Optimal Impulse Control Problems and Quasi-Variational Inequalities Thirty Years Later: a Viability Approach

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1 Foreword

More than thirty years ago, at the very beginning of our collaboration to create the UFR de Mathématiques de la Décision and the CEREMADE, Alain and I split our "territories" of investigations: Alain Bensoussan was exploring management issues whereas I started to dive in the intricacies of the mechanisms governing economic evolution. Optimal control was one of the main tools in a field of issues created by Man's hands and activities[2], whereas economic evolution, guided by Adam Smith's invisible hand, was not in my opinion teleological in nature, but shared with Darwinian evolution several common traits and features.

These economic motivations led some of us to some kind of dissidence away of the mainstream view of the so seducing theory of competitive general equilibrium and to design and build progressively the tools of "viability theory" and "set-valued analysis" for providing mathematical metaphors of dynamic economic issues[3].

However, perversions are as prolific in scientific evolution as in general history. The very tools of viability theory devised for providing an alternative to optimal control theory for aiming at convincing metaphors of evolution under contingent (non-stochastic) uncertainty and viability constraints obeying to an "inertial principle" evolved slowly during the eighties ... and became adequate mathematical instruments for investigating some issues of optimal control theory in the Hamilton-Jacobi tradition !

The bridge between the two approaches was in particular put together by Hélène Frankowska and her collaborators, who did point out the backward invariance and local forward viability properties of the epigraph of the value function of an optimal control problem: She proved that the epigraph of the value function of an optimal control problem — assumed to be only lower semicontinuous — is invariant and backward viable under a (natural) auxiliary system. It allowed her to characterize the value functions as unique solutions of contingent inequalities, and, by duality, to obtain lower semicontinuous (or bilateral) solutions to Hamilton-Jacobi partial differential equations, obtained by other methods in [14, Barron & Jensen] (See also [13, Bardi & Capuzzo-Dolcetta] for more details on this point of view). Furthermore, when the value function is continuous, she demonstrated that its epigraph is viable and its hypograph invariant ([36, 37, 39, Frankowska]). By duality, she proved that the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton-Jacobi equation in the sense of M. Crandall and P.-L. Lions in [22, Crandall & Lions

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[2]Giambattista Vico (1688-1744) insisted in his *New Science* that Man can only understand what he created himself, and in particular, mathematics.

[3]see for instance [4, Aubin] for a review.

P.-L.]. This epigraphical approach in the field of Hamilton-Jacobi equations has since been taken up by other authors.

Recently, a series of papers and books — among which [19, Branicky, Borkar & Mitter], [18, Bensoussan & Menaldi] and [49, Shaft & Schumacher] to quote a few — aims at embedding several classes of "hybrid systems" or "impulse optimal control problems" into a general framework. The paper [12, Aubin, Lygeros, Quincampoix, Sastry & Seube] and the lecture notes [6, Aubin] contribute to this effort, by presenting impulse control and hybrid systems in the framework of the viability of control systems.

Then, I remembered being a privileged witness of the development around Alain Bensoussan and Jacques-Louis Lions of optimal impulse control and quasi-variational Hamilton-Jacobi inequalities (see [15, 16, 17, Bensoussan & Lions J.-L.] for instance for motivations, examples and a review). Hence, thanks to the bridge erected by Hélène Frankowska between viability theory and optimal control, I finally understood thirty years later in my own biased way the first-order theory that was developed during this period by so many of my old-time friends.

For that purpose, I will implicate a general concept of impulse differential inclusions (encompassing the now popular "hybrid systems") for deriving quasi-variational Hamilton-Jacobi inequalities of an optimal impulse control system, whereas recently, [18, Bensoussan & Menaldi] used them as an approach to hybrid systems...

This is then the best opportunity to present these results as a token of "une amitié de trente ans" that required all this time to finally converge.

This application to optimal impulse control is presented not so much for providing the existence of a generalized solution to the quasi variational inequalities to which the value function is a solution, but to convince the reader that the simple framework of impulse differential inclusions we propose allows us to cover many problems arising in the realm of hybrid systems in control theory, of stock management in production theory, of multiple-phase dynamical economies in economics, of the propagation of the nervous influx along axons of neurons triggering spikes in neurosciences, etc. It evidences that the tangential and normal characterizations of the viability of a set or of its "reset kernel" imply the theorems characterizing value functions as (contingent or viscosity) generalized solutions to quasi variational inequalities, and thus, share the same features. This suggests that this approach based on viability theory and set-valued analysis provides some potential for studying those problems which can be formulated as viability problems for impulse differential inclusions.

2 The Objective: Optimal Impulse Control Problems

Let us consider the control problem

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in P(x(t)) \end{cases}$$

"Runs" are associated with a sequence impluse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and initial states ξ_n, satisfying

1. if $t \in [t_{n-1}, t_n[$, then

$$x(t) \;=\; x_0 + \sum_{k=1}^{n-1} \xi_k + \int_0^t f(x(\tau), u(\tau))d\tau$$

2. if $t_n = t_{n-1}$,

$$x_n := x(t_n) \;=\; x_0 + \sum_{k=1}^{n} \xi_k + \int_0^{t_n} f(x(\tau), u(\tau))d\tau$$

Let us introduce a "cost function" $\mathbf{w} : X \mapsto \mathbf{R} \cup \{+\infty\}$ and a intertemporal cost function (Lagrangian) $l : X \times \mathcal{P} \mapsto \mathbf{R}$.

We shall characterize the value function

$$\mathbf{v}(x) := \inf_{(x(\cdot),u(\cdot))\in\mathcal{R}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau \right)$$

of the above control problem and derive that under adequate assumptions, the value function is *the* unique solution $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *of the quasi-variational inequalities*

$$\begin{cases} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \;\; H(x, \mathbf{v}(x), p) \;\leq\; 0 \\ iii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \;\; H(x, \mathbf{v}(x), p)(\mathbf{v}(x) - (\mathbf{v} * \mathbf{w})(x)) \;=\; 0 \end{cases}$$

where

1. $(\mathbf{v} * \mathbf{w})(x) := \inf_{y\in X}(\mathbf{w}(y) + \mathbf{v}(y + x))$ is the inf-convolution of the functions $\mathbf{v}$ and $\mathbf{w}$,

2. $\partial_{-}\mathbf{v}(x)$ denotes the generalized subgradient of $\mathbf{v}$ at x, as it used both is nonsmooth analysis (see for instance [9, Aubin & Frankowska] or [46, Rockafellar & Wets]) and in the theory of viscosity solutions (see for instance [22, Crandall & Lions P.-L.] and [13, Bardi & Capuzzo-Dolcetta]),

3. $H(x, y, p) := \sup_{u\in P(x)}(\langle p, f(x, u)\rangle + l(x, u)) - ay$ denotes the Hamiltonian associated with the control system and the Lagrangian,

4. $\mathbf{v}$ is unique in the class of lower semicontinuous functions.

Knowing the value function $\mathbf{v}$, we introduce the two regulation maps $\mathbf{R}_{(f,P)}$ and $\mathbf{R}_{\mathbf{w}}$ defined by

$$\mathbf{R}_{(f,P)}(x) \;:=\; \{u \in P(x) \mid D_{\uparrow}\mathbf{v}(x)(f(x, u)) + l(x, u) - a\mathbf{v}(x) \leq 0\}$$

where

$$D_{\uparrow}\mathbf{v}(x)(u) \;:=\; \liminf_{h\to 0+,\, u'\to u} \frac{V(x + hu') - V(x)}{h}$$

denotes the contingent epiderivative of $\mathbf{v}$ at x in the direction u and

$$\mathbf{R}_{\mathbf{w}}(x) \;:=\; \{y \in X \mid \mathbf{w}(y) + \mathbf{v}(y + x) = (\mathbf{v} * \mathbf{w})(x)\}$$

Therefore, an optimal run is obtained in the following way: Starting from x_0 such that $\mathbf{v}(x_0) < (\mathbf{v} * \mathbf{w})(x_0)$, we choose a solution to the control system

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in \mathbf{R}_{(f,P,l)}(x(t)) \end{cases}$$

until the time $t_1 \geq 0$ when $\mathbf{v}(x(^-t_1)) = (\mathbf{v} * \mathbf{w})(x(^-t_1))$.

At this stage, we take for reinitialized sate $x_1 \in \mathbf{R}_{\mathbf{w}}(x(^-t_1)$, and we reiterate the process.

We shall show that this optimal impulse control problem with infinite horizon conceals an underlying impulse control system $((f, \mathbf{R}_{(f,P,l)}), \mathbf{R}_{\mathbf{w}})$ that we shall define, the runs of which being the optimal runs of the initial optimal impulse control problem.

3 Impulse Differential Inclusions

3.1 Viable Subsets under Differential Inclusions

Given a control system or a differential game described under the form of a differential inclusion $x' \in F(x)$ and constraints on the states represented by a closed subset K, we say that K *is viable under* F if from any initial state $x_0 \in K$ starts at least one solution of this differential inclusion "viable" in K in the sense that

$$\forall\, t \geq 0, \;\; x(t) \in K$$

There are no reasons why an arbitrary subset K should be viable under the differential inclusion $x' \in F(x)$.

