Path-Dependent Impulse and Hybrid Systems

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Abstract

Path-dependent impulse differential inclusions*, and in particular, path-dependent hybrid control systems, are defined by a path-dependent differential inclusion (or path-dependent control system, or differential inclusion and control systems with memory) and a path-dependent reset map.*

In this paper, we characterize the viability property of a closed subset of paths under an impulse path-dependent differential inclusion using the Viability Theorems for path-dependent differential inclusions.

Actually, one of the characterizations of the Characterization Theorem is valid for any general impulse evolutionary system *that we shall defined in this paper.*

Keywords: hybrid control, impulse control, path-dependent differential inclusion, differential inclusion with memory, functional differential inclusions, viability, run, execution, Kakutani Theorem, contingent cone, Marchaud map.

Introduction

In this paper, we characterize the viability property of a closed subset of paths under an impulse path-dependent differential inclusion using the method of [2, Aubin] or [9, Aubin, Lygeros, Quincampoix, Sastry & Seube], the Path-Dependent Viability Theorems of [13, 14, 15, Haddad].

Actually, one of the characterizations of the Characterization Theorem is true for any general impulse evolutionary system that we shall define in this paper, which is based on recent results of [6, Aubin].

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We recall that hybrid control systems[3] can be embedded in the framework of impulse differential inclusions; in the same way, path-dependent hybrid systems can be regarded as instances of viable path-dependent impulse differential inclusions, and enjoy the same properties.

1 Impulse Path-Dependent Differential Inclusions

Let $X := \mathbf{R}^n$ be a finite dimensional vector space.

1.1 History Spaces

We denote by

$$\mathcal{H}(X) \ := \ \mathcal{C}(-\infty, 0; X)$$

the history (or memory, path) space .

It is supplied with the compact convergence topology. We denote by $\mathcal{H}_\lambda(X)$ the subset of Lipschitz functions with Lipschitz constant λ.

If $\mathbf{K} \subset \mathcal{H}(X)$, we set

$$\forall\, \tau \in]-\infty, 0], \ \ \mathbf{K}(\tau) \ := \ \{\varphi(\tau)\}_{\varphi \in \mathbf{K}}$$

Observe that Ascoli's Theorem states that a *closed subset* $\mathbf{K} \subset \mathcal{H}_\lambda(X)$ *is compact if and only if* $\mathbf{K}(0) := \{\varphi(0)\}_{\varphi \in \mathbf{K}}$ *is bounded*, since it is closed and equicontinuous (by assumption) and pointwise bounded because, for all $\psi \in \mathbf{K}$ and $\tau \leq 0$,

$$\|\psi(\tau)\| \leq \|\psi(\tau) - \psi(0)\| + \|\psi(0)\| \leq \lambda|\tau| + \|\mathbf{K}(0)\|$$

Our study invlves a constrained subset $\mathbf{K} \subset \mathcal{H}(X)$ made of paths or histories and of a target $\mathbf{C} \subset \mathbf{K}$.

A first example of constrained subset of paths or histories and targets associated with subsets $C \subset K \subset X$ of the vector space are given by

$$\mathbf{C} \ := \ \{ \ \varphi \in \mathcal{H}(X) \mid \varphi(0) \in C \ \} \ \subset \ \mathbf{K} \ := \ \{ \ \varphi \in \mathcal{H}(X) \mid \varphi(0) \in K \ \}$$

where the constraints bear only on the present.

Another class is given by Volterra sets defined through a "kernel" $k :]-\infty, 0] \times X \mapsto Y$ and a set-valued map $M : Y \rightsquigarrow X$ by

$$\mathbf{K} \ := \ \left\{ \varphi \in \mathcal{H}(X) \mid \varphi(0) \ \in \ M\left(\int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s) \right) \right\}$$

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[3] See for instance among many papers and books [11, Branicky, Borkar & Mitter], [10, Bensoussan & Menaldi], [16, 17, Matveev & Savkin] and [19, Shaft & Schumacher].

where the constraints involve cumulated consequences of the history.

In the discrete case,

$$\mathbf{K} \ := \ \left\{ \varphi \in \mathcal{H}(X) \mid \varphi(0) \ \in \ M\left(\sum_{j=-\infty}^{0} k(-j, \varphi(\beta_j)) \right) \right\}$$

involves discrete cumulated consequences of the history (delays).

Associated targets can be asociated with set-valued maps $P \subset M$ in the same fashion.

