Detectability through Measurements under Impulse Differential Inclusions

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Abstract

This paper adapts to the case of impulse and hybrid control systems the results obtained by Aubin, Bicchi & Pancanti on "detectability" of solutions of usual control systems. Measurements of the state, described by a informational tube, that may be quantized, are gathered along time. The detector associates at each time with any state satisfying the given measurement the (possibly) empty set of the initial states from which starts a solution that arrives at this state while satisfying the measurements. This detector is then studied by tools of viability theory, and shown to be a solution to a system of Hamilton-Jacobi-Bellman partial differential inclusions satisfying supplementary conditions (that can be regarded as the vectorial analogue of Bensoussan-Lions "quasi-variational inequalities" in impulse optimal control. The derivatives of the detector provide the regulation map governing the motives of the detectable runs.

Keywords: hybrid control, impulse control, observability, detectability, detector, capture basin, run, execution, quasi variational inequalities, system of Hamilton-Jacobi-Bellman partial differential inclusions.

AMS Classification: 93C10, 93C15, 93C55, 49J24, 49J40, 49J53

Introduction

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0,\infty;X)$ the set of evolutions $x(\cdot) \in \mathcal{C}(0,\infty;X)$ starting at x at time $t = 0$ and solutions to the control system

$$\begin{cases} i) & x'(t) \; = \; f(x(t), u(t)) \\ ii) & u(t) \; \in \; U(x(t)) \end{cases}$$

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with state-dependent constraints on the controls.

We follow an idea pioneered in [29, Delchamps] that regards measurements *as a deterministic memoryless entity that gives us a limited amount of information about the states*, whether continuous or "quantized", i.e., when measurements are made in a finite number of instants instead of continuously. The problem is formulated as follows: *How much information about the current state is contained in a long record of past (quantized) measurements of the system's output? Furthermore, how can the inputs to the system be manipulated so as to make the system's output record more informative about the state evolution than might appear possible based on a cursory appraisal?*

This issue was taken up in [9, Aubin, Bicchi & Pancanti] in the case of nonlinear control systems, differential inclusions and more generally, evolutionary systems (defined by set-valued maps $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$) using viability techniques (for another approach of observability regarding measurements as an approximation of the state and using viability techniques, we refer to [31, Doyen & Rapaport]. Detectors have been investigated by the Soviet School. We refer to [35, Kurzhanski & Filippova] and [36, Kurzhanski & Valyi] for surveys of the results obtained by this school.).

We take up in this paper these issues for impulse control systems (of which hybrid control systems are particular cases) and more generally, for impulse evolutionary systems introduced in [18, Aubin & Haddad].

Before proceeding further, we recall that information recorded by measurements is mathematically described in [9, Aubin, Bicchi & Pancanti] by "informational tubes"

$$t \in \mathbf{R}_+ \rightsquigarrow P(t) \ \subset \ X := \mathbf{R}^n$$

that provides the "limited amount of information about the states" at time t. We take $P(t) := X$ when no information is recorded at time t. Hence quantized measurements are obtained when $P(t) \neq X$ only for a finite number of instants t_n.

An example of informational tubes is given by $P(t) := h^{-1}(y(t))$ where $h : X \mapsto Y$ is an observation map and where $t \mapsto y(t)$ is the evolution of the observed output.

The same framework houses also the case when the observation map is set-valued: We set $P(t) := H^{-1}(y(t))$ and the above viability condition reads

$$\forall\, t \in [0, T], \ \ y(t) \ \in \ H(x(t))$$

Quantized or qualitative measurements can also be obtained when the observation map h is constant on a (finite) covering of the state space, as in [29, Delchamps]. This specific case is studied in [12, Aubin & Dordan].

We assume also that we know further that the initial states belong to a subset $C \subset P(0)$.

Once the informational tube $P(\cdot)$ and the subset $C \subset P(0)$ in which the initial are assumed to belong are given, we describe the answer to the question raised in [29, Delchamps] by introducing the detector: *the detector is the set-valued map* $\mathbf{D}_{(P,C)}$ *that associates with any time* T *and any observation* $x \in P(T)$ *at time* T *the (possibly empty) subset of states* $x_0 \in C$ *from which* $x := x(T)$ *can be reached by an evolution* $x(\cdot) \in \mathcal{S}(x_0)$ detectable *by the tube in the sense that*

$$\forall\, t \in [0,T], \;\; x(t) \;\in\; P(t)$$

It is clear that the set-valued map $T \rightsquigarrow \mathrm{Im}(\mathbf{D}_{(P,C)})(T,\cdot)$ is nonincreasing, refining the set of detectable states of C from which at least one evolution is detected ("filtered" so to speak) by the detection tube $P(\cdot)$. The larger T, the more information provided by measurements is gathered, the smaller $\mathrm{Im}(\mathbf{D}_{(P,C)})(T,\cdot)$. This detector has been studied in [9, Aubin, Bicchi & Pancanti] for general evolutionary systems.

We define in this paper the concept of detector under impulse differential inclusions and more generally, impulse evolutionary systems introduced in [18, Aubin & Haddad]. Impulse differential inclusions, and in particular, hybrid control systems, are defined by the solution map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ of a control system and a reset map $\Phi : X \rightsquigarrow X$.

A run is defined by a (finite or infinite) sequence $\vec{x}(\cdot) := (\tau_n, x_n(\cdot))_n$ of cadences $\tau_n \in X$ and of motives $x_n(\cdot) \in \mathcal{C}(0,\tau_n;X)$ describing the evolution $\vec{x}(t)$ at t between two consecutive impulse times t_n and $t_{n+1} := t_n + \tau_n$ (where $t_0 = 0$) by

$$\forall\, n \geq 0,\; \vec{x}(t_n) \;:=\; x_n(0) \;\;\&\;\; \forall\, t \in [t_n, t_{n+1}[,\;\; \vec{x}(t) \;:=\; x_n(t - t_n)$$

A run $\vec{x}(\cdot) := (\tau_n, x_n(\cdot))_n$ is a solution to the impulse evolutionary system starting at x if

$$\forall\, n \geq 0, \begin{cases} i) & \forall\, n \geq 0, \;\; x_n(\cdot) \;\in\; \mathcal{S}(x_n(0)) \\ ii) & \forall\, n \geq 0 \;\text{ such that } \tau_n < +\infty, \;\; x_{n+1}(0) \;\in\; \Phi(x_n(\tau_n)) \end{cases}$$

The elements $x_n := x_n(0)$ are called the reinitialized states.

A first advantage of introducing impulse evolutionary systems is to summarize the usually protracted description of hybrid systems[3] — that can be regarded as instances of impulse differential inclusions — by only two set-valued maps F — the right-hand side of the differential inclusion governing the continuous evolution of a hybrid system — and Φ, describing the reset map reinitializing the system when required. We refer to [3, Aubin] or [20, Aubin, Lygeros, Quincampoix, Sastry & Seube] for more details on that topic.