Hence, the problem of reestablishing viability arises. One can imagine several mechanisms for this purpose:

1. Change either the dynamics or the set of constraints
   (a) either by changing the controls according to feedbacks or dynamic feedbacks that can be constructed (see for instance [2, 4, Aubin]),
   (b) or by changing the dynamics by, for instance, projecting the velocities onto the contingent cones and introducing viability multipliers (see for instance [2, 4, Aubin]),
   (c) or by restricting the constrained set to its viability kernel, which is by definition the largest subset viable under the dynamics,
   (d) or by letting the set of constraints evolve according to mutational equations, as in [5, Aubin].
2. or change the initial conditions by introducing a reset map R mapping any state of K to a (possibly empty) set $R(x) \subset X$ of new "initialized states".

This is the latter strategy we choose to use here: Hence an impulse differential inclusion (and in particular, an impulse control system) is described by a pair (F, R), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the continuous evolution of the system in K and where R, the reset map, governs the discrete switches to new "initial conditions" when the continuous evolution is doomed to leave K.

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called in the "hybrid system" literature a "run" or an "execution".

Let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t-\eta, t[$ where $\eta > 0$, and, for consistency purposes, $x(s) = x(^-t)$ if $s = t$.

Definition 3.1 *Let us consider a finite dimensional vector space X, a closed subset $K \subset X$, a set-valued map $F : X \rightsquigarrow X$ and a set-valued map $R : X \rightsquigarrow X$, regarded as a* reset map. *We denote by $S := R - \mathbf{1}$ the associated* switching map.

We regard the pair (F, R) — or (F, S) — as the dynamics of an impulse differential inclusion.

A run *of the impulse differential inclusion is a map $x(\cdot)$ from $[0, T]$ to X if $T < +\infty$ or from $[0, +\infty[$ to X if $T = +\infty$ which is associated with a non decreasing sequence $\mathcal{T}(x(\cdot)) := \{t_n\}_{n\geq 0}$ of* impulse or switching times *$t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ such that*

1. *$x(t_{n+1}) \in R(x(t_n))$ if $t_{n+1} = t_n$,*

2. *or else, on the interval $[t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ at time t_n until time t_{n+1} at which we take $x(t_{n+1}) \in R(x(^-t_{n+1}))$.*

We denote by $\tau_n := t_n - t_{n-1}$ the nth cadence *of the run and by $x_n(\cdot) := x(\cdot + t_n)$ the nth* motive *of the run, a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ on the interval $[0, \tau_n]$. The sequence of states $x(t_n)$ is called the sequence of* initialized states.

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$.

We shall denote by $\mathcal{R}_{(F,R)}(x_0)$ the set of runs of the impulse differential inclusion starting from x_0 and by $\mathcal{R}^K_{(F,R)}(x_0)$ the set of runs viable in K.

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in R(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

3.2 Examples

Many examples coming from different fields of knowledge fit this framework:

1. multiple-phase economic dynamics in economics (see for instance [27, Day]),

2. stock management in production theory ([15, 16, 17, Bensoussan & Lions J.-L.] for instance),

3. viability theory, for implementing the extreme version of the "inertia principle" ([2, 4, Aubin])),

4. propagation of the nervous influx along axons of neurons triggering spikes in neurosciences and biological neuron networks[4] (See the "Integrate-and-Fire" models

[4]instead of the continuous time Hodgkin-Huxley type of systems of differential equations inspired by the propagation of electrical current which are the subject of an abundant literature. See the pioneering [42, Hodgkin & Huxley].

in [20, Bressloff & Coombes], [21, Burgmann], [28, Destexhe], [50, 51, Shimokawa, Pakdaman & Sato] and [52, Shimokawa, Pakdaman, Takahata, Tanabe & Sato] for instance.),

5. "threshold" impulse control, when "controls jump" when the threshold is about to be trespassed,

6. punctuated evolution as proposed by N. Eldredge and S.J Gould in [31, Eldredge & Gould] in biological evolution,

7. in demographic models, to take into account discontinuous processes such as births and deaths,

8. in issues dealing with "qualitative physics" in Artificial Intelligence and/or "comparative statics" in economics (see for instance [3, Aubin] and [30, Dordan] for a mathematical presentation of these issues and further bibliographical data),

9. in logics, where connections are made between impulse differential inclusions and the μ-calculus (see [23, 24, 25, Davoren], [45, Nerode & Shore] and their references),

10. in manufacturing and economic production systems, when the jumps are governed by Markov processes instead of set-valued maps (see [32, Filar, Gaitsgory & Haurie], [33, Filar & Haurie], [41, 41, Haurie, Leizarowitz & Van Delft]) and the references herein,

11. and, above all, in automatic control theory where a fast growing literature deals with hybrid "systems":

3.3 Hybrid Differential Inclusions

"Hybrid differential inclusions", as they are called by engineers, or "multiple-phase dynamical economies", as they are called by economists, or "Integrate and Fire" models in neurobiology — may be regarded as auxiliary impulse differential inclusions.

Definition 3.2 *An hybrid differential inclusion* (K, F, R) *is defined by*

1. *a finite dimensional vector space* E *of states* e *called* locations,

2. *a set-valued map* $K : E \rightsquigarrow X$ *associating with any location* e *a (possibly empty) subset* $K(e) \subset X$

3. *a set-valued map* $F : \mathrm{Graph}(K) \rightsquigarrow X$ *with which we associate the differential inclusion* $x'(t) \in F(e, x(t))$,

4. *a set-valued map* $R : \mathrm{Graph}(K) \rightsquigarrow E \times X$

A run *of the hybrid differential inclusion is a "map"* $x(\cdot)$ *from* $[0, T]$ *to* X *if* $T < +\infty$ *or from* $[0, +\infty[$ *to* X *if* $T = +\infty$ *which is associated with a non decreasing sequence* $\mathcal{T}(x(\cdot)) := \{t_n\}_{n \geq 0}$ *of* impulse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ *such that*

1. *either* $t_{n+1} = t_n$, $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ *and* $x(t_{n+1}) \in K(e(t_{n+1}))$,

2. *or $t_{n+1} > t_n$, and, for all $t \in [t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x'(t) \in F(e(t_n), x(t))$ viable in $K(e(t_n))$ and we take $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ and $x(t_{n+1}) \in K(e(t_{n+1}))$.*

In other words, $x(\cdot)$ is a run of the hybrid differential inclusions if and only if $(e(\cdot), x(\cdot))$ is a run of the following auxiliary system of impulse differential inclusions

$$\begin{cases} i) & e'(t) \; = \; 0 \\ ii) & x'(t) \; \in \; F(e(t), x(t)) \end{cases}$$

viable in Graph(K).

Indeed the locations remain constant in the intervals $[t_n, t_{n+1}[$ since their velocities are equal to 0. □

3.4 The Characterization Theorem

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 3.3 (Marchaud Map) *We shall say that F is a* Marchaud map[5] *if*

$$\begin{cases} i) & \text{the graph and the domain of } F \text{ are nonempty and closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear:} \\ & \exists\, c > 0 \mid \forall\, x \in X, \;\; \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

We recall the following version of the important Theorem 3.5.2 of Viability Theory, [2, Aubin]:

Theorem 3.4 *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}_F$ is upper semicompact[6] with nonempty values: This means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}_F(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}_F(x)$ uniformly on compact intervals.*

[5] We can replace the linear growth by the more general condition:

$$\exists\, c > 0 \mid \forall\, x \in X, \;\; \sup_{v \in F(x)} \frac{\langle v, x \rangle}{(\|x\| + 1)^2} \;\leq\; c$$

[6] Actually, a sizable part of the following results depend upon few properties: translation and concatenation and upper semicompactness properties of the set-valued map $x \rightsquigarrow \mathcal{S}_F(x)$. Therefore these results are common to other control problems, such as

1. control problems with memory (see the contributions of G. Haddad, some of them being presented in [2, Aubin]) — before known under the name of functional control problems, the new fashion calling them as "path dependent control systems"

2. parabolic type partial differential inclusions (see the contributions of Shi Shuzhong, some of them being presented in [2, Aubin]) — also known as distributed control systems

3. "mutational equations" governing the evolution in metric spaces, including "morphological equations" governing the evolution of sets (see [5, Aubin] for instance).

We denote by $\mathcal{S}_F(x) \subset \mathcal{C}(0,\infty;X)$ the set of absolutely continuous functions $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t))$$

starting at time 0 at x: $x(0) = x$. The set-valued map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is called the solution map (or the set-valued flow) associated with F.

The Viability Theorem[7] and its consequences imply the following

Theorem 3.5 *Let (F,R) be an impulse differential inclusion and $K \subset X$ be a closed subset. Assume that F is Marchaud and that R is upper semicontinuous with compact images[8]. Then the following statements are equivalent*

1. *K is viable under (F,R),*
2. *The subset[9] $K \backslash R^{-1}(K)$ is locally viable under F,*
3. *K, F and R are linked through the tangential condition[10]*

$$\forall\, x \in K \backslash R^{-1}(K), \quad F(x) \cap T_K(x) \neq \emptyset$$

or, equivalently, in dual form, through the normal condition

$$\forall\, x \in K \backslash R^{-1}(K),\ \forall\, p \in N_K(x), \quad \sigma(F(x), -p) \ \geq \ 0$$

3.5 Reset Kernels under Impulse Differential Inclusions

When K is not viable under (F,R), we introduce the following concepts:

Definition 3.6 *Let us consider an impulse differential inclusion (F,R) and a subset K.*

We shall denote by $\mathrm{Reset}_{(F,R)}(K)$ the subset of initial states $x_0 \in K$ from which starts at least one run viable in K and call it the reset kernel *of K under the impulse differential inclusion (F,R).*

We now state the main result:

Theorem 3.7 *Let assume that $F : X \rightsquigarrow X$ is Marchaud, that $R : X \rightsquigarrow X$ is upper semicontinuous and that*

$$\forall\, x \in K, \quad R(x) \cap (K+B) \text{ is compact}$$

[7]See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [2, Aubin].