1.2 Histories or Paths

We associate with any continuous function $x(\cdot) \in \mathbf{C}(-\infty, +\infty; X)$ its *history (or path)*[4] *$T(t)x$ up to time t* defined by:

$$\forall\, \tau \in]-\infty, 0], \ \ T(t)x(\tau) \ := \ x(t+\tau)$$

Then $T(t)$ maps $\mathcal{C}(-\infty, +\infty; X)$ to $\mathcal{H}(X)$ and satisfies the semi-group property

$$T(t+s)x \ = \ T(s)T(t)x$$

We then observe that for any function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$, we have $x(t) = (T(t)x)(0)$.

In this continuous framework, we define the constraints of the history of the evolution through a closed subset $\mathbf{K} \ \subset \ \mathcal{H}(X)$. Viable evolutions $x(\cdot)$ with memory are the ones that satisfy

$$\forall\, t \geq 0, \ \ T(t)x \ \in \ \mathbf{K} \tag{1}$$

and an evolution $x(\cdot)$ reach a target $\mathbf{C}$ at time s if $T(s)x \in \mathbf{C}$.

For instance, an evolution is viable in $\mathbf{K} := \{\varphi \in \mathcal{H}(X) \mid \varphi(0) \in K\}$ if and only if for every $t \in]-\infty, 0], \ \ x(t) \in K$. If

$$\mathbf{K} \ := \ \left\{ \varphi \in \mathcal{H}(X) \mid \varphi(0) \ \in \ M\left(\int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s) \right) \right\}$$

then $x(\cdot)$ is viable in $\mathbf{K}$ if and only if

$$\forall\, t \geq 0, \ \ x(t) \ \in \ M\left(\int_{-\infty}^{t} k(t-s, \varphi(s)) d\mu(s) \right)$$

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[4] often denoted by $x_t := T(t)x$

1.3 Path-Dependent Differential Inclusions

Let us consider a set-valued map $F : \mathcal{H}(X) \mapsto X$ governing the continuous evolution of the state $x(t)$ through the path-dependent differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \; \in \; F(T(t)x)$$

starting at a given $\varphi \in \mathcal{H}(X)$ in the sense that

$$T(0)x \; = \; \varphi$$

i.e., for every $\tau \in]-\infty, 0], \;\; x(\tau) = \varphi(\tau)$.

We denote by $\mathcal{R}_F : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; X)$ the map associating with any initial path $\varphi \in \mathcal{H}(X)$ the set $\mathcal{R}_F(\varphi)$ of solutions $t \mapsto x(t)$ to the path-dependent differential inclusion $x'(t) \in F(T(t)x)$ starting at the initial path φ in the sense that $T(0)x = \varphi$.

Actually, we shall need the properties of the associated set-valued map $\mathcal{S}_F : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; \mathcal{H}(X))$ defined by

$$\mathcal{S}_F(\varphi) \; := \; \{t \mapsto \varphi(t) := T(t)x\}_{x(\cdot) \in \mathcal{R}_F(\varphi)}$$

Definition 1.1 *We shall say that this set-valued map $\mathcal{S}_F : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, +\infty; \mathcal{H}(X))$ is the* solution map *of F.*

The solution map $\mathcal{S}_F$ has the advantage of mapping the set $\mathcal{H}(X)$ into time-dependent functions $t \mapsto \varphi(t) := T(t)x$ that belong to the same histoy space $\mathcal{H}(X)$, even though the traditional view is to call a solution a function $t \mapsto x(t)$, taking its values in X.

This choice of $\mathcal{S}_F$ instead of $\mathcal{R}_F$ is justified by its following properties:

1. the translation property: Let $\varphi(\cdot) \in \mathcal{S}(\varphi)$. Then for all $s \geq 0$, the function $\psi(\cdot)$ defined by $\psi(t) := \varphi(t + s)$ is the history $\psi(\cdot) := T(\cdot)y \in \mathcal{S}(T(s)x)$ of the solution $y(\cdot)$ to the path-dependent differential inclusion starting at $T(s)x$,

2. the concatenation property: Let $\varphi(\cdot) \in \mathcal{S}_F(\varphi)$ be the history of a solution to the path-dependent differential inclusion starting at the path φ and $s \geq 0$. Then for every history $\psi(\cdot) \in \mathcal{S}_F(T(s)x)$ of a solution $y(\cdot)$ to the path-dependent differential inclusion starting at the initial path $T(s)x$, the function $\xi(\cdot)$ defined by

$$\xi(t) \; := \begin{cases} \varphi(t) := T(t)x & \text{if} \;\; t \in [0, s] \\ \psi(t - s) := T(t - s)y & \text{if} \;\; t \geq s \end{cases}$$

is the history of the solution $z(\cdot)$ defined by

$$z(t) \; := \begin{cases} x(t) & \text{if} \;\; t \in [0, s] \\ y(t - T) & \text{if} \;\; t \geq s \end{cases}$$

to the path-dependent differential inclusion starting at the initial path φ, and thus, belongs to $\mathcal{S}_F(\varphi)$.