The definition of a detector under an impulse evolutionary system is the same than the definition of a detector under control systems or differential inclusions: Solutions to control systems or evolutions of evolutionary systems are simply replaced by "runs" of impulse evolutionary systems.

The aims of this paper are to characterize the detector $\mathbf{D}_{(P,C)}$ regarded as a set-valued map from the graph $\mathrm{Graph}(P)$ of the tube P to the set C as

[3] See for instance among many papers and books [25, Branicky, Borkar & Mitter], [24, Bensoussan & Menaldi], [38, 39, Matveev & Savkin] and [41, Shaft & Schumacher].

1. the **unique** set-valued map $W : \text{Graph}(P) \rightsquigarrow C$ satisfying

   (a) for any $x_0 \in W(T,x)$, there exists a run $\vec{x}(\cdot)$ to the impulse evolutionary system $(\mathcal{S},\Phi)$ starting from C, satisfying the viability conditions (3) on $[0,T]$ and reaching x at time T, and any such runs actually satisfies

   $$\forall\, t \in [0,T], \;\; x_0 \in W(t,\vec{x}(t))$$

   (b) for any $x_0 \in C\backslash W(T,x)$, for every run $\vec{x}(\cdot)$ to the impulse evolutionary system $(\mathcal{S},\Phi)$ reaching $x = x(T)$ at time T and satisfying for some $S \in [0,T]$ the conditions

   $$\forall\, t \in [S,T], \;\; \vec{x}(t) \;\in\; P(t)$$

   then

   $$\forall\, t \in [S,T], \;\; x_0 \;\notin\; W(t,\vec{x}(t))$$

   As a consequence, for any $T > 0$ and for any $x_0 \in \partial_C \mathbf{D}_{(P,C)}(T,x)$ in the boundary of $\mathbf{D}_{(P,C)}(T,x)$ relative[4] to C, for detectable every run $\vec{x}(\cdot) \in \mathcal{R}(x_0)$ to the impulse evolutionary system $(\mathcal{S},\Phi)$ reaching x at time T, then actually

   $$\forall\, t \in [0,T], \;\; x_0 \in \partial_C \mathbf{D}_{(P,C)}(t,\vec{x}(t))$$

2. the smallest solution $W : \text{Graph}(P) \rightsquigarrow C$ to the problem

   $$\bigcup_{t\in[0,T]} \;\; \bigcup_{x(\cdot)\in\mathcal{S}(X) \,|\, x(T)=x \;\&\; \forall\, s\in[t,T],\; x(s)\in P(s)} W(t,\Phi(x(t))) \;=\; W(T,x)$$

   and furthermore, satisfies the fixed point condition

   $$\bigcup_{t\in[0,T]} \;\; \bigcup_{x(\cdot)\in\mathcal{S}(X) \,|\, x(T)=x \;\&\; \forall\, s\in[t,T],\; x(s)\in P(s)} \mathbf{D}_{(P,C)}(t,\Phi(x(t))) \;=\; \mathbf{D}_{(P,C)}(T,x)$$

3. the smallest of the set-valued maps $W : \text{Graph}(P) \rightsquigarrow C$ satisfying the initial condition

   $$\forall\, x \in C, \;\; W(0,x) \;=\; \{x\}$$

   the following property

   $$W(t,\Phi(x)) \;\subset\; W(t,x)$$

   and solution to

   $$\forall t > 0, \;\; y \in W(t,x), \;\exists\, u \in U(x) \quad \text{such that} \;\; 0 \in DW(t,x,y)(-1,-f(x,u))$$

[4]The relative boundary $\partial_K D$ to K of a subset $D \subset K$ is equal to $\overline{D} \cap \overline{K\backslash X}$.

where $DW(t,x,y)(-1,-v)$ denotes the (graphical contingent) derivative of the set-valued map $W : \text{Graph}(P) \leadsto C$ at the point (t,x,y) of its graph in the direction $(-1,-v)$ and is the unique solution in the Frankowska sense (see below for precise definitions).

In other words, the detector is a set-valued solution of a "system of Hamilton-Jacobi-Bellman partial differential inclusions" satisfying supplementary conditions, as "quasi variational inequalities" in optimal impulse control are solutions to Hamilton-Jacobi-Bellman partial differential equations satisfying further fixed-point properties.

Knowing the detector and its derivatives, we introduce the associated regulation map $\mathbf{R}_{(P,C)}$ defined by

$$\mathbf{R}(P,C)(t,x,y) \ := \ \{u \in U(x) \ \ 0 \in D\mathbf{D}_{(P,C)}(t,x,y)(-1,-f(x,u))\}$$

that allows to regulate the detected runs: Starting from x_0,

1. either $x_0 \in \Phi^{-1}(C)$ and we take $\tau_0 = 0$ and a next reinitialization state $x_1 \in \Phi(x_0) \cap C$,
2. or $x_0 \notin \Phi^{-1}(C)$ and we take for first motive $x_0(\cdot)$ an evolution governed by

$$\begin{cases} i) & x'(t) \ = \ f(x(t),u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}_{(P,C)}(t,x(t),x_0) \end{cases}$$

until the first time τ_0 (if any) when $x_0 \in \mathbf{D}_{(P,C)}(\tau_0, \Phi(x_0(\tau_0)))$ and we take $x_1 \in \Phi(x_0(\tau_0))$.

We then proceed recursively

The strategy for studying and characterizing detectors under impulse evolutionary system is the same than the one proposed in [9, Aubin, Bicchi & Pancanti]: It involves the concept of impulse capture basin of a target under an impulse evolutionary system, a concept similar to the concept of impulse viability kernel introduced and studied in [27, 28, Cruck]. We prove that the graph of a detector is an impulse capture basin of an adequate target under an adequate auxiliary impulse evolutionary system. We next transfer the properties of these capture basins to properties of the detector.

Outline — We set up the problem in the first section by introducing impulse evolutionary systems, detection tubes and detectors.

We next introduce in the second section the concept of impulse capture basin of a target (viable in a constrained subset) and relate it to the detector.

We reacall in the third section a basic characterization of impulse capture basins, that is translated for the detectors in the fourth section.

The fifth section recalls resutls of viability theory and of [, Aubin & Haddad] relating the impulse capture basin to capture basins of adequate targets under the evolutionary system governing the (continuous) evolution of motives of the run. It thus allows to use the known properties of these capture basins (see for instance [7, 8, Aubin] and [10, Aubin & Catté] for recent results on that topic). These results are thus “transferred” to provide a further characterization of detectors in the sixth section and applied to show that the detector is a solution to a system of Hamilton-Jacobi-Bellman partial differential inclusions satisfying supplementary conditions, that provides the regulation map.