[8]This assumption implies that $R^{-1}(K)$ is closed, which is the property we really need. It remains true when we assume only that the subsets $K \cap (R(x)+B)$ are compact, where B denotes the unit ball.

[9]The subset $K \backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C.

[10]The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n. The (regular) normal cone $N_L(x) := T_L(x)^-$ is the polar cone to the contingent cone $T_L(x)$ (see for instance [9, Aubin & Frankowska]) or [46, Rockafellar & Wets] for more details). We denote by

$$\forall\, p \in X^\star, \quad \sigma(K,p) \ := \ \sup_{x \in K} \langle p, x \rangle$$

the support function of K.

Assume also that K is a closed repeller under both F and R. The reset kernel $\text{Reset}_{(F,R)}(K)$ *is the largest closed subset of K viable under the impulse differential inclusion (F, R). It is the largest closed solution to the "fixed set" problem*

$$\text{Reset}_{(F,R)}(K) \ := \ \text{Capt}_F^K \left(R^{-1}(\text{Reset}_{F,S}(K)) \cap K\right)$$

contained in K.

Furthermore, setting $K_0 := K$ and recursively

$$K_{n+1} \ := \ K_n \cap \text{Cap}_F^K(R^{-1}(K_n) \cap K)$$

it is equal to

$$\text{Reset}_{(F,C)}(K) \ = \ \bigcap_{n \geq 0} K_n$$

(the reset kernel algorithm*).*

Proof — We first observe that the subsets K_n are closed.

Second, we note that any closed subset $L \subset K$ viable under (F, R) is contained in the reset kernel $\text{Reset}_{(F,R)}(K)$, since from any $x_0 \in K$ starts a viable run in L, and thus in K.

The reset kernel $\text{Reset}_{(F,R)}(K)$ is viable under (F, R). Indeed, let $x_0 \in \text{Reset}_{(F,R)}(K)$ and $x(\cdot)$ be a run starting from x_0 viable in K, associated with impulse times $t_0 = 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and reset initial states x_n. Take any $T > 0$ and show that $x(T)$ belongs to the reset kernel $\text{Reset}_{(F,R)}(K)$. We consider the run $y(\cdot)$ defined by $y(t) := x(t + T)$, starting at time 0 from $x(T)$, with impulse times $s_n := t_n + T$ and reset initial states $y_n := x_n \in K$. Furthermore, for every interval $[t_n, t_{n+1}[$ whenever $t_{n+1} > t_n$, $y(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ viable in K. Hence, $y(\cdot)$ is a run of the impulse differential inclusion starting from $x(T)$ and viable in K, and thus, belongs to the reset kernel $\text{Reset}_{(F,R)}(K)$.

Assume next that x_0 belongs to the reset kernel $\text{Reset}_{(F,R)}(K)$ under (F, R) and show that it belongs to to the intersection

$$K_\propto \ := \ \bigcap_{n \geq 0} K_n$$

of the subsets K_n. Indeed, there exists a run $x(\cdot)$ starting from x_0 viable in K associated with a sequence of impulse times $0 \leq t_1 \leq \cdots \leq t_n \leq \cdots$ and of states $\xi_n := x(^-t_n) \in R^{-1}(K) \cap K$ and of initial states $x_n \in R(\xi_n)$. Each element x_n of the sequence of initial states belongs to K. Since $\xi_n \in R^{-1}(K) \cap K$ and $x_n \in \text{Capt}_F^K(R^{-1}(K) \cap K)$ by construction, then each element x_n of the sequence of initial states belongs to K_1. Assume that each element x_n of the sequence of initial states belongs to K_j. Therefore, as for the case $j = 0$, we deduce that $x_n \in \text{Capt}_F^K(R^{-1}(K_j) \cap K)$. Since it belongs to K_j, we infer that it also belongs to K_{j+1}. Therefore, the sequence of initial states x_n ranges over the intersection $K_\propto$ of the subsets K_n, so that $\text{Reset}_{(F,R)}(K) \subset K_\propto$.

Let us prove now that from any $x_0 \in K_\propto$ starts a viable run, i.e., that $K_\propto$ is contained in the reset kernel of K. For any $n \geq 0$, one can find a solution $x_n(\cdot)$ to the differential inclusion $x' \in F(x)$ starting from x_0, a time $t_n \leq \tau_K^{F^\sharp}(x_0)$ such that $\xi_n := x_n(^-t_n)$ and $y_n \in R(\xi_n) \cap K_{n-1}$ since $x_0 \in \text{Cap}_F^K(R^{-1}(K_n) \cap K)$. One can prove that subsequences (again denoted by) t_n, $x_n(\cdot)$ and y_n converge to some $t \leq \tau_K^{F^\sharp}(x_0)$, $x(\cdot)$ and y respectively, where $x(\cdot)$ is a solution to the differential inclusion starting from

x_0, $\xi = x(t)$ and $y \in R(\xi) \cap K$. Since the elements y_n belong to K_{n-1} and since the sequence K_n is decreasing, this limit y belongs to the intersection $K_\propto$ of the subsets K_n. Hence $\xi = x(t)$ belongs to $R^{-1}(K_\propto) \cap K$ and thus, x_0 belongs to $\mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$.

It remains to check that $\mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$ is contained in $K_\propto$. Indeed, starting form $x_0 \in \mathrm{Capt}_F^K(R^{-1}(K_\propto) \cap K)$, there exists a solution to the differential inclusion $x' \in F(x)$ viable in K until it reaches $R^{-1}(K_\propto) \cap K$ at some ξ, when it can be reset to some $x_1 \in R(\xi) \cap K_\propto$. Therefore, a discrete sequence of initial states starts from x_0 and is viable in K, so that x_0 belongs to the reset kernel of K, which is thus equal to $K_\propto$. □

Using the viability theorems providing characterization of local viability and invariance in terms of tangential and/or normal conditions, we derive the following characterizations:

Theorem 3.8 *Let us assume that F is Marchaud and that K is a closed repeller under both F and R. Then the reset kernel* $\mathrm{Reset}_{(F,R)}(K)$ *is the largest closed subset D of K satisfying*

$$\begin{cases} i) & K \cap R^{-1}(\mathrm{Reset}_{(F,R)}(K)) \subset D \subset K \\ ii) & \forall\, x \in D \backslash R^{-1}(\mathrm{Reset}_{(F,R)}(K)),\ \ F(x) \cap T_D(x) \neq \emptyset \end{cases}$$

or, equivalently, in terms of normal cones and support functions, if and only if $\mathrm{Reset}_{(F,R)}(K)$ *is the largest closed subset D satisfying*

$$\begin{cases} i) & K \cap R^{-1}(\mathrm{Reset}_{(F,R)}(K)) \subset D \subset K \\ ii) & \forall\, x \in D \backslash R^{-1}(\mathrm{Reset}_{(F,R)}(K)),\ \forall\, p \in N_D(x),\ \ \sigma(F(x), -p) \geq 0 \end{cases}$$

When K is backward invariant under F, we infer the following consequence:

Theorem 3.9 *Assume that $F : X \rightsquigarrow X$ is Marchaud and Lipschitz and that K is a closed repeller under both F and R and backward invariant under F. Then the reset kernel* $\mathrm{Reset}_{(F,R)}$ *under the impulse differential inclusion (F, R) is the unique Frankowska extension of* $R^{-1}(\mathrm{Reset}_{(F,R)}(K)) \cap K$*, i.e., the unique closed subset D such that*

$$R^{-1}(D) \cap K \subset D \subset K$$

satisfying

$$\begin{cases} i) & \forall\, x \in D \backslash R^{-1}(D),\ \ F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall\, x \in D,\ \ F(x) \subset -T_D(x) \end{cases}$$

or again,

$$\begin{cases} i) & \forall\, x \in D \backslash R^{-1}(D),\ \forall\, p \in N_D(x),\ \ \sigma(F(x), -p) = 0 \\ ii) & \forall\, x \in D,\ \forall\, p \in N_D(x),\ \ \sigma(F(x), -p) \leq 0 \end{cases}$$

Example Let $S \subset X$ be a closed subset (the switching set), with which we associate the reset map $R := \mathbf{1} + S$ defined by

$$\forall\, x \in X,\ \ R(x) := x + S$$

Then a closed subset $L \subset K$ is a viable under $(F, \mathbf{1} + S)$ if and only if

$$L = \mathrm{Capt}_F^K((L - S) \cap K)$$

If not, assuming that the subsets $(K+B-x)\cap S$ when x ranges over K are compact, the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ of K is the largest closed subset contained in K viable under $(F,\mathbf{1}+S)$, equal to the largest fixed set

$$\mathrm{Reset}_{(F,\mathbf{1}+S)}(K) \;=\; \mathrm{Capt}_F^K((\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S)\cap K)$$

and also equal to the intersection of the closed subsets K_n defined recursively by

$$K_{n+1} \;:=\; K_n \cap \mathrm{Capt}_F^K((K_n - S)\cap K)$$

If we assume furthermore that K is backward invariant under F, then the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ is the **unique** closed subset $D\subset K$ containing $(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S)\cap K$ which is backward invariant and such that $D\backslash(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S)$ is locally viable.