These two properties to which we add the upper compactness of the solution map are enough to obtain relevant (and interesting) properties of path-dependent impulse differential inclusions, common to other dynamical systems.

1.4 Runs of Impulse Path-Dependent Differential Inclusions

We now introduce a constrained functional set $\mathbf{K} \subset \mathcal{H}(X)$, a functional target $\mathbf{C} \subset \mathbf{K}$ and a path-dependent reset map $R : \mathbf{C} \rightsquigarrow \mathbf{K}$ with nonempty values $R(\varphi)$.

The pair (F, R) governs the evolution of impulse systems in the following sense.

Definition 1.2 *Let us consider a finite dimensional vector space X, the space $\mathcal{H}(X)$ of histories, a subset $\mathbf{K} \subset \mathcal{H}(X)$, a target $\mathbf{C} \subset \mathbf{K}$, a set-valued map $F : \mathcal{H}(X) \rightsquigarrow X$ and a set-valued map $R : \mathbf{C} \rightsquigarrow \mathcal{H}$ with nonempty values, regarded as a* path-dependent reset map*. We regard the pair (F, R) as the dynamics of a* path-dependent impulse differential inclusion*.*

A run *of the path-dependent impulse differential inclusion (F, R) is defined by*

1. *a finite or infinite sequence $\tau(x(\cdot)) := \{\tau_n\}_n$ of nonnegative* cadences $\tau_n \in [0, \infty[$,
2. *a sequence of reinitialized paths $\varphi_n \in \mathcal{H}(X)$,*
3. *a sequence of motives $\varphi_n(\cdot) := T(\cdot)x_n \in \mathcal{S}_F(\varphi_n)$ where $\varphi_n \in \mathcal{H}(X)$ is the history of a solution $x_n(\cdot)$ to the path-dependent differential inclusion $x'(t) \in F(T(t)x)$ starting at the initial path φ_n and satisfying the end-point condition $T(\tau_n)x_n \in R^{-1}(\varphi_{n+1})$*

by

$$\begin{cases} i) & \text{defining the sequence of impulse times } t_{n+1} := t_n + \tau_n, \\ ii) & \forall\, t \in [t_n, t_{n+1}[,\ \ x(t) \ := \ x_n(t - t_n) \end{cases} \tag{2}$$

If the sequence of cadences is finite[5] *and stops at τ_N, we agree that the Nth motive is defined on $[0, +\infty[$, i.e., that we take $\tau_{N+1} = +\infty$.*

We say that a run $x(\cdot)$ is viable in $\mathbf{K}$ if for any $t \geq 0$, $T(t)x \in \mathbf{K}$ and that $\mathbf{K}$ is locally viable under (F, R) if for any $\varphi \in \mathbf{K}$, there exists at least *one run of the impulse path-dependent differential inclusion viable on a nonempty time interval and (globally) viable if it is viable on $[0, +\infty[$.*

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of paths $\varphi_{n+1} \in R(\varphi_n)$ at the initial time (case when for all $n \geq 0$, the cadences $\tau_n = 0$), or just a (continuous) solution $x(\cdot)$ to the path-dependent differential inclusion $x'(t) \in F(T(t)x)$ (case when $\tau_1 = +\infty$), or an hybrid of these two path-dependent modes, the discrete and the continuous.

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[5] We shall see that we can eliminate this situation by assuming that $R(\mathbf{C}) \cap \mathrm{Viab}_F(\mathbf{K}) = \emptyset$.

Path-dependent hybrid systems can be regarded as instances of viable path-dependent impulse differential inclusions as in the case of usual hybrid systems: we refer to [2, Aubin] or [9, Aubin, Lygeros, Quincampoix, Sastry & Seube] for more details on this topic.

