehat{u}_n, \widehat{v}_n)$ and $(\widehat{\xi}, \widehat{u}, \widehat{v})$ such that

$$\mathcal{L}(\widehat{\xi}_n, \widehat{u}_n, \widehat{v}_n) \;=\; \mathcal{L}(\widetilde{\xi}_n, \widetilde{u}_n, \widetilde{v}_n) \;\&\; \mathcal{L}(\widehat{\xi}, \widehat{u}, \widehat{v}) \;=\; \mathcal{L}(\widetilde{\xi}, \widetilde{u}, \widetilde{v})$$

and a martingale $\widehat{M}$

$$\widehat{M}(t) \;=\; \int_0^t \widehat{v}(s)d\widehat{W}(s)$$

the quadratic variation of which is

$$\langle\langle \widehat{M}(t)\rangle\rangle \;=\; \int_0^t \|\widehat{v}(s)\|^2 ds$$

such that

$$\widehat{\xi}(t) \;=\; \widehat{\xi}(0) + \int_0^t \widehat{u}(s)ds + \widehat{M}(t) \;=\; \widehat{\xi}(0) + \int_0^t \widehat{u}(s)ds + \int_0^t \widehat{v}(s)d\widehat{W}(s)$$

It remains to prove that for almost all $(\omega, t) \in \widehat{\Omega} \times [0,1]$, $(\widehat{u}_\omega(t), \widehat{v}_\omega(t)) \in H(\widehat{\xi}_\omega(t)$.

[3] see for instance criterion II-5-2 of [13, Neveu].

By definition,

$$\mathbf{E}\left(\text{ess sup}_{t\in[0,1]} d^2((\xi_n(t), u_n(t), v_n(t)), \text{Graph}(H))\right)$$

$$= \int_\Phi \text{ess sup}_{t\in[0,1]} d^2((y(t),, u(t), v(t)), \text{Graph}(H))\mathcal{L}(\xi_n(\cdot), u_n(\cdot), v_n(\cdot))(dy, du, dv)$$

$$= \int_\Phi \text{ess sup}_{t\in[0,1]} d^2((y(t), u(t), v(t)), \text{Graph}(H))\mathcal{L}(\widehat{\xi}_n(\cdot), \widehat{u}_n(\cdot), \widehat{v}_n(\cdot))(dy, du, dv)$$

$$= \mathbf{E}\left(\text{ess sup}_{t\in[0,1]} d^2((\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)), \text{Graph}(H))\right)$$

converges to 0 by Lemma 3.3.

Now, we know that the $(\widehat{u}_n(\cdot), \widehat{v}_n(\cdot))$ are bounded in $L^\infty(\widehat{\Omega}\times[0,1]; X\times\mathcal{L}(Y,X))$ so that a subsequence (again denoted by) $(\widehat{u}_n, \widehat{v}_n)$ converges weakly to some $(\widehat{u}, \widehat{v})$ in $L^\infty(\widehat{\Omega}\times[0,1]; X\times\mathcal{L}(Y,X))$, and thus, weakly in $L^1(\widehat{\Omega}\times[0,1]; X\times\mathcal{L}(Y,X))$. On the other hand, for almost all $(\omega,t)\in\widehat{\Omega}\times[0,1]$, $\widehat{\xi}_{n,\omega}(t)$ converges to $\widehat{\xi}_\omega(t)$. By the Chebichev inequality, we deduce that

$$\mathbf{P}\left(\text{ess sup}_{t\in[0,1]} d^2\left((\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)), \text{Graph}(H)\right) \geq \lambda\right) \leq \frac{2}{\lambda n^2}$$

Taking $\lambda := \frac{1}{n}$, we infer that

$$\mathbf{P}\left(\text{ess sup}_{t\in[0,1]} d^2\left((\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)), \text{Graph}(H)\right) \geq \frac{1}{n}\right) \leq \frac{2}{n}$$