When F is assumed to be also Lipschitz, we deduce from the characterization of invariant subsets that the reset kernel $\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)$ is the **unique** closed subset $D\subset K$ containing $(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S)\cap K$ satisfying

$$\left\{\begin{array}{ll} i) & \forall\, x\in D,\;\; F(x) \;\subset\; -T_D(x) \\ ii) & \forall\, x\in D\backslash(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S),\;\; F(x)\cap T_D(x) \;\neq\; \emptyset \end{array}\right.$$

or, in normal form,

$$\left\{\begin{array}{ll} i) & \forall\, x\in D,\; \forall\, p\in N_D(x),\;\; \sigma(F(x),-p) \;\leq\; 0 \\ ii) & \forall\, x\in D\backslash(\mathrm{Reset}_{(F,\mathbf{1}+S)}(K)-S),\;\; \forall\, p\in N_D(x),\;\; \sigma(F(x),-p) \;=\; 0 \end{array}\right.$$

4 Optimal Impulse Control

4.1 The Auxiliary System

The evolution of a control problem (P,f) with a priori feedback map $P: X \rightsquigarrow \mathcal{P}$ from X to some finite dimensional vector space $\mathcal{P}$ is governed by

$$\left\{\begin{array}{lll} i) & x'(t) \;=\; f(x(t),u(t)) \\ ii) & u(t) \;\in\; P(x(t)) \end{array}\right. \tag{1}$$

Denote by $\mathcal{S}_{(P,f)}(x_0)$ the set of pairs $(x(\cdot),u(\cdot))$ solutions to the control problem (1) starting from x_0 at time 0, i.e., such that $x(0)=x_0$.

Let us introduce now a nonnegative lower semicontinuous "Lagrangian"

$$l:(x,v)\in \mathrm{Graph}(P)\mapsto l(x,v)\in \mathbf{R}_+$$

assumed to be convex with respect to u and to have linear growth

$$l(x,u) \;\leq\; c(\|x\|+\|u\|+1)$$

Let us introduce a lower semicontinuous "cost function" $\mathbf{w}: X\mapsto \mathbf{R}\cup\{+\infty\}$ and a Lagrangian $l:\mathrm{Graph}(P)\mapsto \mathbf{R}$.

We shall characterize the value function

$$\left\{\begin{array}{l} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \;:= \\ \displaystyle\inf_{(x(\cdot),u(\cdot))\in\mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau),u(\tau))d\tau\right) \end{array}\right.$$

of the control problem (1) under the Lagrangian l and the cost function $\mathbf{w}$ by proving that its epigraph is the reset kernel of $K \times \mathbf{R}_+$ under the auxiliary impulse differential inclusion (G, R) we now define.

We associate with the control system (P, f) and the Lagrangian l the set-valued map $G : X \times \mathbf{R}_+ \rightsquigarrow X \times \mathbf{R}$ defined by

$$G(x,y) \ := \ \{\{f(x,u)\} \times (ay + [-c(\|x\| + \|u\| + 1), -l(x,u)])\}_{u \in P(x)}$$

which is a Marchaud map whenever (P, f) is Marchaud and the Lagrangian l satisfies the above assumptions.

We shall also associate with the function $\mathbf{w}$ the auxiliary reset map $R : X \times \mathbf{R} \rightsquigarrow X \times \mathbf{R}$ defined by

$$R(x,y) \ := \ (x,y) + \mathcal{H}yp(\mathbf{w})$$

associated with the constant switching map $\mathcal{H}yp(\mathbf{w})$.

In summary, we asociate with the optimal impulse control system the auxiliary impulse differential inclusion (G, R) on the vector space $X \times \mathbf{R}$. We shall prove that the reset kernel of $X \times \mathbf{R}_+$ under (G, R) is the epigraph of the value function of the optimal impulse control problem and translate the properties of the reset kernel for obtaining corresponding properties of the value function.

Since the reset kernel involves the inverse image $R^{-1}(\mathcal{E}p(\mathbf{v}))$ and the capture basin of the epigraph of a function $\mathbf{u}$, we shall first show that they are respectively the epigraph of an inf-convolution and the epigraph of a stopping time.

5 Inf-Convolution of Functions

We associate with any extended functions[11] $\mathbf{u}$ and $\mathbf{w}$ the extended function $\mathbf{u}*\mathbf{w}$ defined by

$$(\mathbf{u} * \mathbf{w})(x) \ := \ \inf_{y \in X} (\mathbf{u}(y) + \mathbf{w}(y - x)) \ := \ \inf_{y \in X} (\mathbf{u}(x + y) + \mathbf{w}(y))$$

We shall prove that under adequate assumptions, the inverse image $R^{-1}(\mathcal{E}p(\mathbf{u}))$ of the epigraph of a lower semicontinuous nonnegative extended function u is the epigraph of $\mathbf{u} * \mathbf{w}$:

Theorem 5.1 *Let us assume that* $\mathbf{u} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *is lower semicontinuous and nonnegative and that* $\mathbf{w} : X \mapsto \mathbf{R} \cup \{\infty\}$ *is lower semicontinuous, nonnegative and coercive in the sense that*

$$c \ := \ \liminf_{\|y\| \to +\infty} \frac{\mathbf{w}(y)}{\|y\|} \ > \ 0$$

Then the function $\mathbf{u} * \mathbf{w}$ *is lower semicontinuous and*

$$\begin{cases} i) & R^{-1}(\mathcal{E}p(\mathbf{u})) \ = \ \mathcal{E}p(\mathbf{u} * \mathbf{w}) \\ ii) & R^{-1}(\mathcal{E}p(\mathbf{u})) \cap \mathcal{E}p(\mathbf{u}) \ = \ \mathcal{E}p(\max(\mathbf{u}, \mathbf{u} * \mathbf{w})) \\ iii) & \mathcal{E}p(\mathbf{u}) \backslash R^{-1}(\mathcal{E}p(\mathbf{u})) \ = \ \mathcal{E}_0 \ := \ \{(x,y) \in X \times \mathbf{R} \quad \textit{such that} \\ & \mathbf{u}(x) \ \leq \ y \ < \ (\mathbf{u} * \mathbf{w})(x)\} \end{cases}$$

[11]The inf-convolution or episum $\mathbf{u} \oplus \mathbf{w}$ of the functions $\mathbf{u}$ and $\mathbf{w}$ is defined by

$$(\mathbf{u} * \mathbf{w})(x) \ := \ \inf_{y \in X} (\mathbf{u}(y) + \mathbf{w}(x - y)) \ := \ \inf_{y \in X} (\mathbf{u}(x - y) + \mathbf{w}(y))$$

The impulse optimal control problem leads instead to the functions $\mathbf{u}$ and $\mathbf{w}$, which have the same kind of properties.

Furthermore, for every $x \in \mathrm{Dom}(\mathbf{u} * \mathbf{w})$, *the set*

$$R_{\mathbf{w}}(x) \;:=\; \{y \in X \mid (\mathbf{u} * \mathbf{w})(x) \;=\; \mathbf{u}(x+y) + \mathbf{w}(y)\}$$

is not empty.

Proof — We first observe that

$$R^{-1}(\mathcal{E}p(\mathbf{u})) \;=\; \mathcal{E}p(\mathbf{u}) - \mathcal{H}yp(-\mathbf{w})$$

and consequently, that

$$R^{-1}(\mathcal{E}p(\mathbf{u})) \;\subset\; \mathcal{E}p(\mathbf{u} * \mathbf{w})$$

Before proving the converse inclusion, let us prove that the function $\mathbf{u} * \mathbf{w}$ is lower semicontinuous. For that purpose, we have to check that its epigraph is closed. Let (x_n, λ_n) be a sequence of the epigraph of $\mathbf{u} * \mathbf{w}$ converging to (x, λ). By definition of the infimum, we can associate y_n such that

$$\forall\, n \geq 0,\;\; \mathbf{u}(y_n + x_n)\mathbf{w}(y_n) \;\leq\; (\mathbf{u} * \mathbf{w})(x_n) + \frac{1}{n} \;\leq\; \lambda + 1 \;=:\; \alpha$$

Since $\mathbf{u}$ is nonnegative, we infer that

$$\frac{\mathbf{w}(y_n)}{\|y_n\|} \;\leq\; \frac{\alpha}{\|y_n\|}$$

and since $\mathbf{w}$ is coercive, that there exists $R > 0$ such that for every $\|y_n\| \geq R$,

$$\frac{c}{2} \;\leq\; \frac{\mathbf{w}(y_n)}{\|y_n\|} \;\leq\; \frac{\alpha}{\|y_n\|}$$

Hence we infer that

$$\forall\, n \geq 0,\;\; \|y_n\| \;\leq\; \max\left(R, \frac{2\alpha}{c}\right)$$

Since X is finite dimensional, this bounded sequence is compact, so that a subsequence (again denoted by) y_n converges to some y. The function $(x, y) \mapsto \mathbf{u}(x+y) + \mathbf{w}(y)$ being lower semicontinuous, we infer by taking the limit that

$$(\mathbf{u} * \mathbf{w})(x) \;\leq\; \mathbf{u}(x+y) + \mathbf{w}(y) \;\leq\; \lambda$$

so that (x, λ) belongs to the epigraph of $\mathbf{u} * \mathbf{w}$, which is then closed.