2 Statement of The Impulse Path-Dependent Viability Theorem

2.1 Marchaud Maps

The Viability Theorems hold true whenever we assume that the dynamics governing the path-dependent evolution is Marchaud:

Definition 2.1 (Marchaud Map) *We shall say that $F : \mathcal{H}(X) \rightsquigarrow X$ is a* Marchaud map *if*

$$\left\{ \begin{array}{ll} i) & F \text{ is upper semicontinuous} \\ ii) & \text{the values } F(\varphi) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear: } \exists\, c > 0 \mid \forall\, \varphi \in \mathcal{H}(X), \\ & \|F(\varphi)\| := \sup_{v \in F(\varphi)} \|v\| \ \leq \ c(\|\varphi(0)\| + 1) \end{array} \right.$$

This covers the case of Marchaud control systems where $(\varphi, u) \mapsto f(\varphi, u)$ is continuous, affine with respect to the controls u and with linear growth and when $P : \mathcal{H}(X) \rightsquigarrow Z$ is Marchaud.

We recall the following version of the important Haddad Theorem 12.4.1 of [1, Aubin]:

Theorem 2.2 *Assume that $F : \mathcal{H}(X) \rightsquigarrow X$ is Marchaud. Then its solution map $\mathcal{S}_F$ is upper semicompact with nonempty values: This means that whenever $\varphi_n \in \mathcal{H}(X)$ converge uniformly on compact intervals to φ in $\mathcal{H}(X)$ and any history $\varphi_n(\cdot) := T(\cdot)x_n \in \mathcal{S}_F(\varphi_n)$ associated to a solution $x_n(\cdot)$ to the path-dependent differential inclusion $x'(t) \in F(T(t)x)$ starting at φ_n, there exists a subsequence (again denoted by) $\varphi_n(\cdot)$ converging uniformly on compact intervals to the history $\varphi(\cdot) := T(\cdot)x$ of a solution $x(\cdot)$ to the path-dependent differential inclusion starting at φ.*

2.2 Contingent Directions

In the case of path-dependent impulse differential inclusions, we shall characterize the viability of a functional constrained set $\mathbf{K}$ in terms of contingent directions to a $\mathbf{K} \subset \mathcal{H}(X)$ be a subset of histories at a path $\varphi \in \mathcal{H}$. Let

$$\mathcal{A}(X) \ := \ \{x(\cdot) \in \mathcal{C}(0, +\infty; X) \quad \text{such that} \ \ x(0) \ = \ 0\}$$

denote the “future space”. We embed the state space X into $\mathcal{A}(X)$ by identifying a vector x with the function $\psi_x(t) := tx$. The image of the ball λB_X of radius λ under this embedding is contained in $\mathcal{A}_\lambda(X)$ of λ-Lipschitz functions.

Definition 2.3 *Let $h > 0$ be given. The h-*concatenation *(or* concatenation *when there are no ambiguities) $\varphi\Diamond_h\psi$ is the bilinear form from $\mathcal{H}(X) \times \mathcal{A}(X) \mapsto \mathcal{H}(X)$ defined by*

$$(\varphi\Diamond_h\psi)(\tau) := \begin{cases} \varphi(\tau+h) & \text{if } \tau \in]-\infty,-h] \\ \varphi(0)+\psi(\tau+h) & \text{if } \tau \in [-h,0] \end{cases}$$

As an example, the concatenation of $\varphi \in \mathcal{H}(X)$ and $x \in X$ is defined by

$$(\varphi\Diamond_h x)(\tau) = \begin{cases} \varphi(\tau+h) & \text{if } \tau \in]-\infty,h] \\ \varphi(0)+(\tau+h)x & \text{if } \tau \in [-h,0] \end{cases}$$

We attach to a function $\varphi \in \mathcal{H}(X)$ and a sequence $v_1,\ldots,v_n$ the Euler concatenation defined by

$$\varphi\Diamond_h v_1\Diamond_h v_2\Diamond_h\cdots\Diamond_h v_n$$

which is piecewise linear function on the interval $[-nh,0]$.