Acknowledgments The authors thank Antonio Bicchi for having introduced them to the detectability problem and for the discussions.

1 Impulse Evolutionary Systems and Detectors

The concept of impulse evolutionary system was introduced in [18, Aubin & Haddad] in order to encompass both impulse systems associated with control systems and differential inclusions, path-dependent or memory-dependent controls systems and differential inclusions, as well as other evolution systems such as parabolic type differential inclusions.

For instance, let $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ denote finite dimensional vector spaces. Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $U : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\left\{ \begin{array}{lll} i) & x'(t) & = & f(x(t), u(t)) \\ ii) & u(t) & \in & U(x(t)) \end{array} \right.$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, U(x)) := \{f(x, u)\}_{u \in U(x)}$.

We denote by $\mathcal{S}_F(x)$ the set of absolutely continuous functions $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \;\; x'(t) \; \in \; F(x(t))$$

starting at time 0 at x: $x(0) = x$.

Let $\mathcal{C}(0, \infty; X)$ denote the space of continuous functions supplied with the compact convergence topology. The set-valued map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the solution map associated with F.

This solution map is the prototype of an evolutionary system:

Definition 1.1 *Let us consider a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ associating with each initial state x the (possibly empty) set $\mathcal{S}(x)$ of* evolutions *$x(\cdot)$ starting from x in the sense that $x(0) = x$. It is said to be an* evolutionary system *if it satisfies*

1. *the* translation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the function $y(\cdot)$ defined by $y(t) := x(t+T)$ is an evolution $y(\cdot) \in \mathcal{S}(x(T))$ starting at $x(T)$,*

2. *the* concatenation property*: Let $x(\cdot) \in \mathcal{S}(x)$ and $T \geq 0$. Then for every $y(\cdot) \in \mathcal{S}(x(T))$, the function $z(\cdot)$ defined by*

$$z(t) \;:=\left\{\begin{array}{lll} x(t) & \text{if} & t \in [0,T] \\ y(t-T) & \text{if} & t \geq T \end{array}\right.$$

belongs to $\mathcal{S}(x)$.

We associate with $\mathcal{S}$ its backward evolutionary system *$\mathcal{S}_- : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ defined by $y(\cdot) \in \mathcal{S}_-(x)$ if and only if there exists an evolution $z(\cdot) \in \mathcal{S}(x)$ such that for every $T \geq 0$, the function $x(\cdot)$ defined by*

$$x(t) \;:=\left\{\begin{array}{lll} y(T-t) & \text{if} & t \in [0,T] \\ z(t-T) & \text{if} & t \geq T \end{array}\right.$$

belongs to $\mathcal{S}(y(T))$.

Differential inclusions with memory ([32, 33, 34, Haddad]), partial differential inclusions (see [42, 44, 45, Shi Shuzhong]), mutational equation $\overset{\circ}{x} \ni f(x)$ on metric spaces (see [2, Aubin]), etc., provide other examples of evolutionary systems.

We identify $\mathcal{C}(0,0;X)$ with X and we define "runs" in the following way:

Definition 1.2 *Let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$. We say that a (finite or infinite) sequence*

$$\vec{x}(\cdot) \;:=\; (\tau_n, x_n(\cdot))_{n\geq 0} \;\in\; \prod_{n\geq 0} \mathbf{R} \times \mathcal{C}(0,\tau_n;X)$$

is a run *$\vec{x}(\cdot)$ made of*

1. *a finite or infinite sequence $\tau(\vec{x}(\cdot)) := \{\tau_n\}_n$ of nonnegative* cadences *$\tau_n \in [0,+\infty[$,*

2. *a sequence of* motives *$x_n(\cdot) \in \mathcal{C}(0,\tau_n;X)$.*

if it is defined by the formulas

$$\left\{\begin{array}{ll} i) & \text{the sequence } \mathcal{T}(\vec{x}(\cdot)) := \{t_n\}_{n\geq 0} \text{ of impulse times } t_{n+1} := t_n + \tau_n,\ t_0 = 0 \\ ii) & \forall\, n \geq 0,\ \vec{x}(t_n) := x_n(0) \ \ \&\ \ \forall\, t \in [t_n, t_{n+1}[,\ \ \vec{x}(t) \;:=\; x_n(t-t_n) \end{array}\right. \tag{1}$$

We say that the sequence of $x_n := x_n(0) \in X$ is the sequence of reinitialized states.

Naturally, if $\tau_n = 0$, i.e., when $t_{n+1} = t_n$, we identify the motive $x_n(\cdot)$ with the reinitialized state $x_n(\cdot) \equiv x_n \in \mathcal{C}(0,0;X) \equiv X$, so that runs can be time-dependent functions, sequences, or hybrids of them. We set

$$\nu_t(\vec{x}(\cdot)) \; := \; \{n \in \mathbf{N} \;\; \text{such that}\; t_n = t\}$$

is the set of impulse times equal to t.

If the sequence of cadences is finite and stops at τ_N, we agree that the Nth motive $(x_N(\cdot), u_N(\cdot))$ is taken on $[0, +\infty[$ and we agree to set $\tau_{N+1} = +\infty$.

We associate with a run $\vec{x}(\cdot)$ its sequence of switches $\mathbf{s}(\vec{x}(\cdot)) := (\mathbf{s}_n(\vec{x}(\cdot)))_{n\geq 0}$ *defined by*

$$\mathbf{s}_n(\vec{x}(\cdot)) \; := \; x_n(0) - x_{n-1}(\tau_{n-1}) \; = \; \vec{x}(t_n) - \vec{x}(^{-}t_n)$$

We denote by $\mathcal{T}(\vec{x}(\cdot);t)$ the set of impulse times $t_n \leq t$ less than or equal to t.

The life expectation $\lambda(\vec{x}(\cdot))$ of a run $\vec{x}(\cdot)$ is defined by $+\infty$ if the run is finite or by

$$\lambda(\vec{x}(\cdot)) \; := \; \sum_{n=0}^{+\infty} \tau_n \; = \; \lim_{n \to +\infty} t_n \; \leq \; +\infty$$

in the opposite case. Hence the domain of definition of a run is the interval $[0, \lambda(\vec{x}(\cdot))]$. We set

$$\lambda^{\flat}(\vec{x}(\cdot)) \; := \; \inf_{n\geq 0} \tau_n \;\; \& \;\; \lambda^{\sharp}(\vec{x}(\cdot)) \; := \; \sup_{n\geq 0} \tau_n$$

The run $\vec{x}(\cdot)$ is said to be

1. discrete *if for some p and for all $n \geq p$, $\tau_n = 0$ (the run ends with a sequence),*

2. exhaustive *if all its cadences τ_n are finite, and thus,* nonexhaustive *if the sequence of cadences is finite and stops at some τ_N (the run ends with a continuous evolution),*

3. a Zeno run *if its life expectation $\lambda(\vec{x}(\cdot)) < +\infty$ is finite, and* NonZeno *in the opposite case,*

4. simple *is all the cadences $\tau_n > 0$ are strictly positive, i.e., if the sequence of impulse times t_n is strictly increasing,*

5. with bounded variations *if it is simple, exhaustive and nonZeno.*

We say that a run $\vec{x}(\cdot)$ is viable in K if for any $t \geq 0$, $\vec{x}(t) \in K$.