In particular, taking $x_n := x$ and $\lambda := (\mathbf{u} * \mathbf{w})(x)$, we deduce that

$$(\mathbf{u} * \mathbf{w})(x) \;=\; \mathbf{u}(x+y) + \mathbf{w}(y)$$

so that the limit y achieves the infimum of the minimization problem $(\mathbf{u} * \mathbf{w})(x)$. This also implies that

$$(x, (\mathbf{u} * \mathbf{w})(x)) \;=\; (x+y, \mathbf{u}(x+y)) - (y, -\mathbf{w}(y))$$

and thus, that $R^{-1}(\mathcal{E}p(\mathbf{u}))$ coincides with the epigraph of $\mathbf{u} * \mathbf{w}$.

The other properties are obvious consequences. $\square$

Corollary 5.2 *We posit the assumptions of Theorem 5.1 and we assume furthermore that* $\inf_{y\in X} \mathbf{w}(y) > 0$, *then the* continuation set *defined by*

$$\mathbf{C} \ := \ \{x \in X \mid \mathbf{u}(x) \ < \ (\mathbf{u} * \mathbf{w})(x)\}$$

is not empty.

Proof — The continuation set is the projection onto X of the set $\mathcal{E}p(\mathbf{u})\backslash R^{-1}(\mathcal{E}p(\mathbf{u}))$ of $\mathcal{E}p(\mathbf{u})$ under the reset map R.

Assume that the continuation subset $\mathbf{C}$ is empty. Then for every $x \in X$, we should have

$$\mathbf{u}(x) \ = \ (\mathbf{u} * \mathbf{w})(x) \ \geq \ \inf_{y\in X} \mathbf{w}(y) + \inf_{y\in X} \mathbf{u}(y)$$

so that, by taking the infimum on the left hand-side of this inequality, we obtain $\inf_{y\in X} \mathbf{w}(y) = 0$, a contradiction. □

We also observe that by defining recursively the extended function $\mathbf{u}_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ by the discrete viability algorithm

$$\mathcal{E}p(\mathbf{u}_{n+1}) \ := \ \mathcal{E}p(\mathbf{u}_n) \cap R^{-1}(\mathcal{E}p(\mathbf{u}_n))$$

we obtain

$$\mathbf{u}_n(x) \ = \ \max\left(\mathbf{u}(x), \inf_{y\in Y}(\mathbf{w}(y) + \mathbf{v}(x+y), \ldots, \inf_{y_j\in X,\, j=1,\ldots n}\left(\sum_{j=1}^{n} \mathbf{w}(y_j) + \mathbf{v}\left(x + \sum_{j=1}^{n} y_j\right)\right)\right)$$

Therefore,

$$\forall\, x \in X, \ \ \mathbf{u}_n(x) \ \geq \ n \inf_{x\in X} \mathbf{w}(x) + \inf_{x\in X} \mathbf{u}(x)$$

If we assume that $\inf_{y\in X} \mathbf{w}(y) > 0$, then

$$\mathbf{u}_\infty(x) \ := \ \sup_{n\geq 0} \mathbf{u}_n(x)$$

is the constant function $+\infty$. In other words, the epigraph of $\mathbf{u}$ is a repeller under the discrete dynamical system defined by $R(x, y) := (x, y) + \mathcal{H}yp(-\mathbf{w})$, since its discrete viability kernel, being the epigraph of $\mathbf{u}_\infty = +\infty$.

This implies that the set $\mathcal{E}p(\mathbf{u})\backslash R^{-1}(\mathcal{E}p(\mathbf{u}))$ is not empty.

Actually, the subsets

$$\mathcal{E}_n \ := \ \{(x, y) \in X \times \mathbf{R} \mid \mathbf{u}_n(x) \ \leq \ y \ < \ \mathbf{u}_{n+1}(x)\}$$

form a partition of the subset

$$\mathcal{E}_0 \ = \ \bigcup_{n\geq 1} \mathcal{E}_n$$

Let us associate with any $x \in X$ the integer $n(x)$ such that $\mathbf{u}_1(x) = \mathbf{u}_{n(x)}(x) < \mathbf{u}_{n(x)+1}(x)$ and the map $\vec{R}_{\mathbf{w}}$ defined by

$$\vec{R}_{\mathbf{w}}(x) \ := \ \left\{\sum_{j=1}^{n(x)} y_j \mid \mathbf{u}_{n(x)}(x) \ = \ \sum_{j=1}^{n(x)} \mathbf{w}(y_j) + \mathbf{u}\left(x + \sum_{j=1}^{n(x)} y_j\right)\right\}$$

Therefore, for any x belonging to the stopping set $\mathbf{S}$ defined by

$$\mathbf{S} \ := \ \{x \in X \ \text{ such that } \ \mathbf{u}(x) \ = \ (\mathbf{u} * \mathbf{w})(x)\}$$

we have

$$\mathbf{u}(\vec{R}_{\mathbf{w}}(x)) \ < \ (\mathbf{u} * \mathbf{w})(\vec{R}_{\mathbf{w}}(x))$$

5.1 Stopping Time Value Function

Let us consider a function[12] $\mathbf{u} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$. We associate with it the stopping time problem

$$\Gamma_{(P,f,l)}(\mathbf{u})(x) \ := \ \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \left(e^{-at}\mathbf{u}(x(t)) + \int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau \right)$$

The function $\Gamma_{(P,f,l)}(\mathbf{u})$ is called the stopping time function associated with $\mathbf{u}$.

We shall characterize its epigraph:

Proposition 5.3 *Let us assume that the control system (P,f) is Marchaud, that the Lagrangian $l : \text{Graph}(P) \leadsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, nonnegative, lower semicontinuous, convex with respect to u and has linear growth*

$$l(x,u) \ \leq \ c(\|x\| + \|u\| + 1)$$

and that $\mathbf{u} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is nontrivial, non negative and lower semicontinuous.

Then the capture basin $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ of the epigraph of $\mathbf{u}$ under G is the epigraph of the stopping time function $\Gamma_{(P,f,l)}(\mathbf{u})$, which is then lower semicontinuous.

Proof — To say that a pair (x,y) belongs to the capture basin $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ means that there exist $t \geq 0$ and a solution $(x(\cdot),u(\cdot)) \in \mathcal{S}_{(P,f)}(x)$ such that

$$\left(x(t), e^{at}y - \int_0^t e^{a(t-\tau)} l(x(\tau),u(\tau))d\tau \right) \ \in \ \mathcal{E}p(\mathbf{u})$$

i.e., if and only if

$$e^{-at}\mathbf{u}(x(t)) + \int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau \ \leq \ y$$

This implies that

$$\begin{cases} \Gamma_{(P,f,l)}(\mathbf{u})(x) \\ := \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \left(e^{-at}\mathbf{u}(x(t)) + \int_0^t l e^{-a\tau}(x(\tau),u(\tau))d\tau \right) \ \leq \ y \end{cases}$$

and thus, that $\text{Capt}_G(\mathcal{E}p(\mathbf{u}))$ is contained in $\mathcal{E}p(\Gamma_{(P,f,l)}(\mathbf{u}))$.

Since F is Marchaud, the subset $\widehat{\mathcal{S}}_{(P,f)}(x)$ of solutions $(x(\cdot),u(\cdot))$ to control system (1) is compact in the space $\mathcal{C}(0,T,X) \times L^1(0,T,X)$ where $\mathcal{C}(0,T,X)$ supplied with the uniform convergence and $L^1(0,T,X)$ with the weakened topologu. On the other hand, the functional

$$(x(\cdot),u(\cdot)) \mapsto \int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau$$

is lower semicontinuous on $\mathcal{C}(0,\infty,X) \times L^1(0,\infty,\mathcal{P})$ when L^1 is supplied with the weakened topology (see for instance Proposition 6.3.4 of [8, Aubin]). Then the infimum

$$\Gamma_{(P,f,l)}(\mathbf{u})(x) \ := \ e^{-a\bar{t}}\mathbf{u}(\bar{x}(\bar{t})) + \int_0^{\bar{t}} e^{-a\tau} l(\bar{x}(\tau),\bar{u}(\tau))d\tau$$

is reached by a solution $(\bar{x}(\cdot),\bar{u}(\cdot)) \in \mathcal{S}_{(P,f)}(x)$ and a time $\bar{t} \geq 0$.

[12]regarded as an "obstacle" in problems of unilateral mechanics.