Lemma 2.4 *For any Lipschitz constant $\lambda > 0$, the h-concatenation maps $\mathcal{H}_\lambda(X)\times\mathcal{A}_\lambda(X)$ to $\mathcal{H}_\lambda(X)$*

Definition 2.5 *We denote by $\mathcal{D}_{\mathbf{K}}(\varphi)$ the set of vectors $v \in X$ such that there exist a sequence $h_n > 0$ converging to 0 and a sequence $v_n \in X$ converging to v satisfying*

$$\forall\, n \geq 0,\;\; \varphi\Diamond_{h_n}v_n \;\in\; \mathbf{K}$$

2.3 The Impulse Path-Dependent Viability Theorem

Theorem 2.6 *Let (F,R) be a path-dependent impulse differential inclusion and $\mathbf{K} \subset \mathcal{H}(X)$ be a closed subset. Assume that F is Marchaud and that $\mathbf{C} \subset \mathbf{K}$ is closed. Then the following statements are equivalent*

1. *K is viable under (F,R),*
2. *The subset $\mathbf{K}\backslash\mathbf{C}$ is locally viable under the path-dependent differential inclusion governed by F,*
3. *$\mathbf{K}$, $\mathbf{C}$, F and R are linked through the tangential condition*

$$\forall\, \varphi \in \mathbf{K}\backslash\mathbf{C},\;\; F(\varphi)\cap\mathcal{D}_{\mathbf{K}}(\varphi) \neq \emptyset$$

Actually, both impulse differential inclusions and path-dependent impulse differential inclusions[6] share the same properties at a higher abstraction level, the level of impulse

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[6] as well as parabolic (or reaction-diffusion type) partial differential inclusions and mutational equations governing the evolution of subset.

evolutionary systems we are about to define. It is at this level that the two first statements are equivalent.

The equivalence between the second and third statement is specific, and provided in our case by the Path-Dependent Viability Theorems of [13, 14, 15, Haddad].

2.4 Examples

Take any path-dependent differential inclusion $x'(t) \in F(T(t)x)$ associated with a Marchaud right-hand side F.

We refer to Chapter 12 of [1, Aubin] for examples of tangential conditions when the constrained set $\mathbf{K}$ and the constrained targets $\mathbf{C} \subset \mathbf{K}$ are Volterra sets defined by kernels, by lack of space. Consider only the simple case when the constrained subset of paths or histories and the target are associated with subsets $C \subset K \subset X$ of the vector space X by

$$\mathbf{C} \;:=\; \{ \varphi \in \mathcal{H}(X) \mid \varphi(0) \in C \} \;\subset\; \mathbf{K} \;:=\; \{ \varphi \in \mathcal{H}(X) \mid \varphi(0) \in K \}$$

Assume that the reset map R is associated with an usual reset map $R_0 : C \rightsquigarrow K$, where $C \subset K$ by the formula

$$\forall\, \varphi \in \mathcal{H}(X),\ \forall\, \tau \in]-\infty, 0],\ \ (R(\varphi))(\tau) \;:=\; R_0(\varphi(\tau))$$

In this case, a run of the path-dependent impulse differential inclusion (F, R_0) is defined by

1. a finite or infinite sequence $\tau(x(\cdot)) := \{\tau_n\}_n$ of nonnegative cadences $\tau_n \in [0, \infty[$,
2. a sequence of reinitialized states $x_n \in K$,
3. a sequence of motives $x_n(\cdot)$ that are solutions to the path-dependent differential inclusion $x'(t) \in F(T(t)x)$ starting at the initial state x_n and satisfying the end-point condition $x_{n+1} \in R(x_n(\tau_n))$

by

$$\left\{ \begin{array}{ll} i) & \text{defining the sequence of impulse times } t_{n+1} := t_n + \tau_n, \\ ii) & \forall\, t \in [t_n, t_{n+1}[,\ \ x(t) \;:=\; x_n(t - t_n) \end{array} \right. \tag{3}$$

Theorem 2.7 *Let F be a path-dependent Marchaud set-valued map, K and $C \subset K$ and $R_0 : C \rightsquigarrow K$ be a reset map. Then K is viable under (F, R_0) if and only if the tangential condition*

$$\forall\, \varphi \in \mathcal{H}(X) \quad \textit{such that } \varphi(0) \in K \backslash C,\ \ F(\varphi) \cap T_K(\varphi(0)) \neq \emptyset$$

2.5 Path-Dependent Hybrid Systems

Definition 2.8 *An path-dependent hybrid differential inclusion (K, F, R_0) is defined by*