We observe that discrete runs are Zeno and that nonexhaustive runs are nonZeno (by definition).

We then define impulse evolutionary systems in the following way:

Definition 1.3 *Let $\Phi : X \rightsquigarrow X$ a set-valued map*[5]*, regarded as a* reset map*, and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an evolutionary system. Then the pair $(\mathcal{S},\Phi)$ governs a run $\vec{x}(\cdot)$ of an impulse evolutionary system if*

$$\forall\, n \geq 0, \begin{cases} i) & x_n(\cdot) \in \mathcal{S}(x_n(0)) \text{ or } (x_n(0), x_n(\cdot)) \in \text{Graph}(\mathcal{S}) \\ ii) & x_{n+1}(0) \in \Phi(x_n(\tau_n)) \text{ or } (x_n(\tau_n), x_{n+1}(0)) \in \text{Graph}(\Phi) \end{cases} \tag{2}$$

We shall denote by $\mathcal{R}(x) := \mathcal{R}^K_{(\mathcal{S},\Phi)}(x)$ the set of runs of the impulse evolutionary system $(\mathcal{S},\Phi)$ starting from $x \in K$ viable in K. We shall say that the impulse evolutionary system is simple (resp. nonZeno, exhaustive) on K if for all $x \in X$, all runs $\vec{x}(\cdot) \in \mathcal{R}(x)$ viable in K are simple (resp. nonZeno, exhaustive).

When $\mathcal{S} := \mathcal{S}_F$ is the solution map of a differential inclusion $x' \in F(x)$, we set $\mathcal{R}^K_{(\mathcal{S}_F,\Phi)} = \mathcal{R}^K_{(F,\Phi)}$.

We now introduce the concept of detector:

Definition 1.4 *Let $P : t \in \mathbf{R}_+ \rightsquigarrow P(t) \subset X$ be a tube regarded as a* informational tube *and $C \subset P(0)$.*

The detector *$\mathbf{D}_{(P,C)} : \text{Graph}(P) \rightsquigarrow C$ associates with any $T \geq 0$ and $x \in P(T)$ the (possibly empty) subset $\mathbf{D}_{(P,C)}(T,x)$ of initial states $x_0 \in C$ from which $x := x(T)$ can be reached by a run $\vec{x}(\cdot) \in \mathcal{R}_{(\mathcal{S},\Phi)}(x_0)$* detectable *(by the tube) in the sense that*

$$\forall\, t \in [0,T], \;\; \vec{x}(t) \;\in\; P(t) \tag{3}$$

The domain $\text{Dom}(\mathbf{D}_{(P,C)})$ is the set of detected states *$x \in P(T)$, in the sense that they can be reached at time T by at least one detectable evolution starting from C, whereas the image $\text{Im}(\mathbf{D}_{(P,C)})$ is the set of* detectable states *$x_0 \in C$ from which starts at least one detectable evolution.*

We shall characterize the graph of a detector as a impulse viable-capture basin under an auxiliary impulse system, a concept closed to the concept of impulse viability kernel introduced and studied in [27, 28, Cruck].

2 Impulse Capture Basins

We associate now with an impulse evolutionary system capture basins of targets defined in [, Aubin & Haddad] in the following way:

[5]When $\Phi : X \rightsquigarrow X$ is defined on X, we associate with it its "graphical restriction" to $K \times K$ (again denoted by) Φ defined on $G := K \cap \Phi^{-1}(K)$ and associating with x the subset $\Phi(x) \cap K$.

Definition 2.1 *Let K and $C \subset K$ be two subsets. The* impulse viable-capture basin

$$\text{ImpCapt}(K, C) := \text{ImpCapt}_{(\mathcal{S},\Phi)}(K, C)$$

of C viable in K under the impulse evolutionary system $(\mathcal{S}, \Phi)$ is the subset of initial states $x \in K$ from which starts at least *one run viable in K until it reaches the target C in finite time.*

We shall say that a subset K captures the target *$C \subset K$ in K under the impulse evolutionary system $(\mathcal{S}, \Phi)$ if $K := \text{ImpViab}_{(\mathcal{S},\Phi)}(K, C)$, i.e., if from any $x \in K$ starts* at least *one run viable in K until it reaches the target C in finite time.*

We also need the concept of backward impulse evolutionary system:

Definition 2.2 *We associate with an exhaustive impulse evolutionary system $(\mathcal{S}, \Phi)$ its* backward impulse evolutionary system *$\mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}$ defined by $\vec{x}(\cdot) \in \mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}(x)$ if and only if there exists a run $\vec{z}(\cdot) \in \mathcal{R}_{(\mathcal{S},\Phi)}(x)$ such that for every $T \geq 0$, the run $\widehat{x}(\cdot)$ defined by*

$$\widehat{x}(t) := \begin{cases} \vec{x}(T-t) & \text{if} \quad t \in [0, T] \\ \vec{z}(t-T) & \text{if} \quad t \geq T \end{cases}$$

belongs to $\mathcal{R}_{(\mathcal{S},\Phi)}(\vec{x}(T))$.

When $\mathcal{S} := \mathcal{S}_F$ is the solution map of a differential inclusion $x' \in F(x)$, we set $\mathcal{R}^K_{(\mathcal{S}_{F_-},\Phi)^{-1}} = \mathcal{R}^K_{(-F,\Phi^{-1})}$.

We observe that $\mathcal{R}_{(\mathcal{S}_{-\,-},(\Phi^{-1})^{-1})} = \mathcal{R}_{(\mathcal{S},\Phi)}$ and that $\widehat{x}(T) = x$ and $\widehat{x}(0) = \vec{x}(T)$.