Therefore, if $y \geq \Gamma_{(P,f,l)}(\mathbf{u})(x)$, then the pair

$$\left(\bar{x}(t), \bar{y}(t) := e^{at}y - \int_0^t e^{a(t-s)} l(\bar{x}(s), \bar{u}(s))ds\right)$$

is a solution to the differential inclusion $(x', y') \in G(x, y)$ that reaches $\mathcal{E}p(\mathbf{u})$ at time $\bar{t}$ because

$$\begin{cases} \bar{y}(\bar{t}) \ \geq \ e^{at}\Gamma_{(P,f,l)}(\mathbf{u})(T, x) - \int_0^{\bar{t}} e^{a(\bar{t}-s)} l(\bar{x}(s), \bar{u}(s))ds \\ = \ e^{a\bar{t}} \left(e^{-a\bar{t}}\mathbf{u}(\bar{x}(\bar{t})) + \int_0^{\bar{t}} e^{-a\tau} l(\bar{x}(\tau), \bar{u}(\tau))d\tau \ = \ u(\bar{x}(\bar{t}))\right) \end{cases}$$

This states that (x, y) belongs to the capture basin of $\mathbf{R}_+ \times \mathcal{E}p(\mathbf{u})$, which is then equal to the epigraph of $\Gamma_{(P,f,l)}(\mathbf{u})$. Being closed when the system is Marchaud, this means that the value function is lower semicontinuous. □

Theorem 5.4 *We posit the assumptions of Propposition 5.3 and we assume that*

$$\forall\, x \in X, \quad \inf_{(x(\cdot),u(\cdot)) \in \mathcal{S}_F(x)} \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau \ = \ +\infty$$

Then the stopping time function $\mathbf{v}_\propto := \Gamma_{(P,f,l)}(\mathbf{u})$ *is characterized as the smallest of the nonnegative lower semicontinuous functions* $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *satisfying for every* x

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \\ & \inf_{u \in P(x)} \left(D_\uparrow \mathbf{v}(x)(f(x, u)) + l(x, u)\right) - a\mathbf{v}(x) \ \leq \ 0 \end{cases}$$

Or, equivalently, in a dual form, denoting by

$$H(x, y, p) \ := \ \sup_{u \in P(x)} \left(\langle p, f(x, u)\rangle + l(x, u)\right) - ay \ = \ \sigma(G(x, y), (p, -1))$$

the Hamiltonian associated with the optimal control problem, the stopping time function is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v} : X \mapsto \mathbf{R} \cup \{+\infty\}$ *satisfying for every* x

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \ \forall\, p \in \partial_- \mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ \geq \ 0 \\ & \text{and } \forall\, p \in \partial_-^\infty \mathbf{v}(x), \ \ \sigma(f(x, P(x)), p) \ \geq \ 0 \end{cases}$$

If we assume furthermore that P*,* f *and* l *are Lipschitz, then the stopping time function is the unique Frankowska solution* $\mathbf{v} \geq 0$ *to the system of differential inequalities: for every* $x \in \mathrm{Dom}(\mathbf{v})$*,*

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \sup_{u \in P(x)} (D_\uparrow \mathbf{v}(x)(-f(x, u)) - l(x, u)) + a\mathbf{v}(x) \ \leq \ 0 \\ iii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \ \ \inf_{u \in P(x)} \left(D_\uparrow \mathbf{v}(x)(f(x, u)) + l(x, u)\right) - a\mathbf{v}(x) \ \leq \ 0 \end{cases}$$

or, equivalently, in dual form,

$$\begin{cases} i) & 0 \ \leq \ \mathbf{v}(x) \ \leq \ \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_- \mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ \leq \ 0 \\ & \text{and } \forall\, p \in \partial_-^\infty \mathbf{v}(x), \ \ \sigma(f(x, P(x)), p) \ \leq \ 0 \\ iii) & \text{if } \mathbf{v}(x) < \mathbf{u}(x), \ \ \forall\, p \in \partial_- \mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ = \ 0 \\ & \text{and } \forall\, p \in \partial_-^\infty \mathbf{v}(x), \ \ \sigma(f(x, P(x)), p) \ = \ 0 \end{cases}$$

which can reformulated in the form of "variational inequalities"[13]*:*

$$\begin{cases} i) & 0 \ \le \ \mathbf{v}(x) \ \le \ \mathbf{u}(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p) \ \le \ 0 \\ & \text{and} \ \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \ \sigma(f(x, P(x)), p) \ \le \ 0 \\ iii) & \forall\, p \in \partial_{-}\mathbf{v}(x), \ \ H(x, \mathbf{v}(x), p)(\mathbf{v}(x) - \mathbf{u}(x)) \ = \ 0 \\ & \text{and} \ \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \ \sigma(f(x, P(x)), p)(\mathbf{v}(x) - \mathbf{u}(x)) \ = \ 0 \end{cases}$$

Knowing the stopping time function, an optimal solution is obtained in the following way. Starting from x_0 *such that* $\mathbf{v}(x_0) < \mathbf{u}(x_0)$*, we choose a solution to the control system*

$$\begin{cases} i) & x'(t) \ = \ f(x(t), u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}_{(P,f,l)}(x(t)) \end{cases}$$

where

$$\mathbf{R}_{(P,f,l)}(x) \ := \ \{u \in P(x) \mid D_{\uparrow}\mathbf{v}_{(P,f,l)}(x)(f(x,u)) + l(x,u) - a\mathbf{v}(x) \le 0\}$$

until the first time $\bar{t} \ge 0$ *when* $\mathbf{v}(x(\bar{t})) = \mathbf{u}(x(\bar{t}))$*.*

5.2 Value Function of an Optimal Impulse Control

We shall characterize the value function

$$\begin{cases} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \ := \\ \inf_{(x(\cdot),u(\cdot)) \in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

of the control problem (1) under the Lagrangian l and the cost function $\mathbf{w}$ by proving that its epigraph is the reset kernel of $X \times \mathbf{R}_+$ under the auxiliary differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ with the constant switching map $S = \mathcal{H}yp(-\mathbf{w})$:

Theorem 5.5 *Let us assume that the control system* (P, f) *is Marchaud, that the Lagrangian* $l : \mathrm{Graph}(P) \rightsquigarrow \mathbf{R}_+ \cup \{+\infty\}$ *is nontrivial, nonnegative, lower semicontinuous, convex with respect to* u *and has linear growth*

$$l(x, u) \ \le \ c(\|x\| + \|u\| + 1)$$

that K *is closed and that* $\mathbf{w} : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *is nontrivial, lower semicontinuous and satisifes* $\inf_{y \in \mathbf{w}(y)} > 0$*.*

Then the epigraph of the value function

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \inf_{(x(\cdot),u(\cdot)) \in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right)$$

for the optimal impulse control problem (1) under the Lagrangian l *and the cost function* $\mathbf{w}$ *is the reset kernel of* $K \times \mathbf{R}_+$ *under the impulse differential inclusion* $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$ *and the optimal runs are the runs of the impulse differential inclusion* $(x'(t), y'(t)) \in G(x(t), y(t))$ *viable in* $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$*.*

[13] See for instance [17, Bensoussan & Lions J.-L.]. The subset $\{x \in X \mid \mathbf{v}(x) < \mathbf{u}(x)\}$ is called the continuation set and the subset $\{x \in X \mid \mathbf{v}(x) = \mathbf{u}(x)\}$ is called the stopping set.

Proof — Let us consider a pair (x_0, y_0) in the reset kernel of $K \times \mathbf{R}_+$ under the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$. Then there exists a run $(x(\cdot), y(\cdot))$ starting from (x_0, y_0) viable in $K \times \mathbf{R}_+$.

A run to the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$ can be written in the form

$$\forall\, t \in [t_n, t_{n+1}[, \begin{cases} x(t) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau), u(\tau)) d\tau \\ y(t) \ \leq \ e^{at} \left(y_0 + \sum_{k=1}^{n} e^{-at_k} \eta_k - \int_0^t e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

and

$$\begin{cases} i) \quad x(t_{n+1}) \ = \ x_0 + \sum_{k=1}^{n+1} \xi_k + \int_0^{t_{n+1}} f(x(\tau), u(\tau)) d\tau \\ ii) \ \ y(t_{n+1}) \ \leq \ e^{at_{n+1}} \left(y_0 - \sum_{k=1}^{n+1} e^{-at_k} \eta_k - \int_0^{t_{n+1}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

where, for every integer k, $\eta_k \leq -\mathbf{w}(\xi_k)$.

Indeed, assuming the above formula to be true, then, for every $t \in [t_{n+1}, t_{n+2}[$,

$$\begin{cases} y(t) \ \leq \ e^{a(t - t_{n+1})} y(t_{n+1}) - \int_{t_{n+1}}^t e^{a(t-\tau)} l(x(\tau), u(\tau)) d\tau \\ = \ e^{at} \left(y_0 + \sum_{k=1}^{n+1} e^{-at_k} \eta_k - \int_0^{t_{n+1}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \\ - \int_{t_{n+1}}^t e^{a(t-\tau)} l(x(\tau), u(\tau)) d\tau \\ = \ e^{at} \left(y_0 + \sum_{k=1}^{n+1} e^{-at_k} \eta_k - \int_0^{t} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

and when $t = t_{n+2}$, we take

$$\begin{cases} x(t_{n+2}) \ = \ x(^-t_{n+2}) + \xi_{n+2} \\ y(t_{n_2}) \ \leq \ y(^-t_{n+2}) + \eta_{n+2} \\ = \ e^{at_{n+2}} \left(y_0 + \sum_{k=1}^{n+1} e^{-at_k} \eta_k - \int_0^{t_{n+2}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) + \eta_{n+2} \\ = \ e^{at_{n+2}} \left(y_0 + \sum_{k=1}^{n+2} e^{-at_k} \eta_k - \int_0^{t_{n+2}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \right) \end{cases}$$

Therefore, a run to the impulse differential inclusion $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$ viable in $K \times \mathbf{R}_+$ satisfies

$$\forall\, t \in [t_n, t_{n+1}[, \begin{cases} x(t) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau), u(\tau)) d\tau \\ e^{-at} y(t) + \sum_{k=1}^{n} e^{-at_k} \mathbf{w}(\xi_k) + \int_0^t e^{-a\tau} l(x(\tau), u(\tau)) d\tau \ \leq \ y_0 \end{cases}$$

and, at switching times t_{n+1},

$$\begin{cases} i) \quad x(t_{n+1}) \ = \ x_0 + \sum_{k=1}^{n+1} \xi_k + \int_0^{t_{n+1}} f(x(\tau), u(\tau)) d\tau \\ ii) \ \ e^{-at_{n+1}} y_{n+1}(t) + \sum_{k=1}^{n+1} e^{-at_k} \mathbf{w}(\xi_k) + \int_0^{t_{n+1}} e^{-a\tau} l(x(\tau), u(\tau)) d\tau \ \leq \ y_0 \end{cases}$$