1. *a finite dimensional vector space E of states e called* locations*,*
2. *a set-valued map $K : E \rightsquigarrow X$ associating with any location e a (possibly empty) subset $K(e) \subset X$ and a set-valued map $C : E \rightsquigarrow X$ associating with any location e a (possibly empty) subset $C(e) \subset K(e)$,*
3. *a set-valued map $F : E \times \mathcal{H}(X) \rightsquigarrow X$ with which we associate the path-dependent differential inclusion $x'(t) \in F(e, T(t)x)$,*
4. *a set-valued map (reset map) $R_0 : \mathrm{Graph}(C) \rightsquigarrow \mathrm{Graph}(K)$.*

A run *of the path-dependent hybrid system is defined by*

1. *a finite or infinite sequence $\tau(\vec{e}, x(\cdot)) := \{\tau_n\}_n$ of nonnegative* cadences *$\tau_n \in [0, \infty[$,*
2. *a sequence of locations e_n and of reinitialized states $x_n \in K(e_n)$,*
3. *a sequence of motives $x_n(\cdot)$ that are solutions to the path-dependent differential inclusion $x'(t) \in F(e_n, T(t)x)$ starting at the initial state x_n and satisfying the end-point condition $x_{n+1} \in R_0(e_n, x_n(\tau_n))$*

by (3).

We observe right away that a map $(\vec{e}, x(\cdot))$ is a run of the hybrid differential inclusions if and only if $(e(\cdot), x(\cdot))$ is a run of

$$\begin{cases} i) & e'(t) \;=\; 0 \\ ii) & x'(t) \;\in\; F(e(t), T(t)x) \end{cases}$$

"viable" in $\mathrm{Graph}(K)$ until it reaches the graph of the map C. Indeed the locations remain constant in the intervals $[t_n, t_{n+1}[$ since their velocities are equal to 0.

Since the existence of solutions to path-dependent hybrid differential inclusions amounts to the viability of the graph of the set-valued map K under an associated auxiliary path-dependent impulse differential inclusion, we obtain a necessary and condition for the existence of solutions to hybrid differential inclusions thanks to Theorem 2.7. For that purpose, we need the definition of the contingent derivative $DK(e, x) : E \rightsquigarrow X$ of a set-valued map $K : E \rightsquigarrow X$ at a point (e, x) of its graph: It can be defined by

$$\mathrm{Graph}(DK(e, x)) \;:=\; T_{\mathrm{Graph}(K)}(e, x)$$

Theorem 2.9 *Let (K, F, R_0) be a path-dependent hybrid differential inclusion. Assume that F is Marchaud. Then the path-dependent hybrid differential inclusion has a solution for every initial state if and only if*

$$\forall\, e \in E, \forall\, \varphi \in \mathcal{H}(X) \;\; \textit{such that}\; \varphi(0) \in K(e) \backslash K(e) \backslash C(e), \;\; F(e, \varphi) \cap DK(e, \varphi(0))(0) \neq \emptyset$$

3 Impulse Evolutionary Systems

Therefore, it costs nothing to prove the equivalence between the two first statements in the general case of impulse evolutionary systems:

3.1 Impulse Evolutionary Systems

Definition 3.1 *An evolutionary system is a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ satisfying*

1. *the* translation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the function $y(\cdot)$ defined by $y(t) := x(t + T)$ is a solution $y(\cdot) \in \mathcal{S}(x(T))$ starting at $x(T)$,*
2. *the* concatenation property*: Let $x(\cdot) \in \mathcal{S}(x)$ and $T \geq 0$. Then for every $y(\cdot) \in \mathcal{S}(x(T))$, the function $z(\cdot)$ defined by*

$$z(t) \;:= \begin{cases} x(t) & \text{if} \;\; t \in [0, T] \\ y(t - T) & \text{if} \;\; t \geq T \end{cases}$$

   belongs to $\mathcal{S}(x)$.

We can define impulse evolutionary systems in the following way:

Definition 3.2 *Let $K \subset X$, $C \subset K$ be two nonempty subsets and $R : C \rightsquigarrow K$ a set-valued map*[7] *with nonempty values, regarded as a reset map, and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system. Then the pair $(\mathcal{S}, R)$ governs a run $x(\cdot)$ of an impulse evolutionary system defined by*

1. *a finite or infinite sequence $\tau(x(\cdot)) := \{\tau_n\}_n$ of nonnegative* cadences *$\tau_n \in [0, +\infty[$,*
2. *a sequence of reinitialized states x_n,*
3. *a sequence of motives $x_n(\cdot) \in \mathcal{S}(x_n)$ satisfying the end-point condition $x_n(\tau_n) \in R^{-1}(x_{n+1})$*