Indeed, a backward run $\vec{x}(\cdot) = (\tau_n, x_n(\cdot))_n \in \mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}(x)$ of the impulse system is the run defined by a sequence of cadences τ_n and motives $x_n(\cdot) \in \mathcal{S}(x_n)$ if and only if for every $T \geq 0$, the run $\widehat{x}(\cdot) = (\widehat{\tau}_p, \widehat{x}_p(\cdot)) \in \mathcal{R}_{(\mathcal{S},\Phi)}(\vec{x}(T))$ where, N denoting the largest integer such that $t_N < T$,

1. the cadences $\widehat{\tau}_p$ are defined by

$$\begin{cases} i) & \widehat{\tau}_0 := T - t_N \\ ii) & \widehat{\tau}_p := \tau_{N-p}, \quad p = 1, \ldots, N, \\ iii) & \widehat{\tau}_p := 0, \quad p \geq N+1 \end{cases}$$

and the corresponding sequence of switching times $\widehat{t}_p$ is defined by

$$\begin{cases} i) & \widehat{t}_0 := 0 \\ ii) & \widehat{t}_p := T - t_{N-p+1}, \quad p = 1, \ldots, N, \\ iii) & \widehat{t}_p := T, \quad p \geq N+1 \end{cases}$$

2. and the motives by

$$\begin{cases} i) & \widehat{x}_p(\tau) \ := \ x_{N-p}(T-t), \ \ p = 0, \ldots, N \\ ii) & \widehat{x}_{N+1}(t) \ := \ \vec{z}(T-t) \end{cases}$$

satisfy $\widehat{x}_p(\cdot) \in \mathcal{S}(\widehat{x}_p)$ and the end-point condition $\widehat{x}_p(\widehat{\tau}_p) \in \Phi(\widehat{x}_{p+1})$.

The reinitialized states are equal to $\widehat{x}_p := x_{N-p-1}(\tau_{N-p-1}) \in \Phi^{-1}(\widehat{x}_{p-1}(\widehat{\tau_{p-1}}))$

We now relate the graph of a detector with the capture basin of an auxiliary system:

Lemma 2.3 *The graph of the detector* $\mathbf{D}_{(P,C)}$ *is the impulse viable-capture basin of* $\{0\} \times \mathrm{Graph}(\mathbf{I}|_C)$ *viable in* $\mathrm{Graph}(P) \times C$ *under the auxiliary impulse evolutionary system* $(\{-1\} \times \mathcal{S}_- \times \{0\}, \mathbf{I} \times \Phi^{-1} \times \mathbf{I})$*:*

$$\mathrm{Graph}(\mathbf{D}_{(P,C)}) \ := \ \mathrm{ImpCapt}_{(\{-1\}\times\mathcal{S}_-\times\{0\},\mathbf{I}\times\Phi^{-1}\times\mathbf{I})}(\mathrm{Graph}(P) \times C, \{0\} \times \mathrm{Graph}(\mathbf{I}|_C))$$

Proof — Indeed, to say that (T, x, x_0) belongs to the viable-capture basin of $\{0\} \times \mathrm{Graph}(\mathbf{I}|_C)$ viable in $\mathrm{Graph}(P)\times C$ under the auxiliary system $(\{-1\}\times\mathcal{S}_-\times\{0\}, \mathbf{I}\times\Phi^{-1}\times\mathbf{I})$ means that there exists a run $\widehat{x}(\cdot) \in \mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}(x)$ and a time $t^* \geq 0$ such that

$$\begin{cases} i) & \forall\, t \in [0, t^*], \ \ (T-t, \widehat{x}(t), x_0) \ \in \ \mathrm{Graph}(P) \times C \\ ii) & \qquad\qquad\qquad (T-t^*, \widehat{x}(t^*), x_0) \ \in \ \{0\} \times \mathrm{Graph}(\mathbf{I}|_C) \end{cases}$$

The second condition means that $t^* = T$ and that $\widehat{x}(T) = x_0$ belongs to C. The first one means that for every $t \in [0, T]$, $\widehat{x}(t) \in P(T-t)$. This amounts to saying that the run $\vec{x}(\cdot) := \widehat{x}(T-\cdot) \in \mathcal{R}_{(\mathcal{S},\Phi)}(x_0)$ where x_0 belongs to C satisfies the viability conditions (3), i.e., that x_0 belongs to $\mathbf{D}_{(P,C)}(T, x)$. □

3 Characterization of Impulse Capture Basins

From now on, *we shall assume that the impulse evolutionary system* $(\mathcal{S}, \Phi)$ *is nonZeno* (this means that all runs solutions to the impulse evolutionary system have an infinite life expectation $\lambda(\vec{x}(\cdot))$. In particular, they cannot end with an infinite sequence.)

Let us consider a target $C \subset K$ and the family $\mathcal{D}(K, C)$ of subsets D satisfying $C \subset D \subset K$.

We are led to characterize impulse viable-capture basins of targets in order to derive characterizations of the detector. We begin by quoting reults of [, Aubin & Haddad] (derived from results of [10, Aubin & Catté]) characterizing impulse viable-capture basins as unique common fixed points of the two maps $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K, \cdot) : \mathcal{D}(K, C) \mapsto \mathcal{D}(K, C)$ and $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(\cdot, C) : \mathcal{D}(K, C) \mapsto \mathcal{D}(K, C)$ associating with any subset $D \in \mathcal{D}(K, C)$ the impulse viability kernels $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K, D)$ and $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(D, C)$.

Theorem 3.1 *The impulse capture basin* $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ *of a target* $C \subset K$ *viable in* K *is*

1. *the* largest *subset* D *satisfying* $C \subset D \subset K$ *and* $D \subset \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(D,C)$,
2. *the* smallest *subset* D *satisfying* $C \subset D \subset K$ *and* $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,D) \subset D$,
3. *the* unique *subset* D *satisfying* $C \subset D \subset K$ *and*

$$D \;=\; \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,D) \;=\; \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(D,C)$$

4 First Characterization of Detectors

We derive a first characterization of the detector from the above characterization of the viable-capture basin:

Theorem 4.1 *The detector* $\mathbf{D}_{(P,C)}$ *is the* **unique** *set-valued map satisfying*

$$\forall\, x \in C,\;\; \mathbf{D}_{(P,C)}(0,x) \;:=\; \{x\}$$

and, for any $T > 0$

1. *for any* $x_0 \in \mathbf{D}_{(P,C)}(T,x)$*, there exists a run* $\vec{x}(\cdot)$ *to the impulse evolutionary system* $(\mathcal{S},\Phi)$ *starting from* C*, satisfying the viability conditions (3) on* $[0,T]$ *and reaching* x *at time* T*, and any such run actually satisfies*

$$\forall\, t \in [0,T],\;\; x_0 \in \mathbf{D}_{(P,C)}(t,\vec{x}(t)) \tag{4}$$

2. *for any* $x_0 \in C \backslash W(T,x)$*, for every run* $x(\cdot)$ *to the impulse evolutionary system* $(\mathcal{S},\Phi)$ *reaching* $x = x(T)$ *at time* T *and satisfying for some* $S \in [0,T]$ *the conditions*

$$\forall\, t \in [S,T],\;\; x(t) \;\in\; P(t)$$

   then

$$\forall\, t \in [S,T],\;\; x_0 \;\notin\; W(t,x(t))$$

As a consequence, for any $T > 0$ *and for any* $x_0 \in \partial_C \mathbf{D}_{(P,C)}(T,x)$ *in the boundary of* $\mathbf{D}_{(P,C)}(T,x)$ *relative to* C*, for every run* $\vec{x}(\cdot)$ *to the impulse evolutionary system* $(\mathcal{S},\Phi)$ *reaching* x *at time* T *and satisfying (4), then*

$$\forall\, t \in [0,T],\;\; x_0 \in \partial_C \mathbf{D}_{(P,C)}(t,\vec{x}(t))$$