Since the run $(x(\cdot), y(\cdot))$ is viable in $K \times \mathbf{R}_+$, we infer that $y(t) \geq 0$, and thus, that

$$\forall\, t \in [t_n, t_{n+1}[,\ \ \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau} l(x(\tau), u(\tau))d\tau \ \leq\ y_0$$

The run cannot stop in finite time t_n where $t_{n+j} = t_n$ for any $j \in \mathbf{N}$. Indeed, in this case, we would obtain

$$\sum_{k=1}^{n-1} e^{-at_k}\mathbf{w}(\xi_k) + e^{-at_n}\sum_{j=1}^{+\infty}\mathbf{w}(\xi_{n+j}) + \int_0^{t_n} e^{-a\tau} l(x(\tau), u(\tau))d\tau \ \leq\ y_0$$

Since $\mathbf{w}(\xi_{n_j}) \geq c := \inf_y \mathbf{w}(y) > 0$, the left hand-side of the above inequality would go to $+\infty$ whereas the right hand-side is bounded.

We denote by $n[t]$ the largest switching time smaller than or equal to t. Since the cost function $\mathbf{w}$ and the Lagrangian l are nonnegative, the function

$$\sum_{k=1}^{n[t]} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau} l(x(\tau), u(\tau))d\tau$$

is non decreasing, so that, taking the limit when $t \to +\infty$, we infer that the optimal cost function

$$\begin{cases} \mathbf{v}_{(P,f,l,\mathbf{w})}(x) \ := \\ \displaystyle\inf_{(x(\cdot),u(\cdot)) \in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau \right) \end{cases}$$

satisfies

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) \ \leq\ y_0$$

Therefore, the reset kernel of $K \times \mathbf{R}_+$ is contained in the epigraph of the value function $\mathbf{v}_{(P,f,l,\mathbf{w})}$.

Under the assumptions of the theorem, the infimum in the optimal control problem

$$\mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\bar{\xi}_k) + \int_0^{+\infty} e^{-a\tau} l(\bar{x}(\tau), \bar{u}(\tau))d\tau$$

is achieved at some run $(\bar{x}(\cdot), \bar{u}(\cdot))$. We set for all $t \in [t_n, t_{n+1}[$,

$$\bar{y}(t) \ :=\ e^{at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x) - \sum_{k=1}^{n} e^{a(t-t_k)}\mathbf{w}(\bar{\xi}_k) - \int_0^t e^{a(t-\tau)} l(\bar{x}(\tau), \bar{u}(\tau))d\tau$$

and we deduce that the pair $(\bar{x}(\cdot), \bar{y}(\cdot))$ is a run of the impulse differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ viable in $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$. This implies that the graph of the value function $\mathbf{v}_{(P,f,l,\mathbf{w})}$ is contained in the reset kernel of $K \times \mathbf{R}_+$, so that the epigraph of the value function is actually equal to this reset kernel.

We also deduce that the runs viable in the epigraph of the value function are the runs satisfying

1. for any switching time t_n, we have $y(^-t_n) = \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n))$ and

$$\begin{cases} \mathbf{v}_{(P,f,l,\mathbf{w})}(x(t_n)) \ =\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n) + \xi_n) \\ \leq\ y(t_n) \ =\ y(^-t_n) + \eta_n \ \leq\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)) - \mathbf{w}(\xi_n) \end{cases}$$

and thus

$$\begin{cases} (\mathbf{v}_{(P,f,l,\mathbf{w})} * \mathbf{w})(x(^-t_n)) \\ \leq\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n) + \xi_n) + \mathbf{w}(\xi_n) \ \leq\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x(^-t_n)) \end{cases}$$

2. for any $t \in [t_n, t_{n+1}[$,

$$\begin{cases} e^{-at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x(t)) \ \leq \ e^{-at}y(t) \ = \\ \mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) - \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\eta_k) - \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau \end{cases}$$

This amounts to saying that for every $t \in [t_n, t_{n+1}[$,

$$\begin{cases} x(t) \ = \ x_0 + \sum_{k=1}^{n} \xi_k + \int_0^t f(x(\tau),u(\tau))d\tau \\ e^{-at}\mathbf{v}_{(P,f,l,\mathbf{w})}(x(t)) + \sum_{k=1}^{n} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^t e^{-a\tau}l(x(\tau),u(\tau))d\tau \\ \leq \ \mathbf{v}_{(P,f,l,\mathbf{w})}(x_0) \end{cases}$$

Therefore, the optimal runs are the runs of the of the impulse differential inclusion $(x'(t), y'(t)) \in G(x(t), y(t))$ viable in $\mathcal{E}p(\mathbf{v}_{(P,f,l,\mathbf{w})})$. $\square$

The epigraph of the value function of an optimal impulse control problem being the reset kernel under an impulse differential inclusion, it enjoys the characterizations and the properties of reset kernels which we now translate in the control framework.

5.3 Characterizations of the Value Function

Since under the assumptions of Theorem 5.5, the epigraph of the value function of an optimal impulse control problem is the reset kernel of $K \times \mathbf{R}_+$ under $(G, \mathbf{1} + \mathcal{H}yp(-\mathbf{w}))$, we can translate the properties of reset kernels into corresponding properties of the value function.

We begin by introducing notations of nonsmooth analysis: We shall denote by $\mathbf{u}|_K$, the restriction of $\mathbf{u}$ to K, the function equal to $\mathbf{u}$ on K and to $+\infty$ outside K. We recall that the epiderivative of the restriction of $\mathbf{u}$ to K at $x \in K$ is the restriction of the epiderivative to the contingent cone

$$\forall\, x \in K, \ \ D_{\uparrow}\mathbf{u}|_K(x) \ = \ D\mathbf{u}(x)|_{T_K(x)}$$

under constraint qualification assumptions when $\mathbf{u}$ and K are convex and under transversality conditions in the nonconvex case. See for instance Chapter 6 of [9, Aubin & Frankowska] for more details. We also recall that the subdifferential of the restriction of $\mathbf{u}$ to K at $x \in K$ is the sum of the subdifferential and of the normal cone

$$\forall\, x \in K, \ \ \partial_{-}\mathbf{u}|_K(x) \ = \ \partial_{-}\mathbf{u}(x) + N_K(x)$$

under constraint qualification assumptions when $\mathbf{u}$ and K are convex and under transversality conditions in the nonconvex case.

We also need to assume that $X \times \mathbf{R}_+$ is a repller under G: This is the topic of the following

Lemma 5.6 *Assume that the Lagrangian is nonnegative. The closed subset $X \times \mathbf{R}_+$ is backward invariant under G and the closed subset $K \times \mathbf{R}_+$ is backward invariant under G if and only if K is backward invariant under the control problem (1). It is a repeller under G whenever*

$$\forall\, x \in X, \ \ \inf_{(x(\cdot),u(\cdot)) \in \mathcal{S}_F(x)} \int_0^{+\infty} e^{-a\tau}l(x(\tau),u(\tau))d\tau \ = \ +\infty$$

One can provide a sufficient condition for $X \times \mathbf{R}_+$ to be a repeller under the auxilairy differential inclusion G:

Lemma 5.7 *Let us assume that*

$$\left\{ \begin{array}{lll} i) & \inf_{x\in X} \inf_{u\in P(x)} \frac{\langle x, f(x,u)\rangle}{\|x\|} & \geq \gamma(\|x\|+1) \\ ii) & \inf_{x\in X} \inf_{u\in P(x)} l(x,u) & \geq \delta(\|x\|+1) \end{array} \right. \tag{2}$$

If $a < \gamma$, then $X \times \mathbf{R}_+$ is a repeller.