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[7] When $R : X \rightsquigarrow X$ is defined on X, we associate with it its "graphical restriction" to $K \times K$ (again denoted by) R where $C := K \cap R^{-1}(K)$ and $R(x)$ is replaced by $R(x) \cap K$.

by

$$\left\{\begin{array}{ll} i) & \text{defining the sequence of impulse times } t_{n+1} := t_n + \tau_n, \\ ii) & \forall\, t \in [t_n, t_{n+1}[, \;\; x(t) \; := \; x_n(t - t_n) \end{array}\right. \tag{4}$$

If the sequence of cadences is finite[8] *and stops at τ_N, we agree that the Nth motive $x_N(\cdot) \in \mathcal{S}(x_N)$ is taken on $[0, +\infty[$, i.e., and we agree to set $\tau_{N+1} = +\infty$.*

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$ and that a closed subset K is viable under an impulse evolutionary system $(\mathcal{S}, R)$ if from any $x \in K$ starts at least *one run viable in K.*

In order to characterize the viability of K under an evolutionary system, we also need the following definitions:

Definition 3.3 *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ be a set-valued evolutionary system and $K \subset X$ be a subset regarded as a constrained set.*

The subset K is said locally viable *under $\mathcal{S}$ if from any initial state $x \in K$ starts at least one solution viable in K on a nonempty interval and* viable *if this solution is viable on $[0, +\infty[$.*

The viability kernel $\mathrm{Viab}(K)$ is the subset of initial states $x_0 \in K$ such that one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$. A subset K is a repeller *under $\mathcal{S}$ if its viability kernel is empty.*

3.2 Characterization of Impulse Evolutionary Systems Viability

We can adapt the proof of [2, Aubin] or [9, Aubin, Lygeros, Quincampoix, Sastry & Seube] for characterizing the viability of a closed subset under an impulse evolutionary system:

Theorem 3.4 *Let $(\mathcal{S}, R)$ be an impulse evolutionary system and $K \subset X$ and $C \subset K$ be closed subsets. Assume that $\mathcal{S}$ is upper semicompact. Then the following statements are equivalent*

1. *K is viable under $(\mathcal{S}, R)$,*
2. *The subset $K \backslash C$ is locally viable under $\mathcal{S}$,*

3.3 Prerequisites of Viability Theory

For proving this characterization theorem, we need some results of viability theory.

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[8] We shall see that we can eliminate this situation by assuming that $R(C) \cap \mathrm{Viab}_{\mathcal{S}}(K) = \emptyset$.

Definition 3.5 *Let $K \subset X$ be a subset. The functional $\tau_K : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its* exit time *$\tau_K(x(\cdot))$ defined by*

$$\tau_K(x(\cdot)) \ := \ \inf\{t \in [0,\infty[\mid x(t) \notin K\} \ := \ \varpi_{X\backslash K}(x(\cdot))$$

is called the exit functional*.*

Let $C \subset K$ be a target. We introduce the (constrained) hitting functional *$\varpi_{(K,C)}$ defined by*

$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \ \& \ \forall s \in [0,t], \ x(s) \in K \ \}$$

associating with $x(\cdot)$ its hitting time*, introduced in [12, Cardaliaguet, Quincampoix & Saint-Pierre]).*

Consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$. Let $C \subset K$ and K be two subsets.

The function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\sharp(x) \ := \ \sup_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot))$$

is called the upper exit function*.*

The function $\varpi^\flat_{(K,C)} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\varpi^\flat_{(K,C)}(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$

is called the lower constrained hitting function*.*

We shall need the following

Theorem 3.6 *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be a strict upper semicompact map and C and K be two closed subsets such that $C \subset K$. Then the hitting function $\varpi^\flat_{(K,C)}$ is lower semicontinuous and the exit function $\tau_K^\sharp$ is upper semicontinuous. Furthermore, for any $x \in \mathrm{Dom}(\varpi^\flat_{(K,C)})$, there exists at least one solution $x^\flat(\cdot) \in \mathcal{S}(x)$ which hits C as soon as possible before possibly leaving K*

$$\varpi^\flat_{(K,C)}(x) \ = \ \varpi_{(K,C)}(x^\flat(\cdot))$$

and for any $x \in \mathrm{Dom}(\tau_K^\sharp)$, there exists at least one solution $x^\sharp(\cdot) \in \mathcal{S}(x)$ which remains viable in K as long as possible:

$$\tau_K^\sharp(x) \ = \ \tau_K(x^\sharp(\cdot))$$

(See [6, Aubin] for a proof and more details on evolutionary systems).