Therefore, $\text{ess sup}_{t\in[0,1]} d^2\left((\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)), \text{Graph}(H)\right)$ converges to 0 in probability, and thus, there exists a subsequence (again denoted by) $(\widehat{\xi}_n, \widehat{u}_n, \widehat{v}_n)$ such that

$$\text{almost surely,}\quad \text{ess sup}_{t\in[0,1]} d^2\left((\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)), \text{Graph}(H)\right) \to 0$$

Hence,

$$\text{almost surely,}\quad (\widehat{\xi}_n(t), \widehat{u}_n(t), \widehat{v}_n(t)) \in B(\text{Graph}(H), \frac{1}{n})$$

Now, by using the Convergence Theorem (see for instance Theorem 7.2.1 of [6, Aubin & Frankowska]), we infer from the fact that F is a Marchaud set–valued map, that

$$\text{for almost all } (\omega,t)\in\widehat{\Omega}\times[0,1],\quad (\widehat{\xi}(t), \widehat{u}(t), \widehat{v}(t)) \in \text{Graph}(H)$$

Let us prove now that the stochastic process $\widehat{\xi}(\cdot)$ is viable in K. By definition,

$$\mathbf{E}\left(\sup_{t\in[0,1]} d^2(\xi_n(t), K)\right)$$
$$= \int_\Phi \sup_{t\in[0,1]} d^2(y(t), K)\mathcal{L}(\xi_n(\cdot), u_n(\cdot), v_n(\cdot))(dy, du, dv)$$

Since the left-hand side converges to 0, we observe that

$$0 \;=\; \int_{\Phi} \sup_{t\in[0,1]} d^2(y(t),K)\mathcal{L}(\widehat{\xi}(\cdot),\widehat{u}(\cdot),\widehat{v}(\cdot))(dy,du,dv)$$

thanks to Lemma 3.3. We thus deduce that

$$\int_{\widehat{\Omega}} \sup_{t\in[0,1]} d^2(\widehat{\xi}(t),K)\widehat{\mathbf{P}}(d\omega) \;=\; 0$$

thanks to (3.14). Hence, for every $t \geq 0$, $\widehat{\xi}(t) \in K$ almost surely. ∎

3.4 Tightness

Lemma 3.5 *if the expectations of the norms* $\mathbf{E}(\|u_n(\cdot)\|_\infty) \leq \gamma$, $\mathbf{E}(\|v_n(\cdot)\|_\infty) \leq \gamma$ *and* $\mathbf{E}(\|\xi_n(\cdot)\|_\alpha) \leq \gamma$ *are bounded, the probability laws* $\mathcal{L}(\xi_n, u_n, v_n)$ *are tight.*

Proof — We have to associate with any $\varepsilon > 0$ a compact subset K_ε such that

$$\mathbf{P}((\xi_n,u_n,v_n) \in K_\varepsilon) \;\geq\; 1-\varepsilon$$

Since the expectations of the norms are bounded, we deduce from the Chebichev inequality that

$$\mathbf{P}\left(\|\xi\|_\alpha \geq \frac{\gamma}{\varepsilon}\right) \;\leq\; \varepsilon \;,\; \mathbf{P}\left(\|u\|_\infty \geq \frac{\gamma}{\varepsilon}\right) \;\leq\; \varepsilon \;\&\; \mathbf{P}\left(\|v\|_\infty \geq \frac{\gamma}{\varepsilon}\right) \;\leq\; \varepsilon$$

Therefore, it is enough to take

$$K_\varepsilon \;:=\; \frac{\gamma}{\varepsilon}(B_\alpha \times B_\infty \times B_\infty)$$

which is compact thanks to Ascoli's Theorem and the weak compactness of the unit ball of L^∞ in L^1. Indeed, the complement of K_ε is the product of the sets $\{\|\xi\|_\alpha \geq \frac{\gamma}{\varepsilon}\}$, $\{\|u\|_\infty \geq \frac{\gamma}{\varepsilon}\}$ and $\{\|v\|_\infty \geq \frac{\gamma}{\varepsilon}\}$, whose measure is larger than or equal to ε. ∎

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