Proof — Let $(x(\cdot), y(\cdot))$ be a solution to the differential inclusion $(x', y') \in G(x,y)$ starting from (x_0, y_0). Therefore

$$\frac{d}{dt}\|x(t)\| = \left\langle x'(t), \frac{x(t)}{\|x(t)\|} \right\rangle = \left\langle f(x(t), u(t)), \frac{x(t)}{\|x(t)\|} \right\rangle \geq \gamma(\|x(t)\|+1)$$

so that

$$\forall\, t \geq 0, \ \ \|x(t)\| \geq e^{\gamma t}(\|x_0\|+1) - 1$$

Furthermore, since

$$l(x(\tau), u(\tau)) \geq \delta(\|x(\tau)\|+1) \geq \delta(\|x_0\|+1)e^{\gamma t}$$

and since

$$e^{-at}y(t) = y_0 - \int_0^t e^{-a\tau} l(x(\tau), u(\tau))d\tau$$

we infer that

$$e^{-at}y(t) \leq y_0 - \delta(\|x_0\|+1)\int_0^t e^{(\gamma-a)\tau} = y_0 - \frac{\delta(\|x_0\|+1)}{\gamma - a}\left(e^{(\gamma-a)t} - 1\right)$$

Consequently, if $a < \gamma$

$$e^{-at}y(t) \leq y_0 + \frac{\delta(\|x_0\|+1)}{\gamma - a} - \frac{\delta(\|x_0\|+1)}{\gamma - a}e^{(\gamma-a)t}$$

so that $y(t)$ becomes negative in finite time. □

We now have in our hands all what we need to prove the following

Theorem 5.8 *We posit the assumptions of Theorem 5.5 and we assume that* $\mathbf{w}$ *is coercive, that* $\inf_{y\in X} \mathbf{w}(y) > 0$ *and that*

$$\forall\, x \in K, \ \ \inf_{(x(\cdot),u(\cdot))\in \mathcal{S}_F(x)} \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau = +\infty$$

Then the value function $\mathbf{v}_\propto := \mathbf{v}_{(P,f,l,\mathbf{w})}$ *of the optimal impulse control problem (1) under the Lagrangian* l *and the cost function* $\mathbf{w}$ *defined by*

$$\left\{ \begin{array}{l} \mathbf{v}_\propto(x) := \mathbf{v}_{(P,f,l,\mathbf{w})}(x) := \\ \displaystyle\inf_{(x(\cdot),u(\cdot))\in \mathcal{R}^K_{(P,f,l,\mathbf{w})}(x)} \left(\sum_{k=1}^{+\infty} e^{-at_k}\mathbf{w}(\xi_k) + \int_0^{+\infty} e^{-a\tau} l(x(\tau), u(\tau))d\tau \right) \end{array} \right.$$

is the solution to the problem

$$\mathbf{v}_\propto \;=\; \Gamma_{(P,f,l)}(\mathbf{v}_\propto * \mathbf{w})$$

i.e., for every $x \in \mathrm{Dom}(\mathbf{v}_\propto)$

$$\left\{ \begin{array}{l} \mathbf{v}_\propto(x) \;=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \\ \left(e^{-at}\left(\mathbf{v}_\propto(x(t)+y)+\mathbf{w}(y)\right)+\int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau\right) \end{array} \right.$$

It is equal to the epigraphical limit of the increasing sequence of functions $\mathbf{v}_n$ *defined recursively*[14] *by* $\mathbf{v}_0 := 0$ *and*

$$\left\{ \begin{array}{l} \mathbf{v}_{n+1}(x) \;:=\; \max\left(\mathbf{v}_n(x), \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \right.\\ \left.\left(e^{-at}\left(\mathbf{v}_n(x(t)+y)+\mathbf{w}(y)\right)+\int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau\right)\right) \end{array} \right.$$

It is the smallest of the nonnegative lower semicontinuous functions $\mathbf{v}: K \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying the inequalities*

$$\left\{ \begin{array}{l} \mathbf{v}(x) \;\geq\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{S}_{(P,f)}(x)} \inf_{t\geq 0} \inf_{y\in X} \\ \left(e^{-at}\left(\mathbf{v}(x(t)+y)+\mathbf{w}(y)\right)+\int_0^t e^{-a\tau} l(x(\tau),u(\tau))d\tau\right) \end{array} \right.$$

It is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v}: K \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying for every* $x \in K$*:*

$$\left\{ \begin{array}{ll} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \text{if } \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x), \\ & \inf_{u\in P(x)}\left(D_\uparrow \mathbf{v}(x)(f(x,u)) + l(x,u)\right) - a\mathbf{v}(x) \;\leq\; 0 \end{array} \right.$$

Introducing the Hamiltonian defined by

$$H(x,y,p) \;:=\; \sup_{u\in P(x)} \left(\langle p, f(x,u)\rangle + l(x,u)\right) - ay \;=\; \sigma(G(x,y),(p,-1))$$

the value function is also characterized as the smallest of the nonnegative lower semicontinuous functions $\mathbf{v}: K \mapsto \mathbf{R}\cup\{+\infty\}$ *satisfying for every* $x \in K$

$$\left\{ \begin{array}{ll} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \text{if } \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x),\ \forall\, p \in \partial_- \mathbf{v}(x),\ \ H(x,\mathbf{v}(x),p) \;\geq\; 0 \\ & \text{and } \forall\, p \in \partial_-^\infty \mathbf{v}|_K(x),\ \ \sigma(f(x,P(x)),p) \;\geq\; 0 \end{array} \right.$$

If we assume furthermore that P, f *and* l *are Lipschitz and that* K *is backward invariant under the control system (1), then the value function is the unique Frankowska solution* $\mathbf{v}: K \mapsto \mathbf{R}_+\cup\{+\infty\}$ *to the system of "differential inequalities": for every* $x \in K$*,*

$$\left\{ \begin{array}{ll} i) & 0 \;\leq\; \mathbf{v}(x) \;\leq\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \sup_{u\in P(x)}(D_\uparrow \mathbf{v}|_K(x)(-f(x,u)) - l(x,u)) + a(x) \;\leq\; 0 \\ iii) & \text{if } \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x), \\ & \inf_{u\in P(x)}\left(D_\uparrow \mathbf{v}|_K(x)(f(x,u)) + l(x,u)\right) + a\mathbf{v}(x) \;\leq\; 0 \end{array} \right.$$

[14] Compare with [53, Tartar], reproduced in Chapter 15 of [1, Aubin].

Or, equivalently, it is the unique solution to the system

$$\begin{cases} i) & 0 \;\le\; \mathbf{v}(x) \;\le\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}|_K(x), \;\; H(x, \mathbf{v}(x), p) \;\le\; 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \;\; \sigma(f(x, P(x)), p) \;\le\; 0 \\ iii) & \text{if } \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x), \;\; \forall\, p \in \partial_{-}\mathbf{v}|_K(x), \;\; H(x, \mathbf{v}(x), p) \;=\; 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}|_K(x), \;\; \sigma(f(x, P(x)), p) \;=\; 0 \end{cases}$$

which can reformulated in the form of "quasi variational inequalities":

$$\begin{cases} i) & 0 \;\le\; \mathbf{v}(x) \;\le\; (\mathbf{v} * \mathbf{w})(x) \\ ii) & \forall\, p \in \partial_{-}\mathbf{v}|_K(x), \;\; H(x, \mathbf{v}(x), p) \;\le\; 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}(x), \;\; \sigma(f(x, P(x)), p) \;\le\; 0 \\ iii) & \forall\, p \in \partial_{-}\mathbf{v}|_K(x), \;\; H(x, \mathbf{v}(x), p)(\mathbf{v}(x) - (\mathbf{v} * \mathbf{w})(x)) \;=\; 0 \\ & \text{and } \forall\, p \in \partial_{-}^{\infty}\mathbf{v}|_K(x), \;\sigma(f(x, P(x)), p)(\mathbf{v}(x) - (\mathbf{v} * \mathbf{w})(x)) \;=\; 0 \end{cases}$$

Knowing the value function $\mathbf{v}$*, we introduce the two regulation maps* $\mathbf{R}_{(f,P)}$ *and* $\mathbf{R}_{\mathbf{w}}$ *defined by*

$$\mathbf{R}_{(f,P)}(x) \;:=\; \{u \in P(x) \mid D_{\uparrow}\mathbf{v}(x)(f(x, u)) + l(x, u) - a\mathbf{v}(x) \le 0\}$$

and

$$\mathbf{R}_{\mathbf{w}}(x) \;:=\; \{y \in X \mid \mathbf{v}(y) + \mathbf{w}(y - x) = (\mathbf{v} * \mathbf{w})(x)\}$$

Therefore, an optimal run is obtained in the following way[15]*: Starting from* x_0 *such that* $\mathbf{v}(x_0) < (\mathbf{v} * \mathbf{w})(x_0)$*, we choose a solution to the control system*

$$\begin{cases} i) & x'(t) \;=\; f(x(t), u(t)) \\ ii) & u(t) \;\in\; \mathbf{R}_{(f,P)}(x(t)) \end{cases}$$

until the time $t_1 \ge 0$ *when* $\mathbf{v}(x(^{-}t_1)) = (\mathbf{v} * \mathbf{w})(x(^{-}t_1))$.

At this stage, we take for reinitialized sate $x_1 \in \mathbf{R}_{\mathbf{w}}(x(^{-}t_1)$*, and we reiterate the process.*

When the value function is continuous and when P, f and l are Lipschitz, one can deduce from [7, Aubin] that the backward invariance of the epigraph of $\mathbf{v}$ under G is equivalent to the invariance of the hypograph of $\mathbf{v}$ under G. In normal form, this amounts to saying that

$$\forall\, p \in \partial_{-}\mathbf{v}(x), \;\; H(x, \mathbf{v}(x), p) \;\le\; 0$$

is equivalent to

$$\forall\, p \in \partial_{+}\mathbf{v}(x), \;\; H(x, \mathbf{v}(x), -p) \;\ge\; 0$$

where $\partial_{+}\mathbf{v}(x) := -\partial_{-}(-\mathbf{v})(x)$. This means that the value function is a viscosity solution (in the sense of [22, Crandall & Lions P.-L.]) to the quasi variational inequalities. For proofs of abstract theorems on existence of solutions to quasi variational inequalities, we mention the theorem [43, Joly & Mosco], derived (simply) from the Ky Fan inequality in [1, Aubin], and naturally, the book [16, 17, Bensoussan & Lions J.-L.].

[15] The subset $\{x \in K \mid \mathbf{v}(x) < (\mathbf{v} * \mathbf{w})(x)\}$ is called the continuation set and the subset $\{x \in K \mid \mathbf{v}(x) = (\mathbf{v} * \mathbf{w})(x)\}$ is called the stopping set.

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