Theorem 3.1 states that the impulse viable-capture basin

$$\text{Graph}(\mathbf{D}_{(P,C)}) \;=\; \text{Capt}_{(\{-1\}\times\mathcal{S}_-\times\{0\},\mathbf{I}\times\Phi^{-1}\times\mathbf{I})}\,(\text{Graph}(P)\times C,\{0\}\times\text{Graph}(\mathbf{I}|_C))$$

is the unique subset $\mathcal{V}\subset\text{Graph}(P)\times C$ containing $\{0\}\times\text{Graph}(\mathbf{I}|_C)$ satisfying

$$\mathcal{V} \;=\; \text{ImpCapt}_{(\{-1\}\times\mathcal{S}\times\{0\},\mathbf{I}\times\Phi^{-1}\times\mathbf{I})}(\mathcal{V},\{0\}\times\text{Graph}(\mathbf{I}|_C))$$

and

$$\text{ImpCapt}_{(\{-1\}\times\mathcal{S}\times\{0\},\mathbf{I}\times\Phi^{-1}\times\mathbf{I})}(\text{Graph}(P)\times C,\mathcal{V}) \;=\; \mathcal{V}$$

The first statement means that whenever (T,x,x_0) belongs to $\mathcal{V}$, there exists a run $\widehat{x}(\cdot)\in\mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}(x_0)$ such that $(T-t,\widehat{x}(t),x_0)$ belongs to $\mathcal{V}$ until it reaches $\{0\}\times\text{Graph}(\mathbf{I}|_C)$ at time T. This is equivalent to saying that

$$\forall\, t\in[0,T],\;\; x_0\in\mathbf{D}_{(P,C)}(T-t,\widehat{x}(t)) \;=\; \mathbf{D}_{(P,C)}(T-t,\vec{x}(T-t))$$

where $\vec{x}(\cdot):=\widehat{x}(T-\cdot)\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)$.

The second statement means that whenever (T,x,x_0) does not belong to $\mathcal{V}$, all runs $\widehat{x}(\cdot)\in\mathcal{R}_{(\mathcal{S}_-,\Phi^{-1})}(x)$ are such that $(T-t,\widehat{x}(t),x_0)$ do not belong to $\mathcal{V}$ whenever $(T-t,\widehat{x}(t),x_0)\in\text{Graph}(P)\times C$, i.e., whenever $\widehat{x}(t)\in P(T-t)$ for every $t\in[0,T]$. This is equivalent to saying that for all runs $\vec{x}(\cdot)\in\mathcal{R}_{(\mathcal{S},\Phi)}(x_0)$ reaching x at time T and satisfying for some $S\in[0,T]$ the conditions

$$\forall\, t\in[S,T],\;\; x(t)\;\in\;P(t)$$

then

$$\forall\, t\in[S,T],\;\; x_0\;\notin\;\mathbf{D}_{(P,C)}(t,x(t))$$

Let us consider now $x_0\in\partial_C\mathbf{D}_{(P,C)}(T,x)$ where $T>0$. This means that there exists a sequence $y_n\in C\backslash\mathbf{D}_{(P,C)}(T,x)$ converging to x_0. Hence (T,x,y_n) does not belong to the capture basin of $\{0\}\times\text{Graph}(\mathbf{I}|_C)$ viable in $\text{Graph}(P)\times C$. Therefore we know that for any run $\vec{x}(\cdot)\in\mathcal{R}_{(\mathcal{S},\Phi)}(x_0)$ satisfying detectability conditions (3), for any $t\in[0,T]$, $y_n\in C\backslash\mathbf{D}_{(P,C)}(t,\vec{x}(t))$. Taking any run $\vec{x}(\cdot)\in\mathcal{S}(x)$ satisfying (4) and passing to the limit when $n\to+\infty$, we infer that

$$\forall\, t\in[0,T],\;\; x_0\in\partial_C\mathbf{D}_{(P,C)}(t,\vec{x}(t))$$

5 Prerequisites on Impulse Capture Basins

5.1 Prerequisites from Viability Theory

We shall need some the following definitions and results from Viability Theory:

Definition 5.1 *Let K and $C \subset K$ be two subsets, C being regarded as a target, K as a constrained set.*

1. *The subset*[6] $\text{Viab}(K, C)$ *of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

   A subset $C \subset K$ is said to be isolated in K by $\mathcal{S}$ *if it coincides with its viability kernel:*

   $$\text{Viab}(K, C) \; = \; C$$

2. *The subset* $\text{Capt}(K, C)$ *of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K and*

   $$\text{Capt}(C) \; := \; \text{Capt}(X, C)$$

   is said to be the capture basin *of C.*

3. *When the target $C = \emptyset$ is the empty set, we set*

   $$\text{Viab}(K) \; := \; \text{Viab}(K, \emptyset) \;\; \& \;\; \text{Capt}(K, \emptyset) \; = \; \emptyset$$

   and we say that $\text{Viab}(K)$ *is the viability kernel of K.*

   A subset K is a repeller *under $\mathcal{S}$ if its viability kernel is empty, or, equivalently, if the empty set is isolated in K.*

In other words, the viability kernel $\text{Viab}(K)$ is the subset of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$. Furthermore, we observe that

$$\text{Viab}(K, C) \; = \; \text{Viab}(K \backslash C) \cup \text{Capt}(K, C) \tag{5}$$

[6]When C is not contained in K, we naturally set

$$\text{Viab}(K, C) \; := \; \text{Viab}(K, K \cap C) \;\; \& \;\; \text{Capt}(K, C) \; := \; \text{Capt}(K, K \cap C)$$

Consequently, the viability kernel $\text{Viab}(K, C)$ of K with target C coincides with the capture basin $\text{Capt}(K, C)$ of C viable in K whenever the viability kernel $\text{Viab}(K\backslash C)$ is empty, i.e., whenever $K\backslash C$ is a repeller:

$$\text{Viab}(K\backslash C) \;=\; \emptyset \;\Rightarrow\; \text{Viab}(K, C) \;=\; \text{Capt}(K, C) \tag{6}$$

This happens in particular when K is a repeller, or when the viability kernel $\text{Viab}(K)$ of K is contained in the target C.