3.4 Proof of the Characterization of Impulse Evolutionary Systems Viability

Indeed, if K is viable under $(\mathcal{S}, R)$, then from any $x_0 \in K \backslash C$ starts at least a solution $x(\cdot) \in \mathcal{S}(x)$ viable in K

1. either forever if x_0 belongs to the viability kernel $\mathrm{Viab}(K)$ of K
2. or until it reaches at some time $t_1 > 0$ a state $x(^{-}t_1)$ in C.

This shows that $K \backslash C$ is locally viable.

Conversely, let us assume that $K \backslash C$ is locally viable and take an initial state $x_0 \in K$. If x_0 belongs to C, we may take $\tau_0 = 0$ and $x_1 \in R(x_0)$. Consider now the case when $x_0 \in K \backslash C$.

If x_0 belongs to $\mathrm{Viab}(K)$, then at least one solution starting from x_0 is viable in K, and thus, defines a run viable in K: We may take the cadence $\tau_0 = +\infty$ and for motive a solution $x_0(\cdot) \in \mathcal{S}(x_0)$.

If x_0 does not belong to $\mathrm{Viab}(K)$, all solutions leave K in finite time before (possibly) reaching the viability kernel. It is then enough to prove that at least one of them reaches C before leaving K. This is the case of a solution $x^{\sharp}(\cdot) \in \mathcal{S}(x)$ which maximizes $\tau_K(x(\cdot))$, i.e., which satisfies

$$\tau_K^{\sharp}(x) \ := \sup_{x(\cdot) \in \mathcal{S}(x)} \tau_K(x(\cdot)) \ = \ \tau_K(x^{\sharp}(x))$$

leaves $K \backslash (\mathrm{Viab}(K) \cup C)$ through C. This solution exists by Theorem 3.6 since K is closed and $\mathcal{S}$ is upper semicompact. Next, we claim that $x^{\sharp} := x^{\sharp}(\tau_K^{\sharp}(x)) \in K \backslash \mathrm{Viab}(K)$. Otherwise, if $x^{\sharp}$ would belong to the viability kernel, it could be concatenated with a solution viable in K for ever, so that the initial state x_0 would belong the viability kernel, which is not the case.

Furthermore, $x^{\sharp}$ belongs to C. If not, $x^{\sharp}$ would belong to $K \backslash C$ which is assumed to be locally viable. Then one could associate with $x^{\sharp} \in K \backslash (\mathrm{Viab}(K) \cup C)$ a solution $y(\cdot) \in \mathcal{S}(x^{\sharp})$ and $T > 0$ such that $y(\tau) \in K \backslash (\mathrm{Viab}(K) \cup C)$ for all $\tau \in [0, T]$. Concatenating this solution to $x^{\sharp}(\cdot)$, we obtain a solution viable in K on an interval $[0, \tau_K^{\sharp}(x) + T]$, which contradicts the definition of $x^{\sharp}(\cdot)$.

Therefore $x^{\sharp}$ belongs to $K \cap C$ so that there exists $x_1^{\sharp} \in K \cap R(x^{\sharp})$. $\square$

4 The Path-Dependent Viability Theorem

Since the solution map of a Marchaud map $F : \mathcal{H}(X) \rightsquigarrow X$ is upper semicompact by Theorem 2.2, the equivalence between the first and second statements of Theorem 3.4 holds true.

The equivalence between the second and the third statement follow from the following Haddad's Path-Dependent Viability Theorem:

Theorem 4.1 *Assume that F is Marchaud and take $\lambda > 0$. The two following statements hold true:*

1. *If $\mathbf{K} \subset \mathcal{H}_\lambda(X)$ is closed, then $\mathbf{K}$ is (globally) viable under F if and only if*

$$\forall\, \varphi \in \mathbf{K}, \quad F(\varphi) \cap \mathcal{D}_{\mathbf{K}}(\varphi) \neq \emptyset$$

2. *If $\mathbf{C} \subset \mathbf{K}$ is closed, then $\mathbf{K} \backslash \mathbf{C}$ is locally viable under F if and only if*

$$\forall\, \varphi \in \mathbf{K} \backslash \mathbf{C}, \quad F(\varphi) \cap \mathcal{D}_{\mathbf{K}}(\varphi) \neq \emptyset$$

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