When the evolutionary system $\mathcal{S} := \mathcal{S}_F$ comes from a differential inclusion $x' \in F(x)$, we introduce the following Frankowska property that we need for deriving the system of Hamilton-Jacobi-Bellman equations of which the detector is a solution:

Definition 5.2 *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets K and $C \subset K$. We shall say that a subset D between C and K satisfies the* Frankowska property *with respect to F if*[7]

$$\begin{cases} i) & \forall\, x \in D\backslash C,\;\; F(x) \cap T_D(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in D \cap \text{Int}(K),\;\; -F(x) \;\subset\; T_D(x) \\ iii) & \forall\, x \in D \cap \partial K,\;\; -F(x) \;\subset\; T_D(x) \cup T_{X\backslash K}(x) \end{cases} \tag{7}$$

Remark that when K is assumed further to be backward invariant and F to be Lipschitz, the above conditions (7) boil down to

$$\begin{cases} i) & \forall\, x \in D\backslash C,\;\; F(x) \cap T_D(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in D,\;\; -F(x) \subset T_D(x) \end{cases} \tag{8}$$

Viability[8] and Invariance[9] Theorems imply

Theorem 5.3 *Let us assume that F is Marchaud*[10]*, that K is closed and that a closed subset C satisfies $\text{Viab}_F(K\backslash C) = \emptyset$. Then the viable-capture basin $\text{Capt}_F(K, C)$ is*

[7]The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [13, Aubin & Frankowska]) or [40, Rockafellar & Wets] for more details).

[8]See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [1, Aubin].

[9]See for instance Theorems 5.3.4 of [1, Aubin].

[10]A set-valued map F is a Marchaud map if

$$\begin{cases} i) & \text{the graph of } F \text{ is closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of F is linear: } \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

1. *the* **largest** *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall\, x \in D \backslash C, \quad F(x) \cap T_D(x) \;\neq\; \emptyset \tag{9}$$

2. *if F is Lipschitz, the* **unique** *closed subset D satisfying the Frankowska property (7).*

5.2 Links between Capture Basins and Impulse Capture Basins

We recall the follwowing important characterization of the impulse capture basin given in [, Aubin & Haddad]:

Theorem 5.4 *Let us assume that the impulse differential inclusion is exhaustive. The impulse capture basin* $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ *is a fixed point*

$$\mathrm{Capt}_{\mathcal{S}}(K, \Phi^{-1}(\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C))) \;=\; \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$$

of the map $D \mapsto \mathrm{Capt}_{\mathcal{S}}(K,D)$ and actually, the smallest of such fixed points or even, of the subsets D between C and K such that

$$\mathrm{Capt}_{\mathcal{S}}(K, \Phi^{-1}(D)) \;\subset\; D$$

5.3 Regulation of Viable Runs

Since the impulse capture basin $\overline{D} := \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ is equal to $\mathrm{Capt}(K, \Phi^{-1}(\overline{D}))$ by Theorem 5.4, we infer that starting from any initial state $x_0 \in \overline{D}$, a run $\vec{x}(\cdot) = (\tau_n, x_n(\cdot))_{n\geq 0}$ is viable in K until it reaches the target C in finite time is regulated in the following way:

1. if $x_0 \in C$, then the runs stops, and otherwise,
2. if $x_0 \in \overline{D} \cap \Phi^{-1}(\overline{D})$, then $\tau_0 = 0$ and $x_1 \in \Phi(x_0)$
3. if $x_0 \in \overline{D} \backslash \Phi^{-1}(\overline{D})$, then the first motive $x_0(\cdot) \in \mathcal{S}(x_0)$ is viable[11] in $\overline{D}$ until the first time $\tau_0 > 0$ when either $x_0(^{-}\tau_0) \in C$, and then, the run stops, or $x_0(^{-}\tau_0) \in \Phi^{-1}(\overline{D})$, and then, we take $x_1 \in \Phi(x_0(^{-}\tau_0))$ as the next reinitialization state.

[11]and thus, regulated by the differential inclusion $x'(t) \in F(x(t) \cap T_{\overline{D}}(x))$ when the evolutionary system is the solution map of a differential inclusion $x' \in F(x)$.

5.4 The Frankowska Property of Impulse Capture Basin

When the evolutionary system $\mathcal{S} := \mathcal{S}_F$ comes from a differential inclusion $x' \in F(x)$, Theorem 5.3 characterizing capture basins by tangential properties, Theorem 5.4 relating them with impulse capture basins and Lemma 2.3 relating them to the graphs of detectors imply the following characterization of the impulse capture basins:

Theorem 5.5 *Let us assume that F is Marchaud, that K and $\Phi^{-1}(K)$ are closed and that $K \backslash \Phi^{-1}(K)$ is a repeller. The impulse capture basin* $\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$ *is the* smallest of the subsets *D between C and K satisfying*

1. *D is the* **largest** *closed subset L satisfying*
$$\Phi^{-1}(D) \subset L \tag{10}$$
and
$$\forall\, x \in L \backslash C, \;\; F(x) \cap T_L(x) \neq \emptyset \tag{11}$$
2. *if F is Lipschitz, D is the* **unique** *closed subset L satisfying (10) and the Frankowska property (7).*

Let us define the regulation map $\mathbf{R}((K,C))$ by

$$\mathbf{R}(K,C)(x) := \{u \in F(x) \mid F(x) \cap T_{\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)}(x) \neq \emptyset\}$$

In this case, we obtain the following regulation law:

Theorem 5.6 *Let us assume that F is Marchaud, that K and $\Phi^{-1}(K)$ are closed and that $K \backslash \Phi^{-1}(K)$ is a repeller.*

the motives $x_n(\cdot)$ of a run $\vec{x}(\cdot) := (\tau_n, x_n(\cdot))_{n \in \{0,\ldots,N\}}$ viable in K and reaching the target C in finite time are regulated by

$$\begin{cases} \text{whenever } x_n(t) \in \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C) \backslash \Phi^{-1}(\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)), \\ x_n'(t) \in \mathbf{R}(K,C)(x_n(t)) \;\; \& \;\; x_n(0) = 0 \end{cases}$$

and the reinitialized states by

whenever $x_n(\tau_n) \in \Phi^{-1}(\mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C))$, $x_{n+1} \in \Phi(x_n(\tau_n)) \cap \mathrm{ImpCapt}_{(\mathcal{S},\Phi)}(K,C)$

6 Second Characterization of Detectors

We now use Theorem 5.4 to prove a formula analogous to the formula for the valuation function of an optimal impulse control problem established in [4, Aubin]:

Proposition 6.1 *The detector* $\mathbf{D}_{(P,C)} : \text{Graph}(P) \rightsquigarrow C$ *under an impulse differential inclusion is the smallest solution* $W : \text{Graph}(P) \rightsquigarrow C$ *to the problem*

$$\bigcup_{t\in[0,T]} \bigcup_{x(\cdot)\in\mathcal{S}(X) \mid x(T)=x \ \&\ \forall\, s\in[t,T],\ x(s)\in P(s)} W(t,\Phi(x(t))) \ \subset\ W(T,x)$$

and thus, satisfies the condition

$$\bigcup_{t\in[0,T]} \bigcup_{x(\cdot)\in\mathcal{S}(X) \mid x(T)=x \ \&\ \forall\, s\in[t,T],\ x(s)\in P(s)} \mathbf{D}_{(P,C)}(t,\Phi(x(t))) \ \subset\ \mathbf{D}_{(P,C)}(T,x)$$

Proof — By Theorem 5.4, we need to characterize subsets of the form

$$\text{Capt}_{\{-1\}\times\mathcal{S}_-\times\{0\}}(\text{Graph}(P)\times C, (\mathbf{I}\times\Phi\times\mathbf{I})^{-1}(\text{Graph}(W)))$$

First, we observe that

$$(\mathbf{I}\times\Phi\times\mathbf{I})^{-1}(\text{Graph}(W)) \ =\ \text{Graph}(W\circ\Phi)$$

where $W\circ\Phi$ is defined by

$$(W\circ\Phi)(t,x) \ =\ W(t,\Phi(x))$$

Indeed, (t,x,y) belongs to $(\mathbf{I}\times\Phi\times\mathbf{I})^{-1}(\text{Graph}(W))$ if and only if there exists $x_*\in\Phi(x)$ such that $(t,x_*,y)\in\text{Graph}(W)$, i.e., if and only if $y\in W(t,x_*)\subset W(t,\Phi(x))$.

Second, let us consider two set-valued maps $U : \mathbf{R}_+\times X \rightsquigarrow X$ and $V : \mathbf{R}_+\times X \rightsquigarrow X$ contained in U, with which we associate the set-valued map $\mathbf{V}_{(U,W)}$ defined by

$$\text{Graph}(\mathbf{V}_{(U,W)}) \ =\ \text{Capt}_{\{-1\}\times\mathcal{S}_-\times\{0\}}(\text{Graph}(U),\text{Graph}(W))$$

that plays the role of a value function in optimal control.

We deduce from [8, Aubin] the following formula for the set-valued map $\mathbf{V}_{(U,W)}$:

$$\mathbf{V}_{(U,W)}(T,x) \ :=\ \bigcup_{x(\cdot)\in\mathcal{S}(X)\mid x(T)=x}\ \bigcup_{t\in[0,T]} \left(W(t,x(t)) \cap \bigcap_{s\in[t,T]} U(s,x(s)) \right)$$

In our case, the set-valued map U is defined by

$$\text{Graph}(U) \ :=\ \text{Graph}(K)\times C$$

so that $y \in U(t,x)$ if and only if $x \in P(t)$ and $y \in C$. Therefore

$$y \ \in \ \bigcap_{s\in[t,T]} U(s,x(s)) \text{ if and only if } y \in C \text{ and } \forall\, s \in [t,T],\ x(s) \in P(s)$$

In other words, the detector under an impulse differential inclusion is the smallest solution to the problem

$$\bigcup_{t\in[0,T]} \ \bigcup_{x(\cdot)\in\mathcal{S}(X)\ |\ x(T)=x\ \&\ \forall\, s\in[t,T],\ x(s)\in P(s)} W(t,\Phi(x(t))) \ \subset \ W(T,x)$$

7 Hamilton-Jacobi Characterization of the Detector

Finally, thanks to the concept of contingent derivative of a set-valued map, we also deduce from Theorem 5.5 as in [8, Aubin] that the detector is a Frankowska solution to an adequate "quasi" system of Hamilton-Jacobi-Bellman partial differential equations, "quasi" in the sense of quasi variational Hamilton-Jacobi inequalities introduced by Alain Bensoussan and Jacques-Louis Lions in optimal impulse control (see for instance [22, 23, Bensoussan & Lions J.-L.]for motivations, examples and a review and [4, Aubin]).

We refer to [1, 5, 8, Aubin], [16, Aubin & Frankowska] and their references for set-valued solutions to systems of Hamilton-Jacobi inclusions. For that purpose, we recall that the (graphical contingent) derivative of a set-valued map $V : \text{Graph}(P) \rightsquigarrow C$ may be defined by the relation

$$\text{Graph}(DV(t,x,y)) \ := \ T_{\text{Graph}(V)}(t,x,y)$$

(see for instance [13, Aubin & Frankowska]) or [40, Rockafellar & Wets] for more details on derivatives of set-valued maps and set-valued analysis).

Definition 7.1 *We shall say that a set-valued map* $V : \text{Graph}(P) \rightsquigarrow C$ *is a* Frankowska solution *to the Hamilton-Jacobi system of first-order partial differential inclusions*

$$0 \ \in \ \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} \cdot F(x) \tag{12}$$

satisfying the initial condition $V(0,x) = x$ *if its graph is closed, if*

$$\forall\, t > 0,\ \forall\, y \in V(t,x),\ \exists\, v \in F(x) \quad \textit{such that } 0 \in DV(t,x,y)(-1,-v)$$

and if for every $v \in F(x)$

$$\forall\, t \geq 0,\ \forall\, y \in V(t,x), \quad 0 \in DV(t,x,y)(1,v)$$

or

$$\text{whenever } (t,x) \in \partial\text{Graph}(P), \ \ v \notin DP(t,x)(1)$$

Theorem 5.5 and the above definition of contingent derivative of a set-valued map imply the following

Theorem 7.2 *Let us assume that F is Marchaud, that C is closed and the graph of the tube $P(\cdot)$ are closed and that for all $T \geq 0$, $P(T) \cap \Phi(P(T))$ is closed.*

The detector $\mathbf{D}_{(P,C)}$: $\mathrm{Graph}(P) \rightsquigarrow C$ is the smallest of the set-valued maps W : $\mathrm{Graph}(P) \rightsquigarrow C$ satisfying the initial condition

$$\forall\, x \in C,\;\; W(0,x) \;=\; \{x\}$$

the following property:

1. *the* **largest** *set-valued map $U : \mathrm{Graph}(P) \rightsquigarrow C$ with closed graph satisfying*

$$(W \circ \Phi)(x) \;\subset\; U(t,x) \tag{13}$$

   and

$$\forall\, t > 0,\; y \in U(t,x),\; \exists\, v \in F(x) \quad \text{such that } 0 \in DU(t,x,y)(-1,-v)$$

2. *If furthermore, F is assumed to be Lipschitz, the* **unique** *Frankowska solution V : $\mathrm{Graph}(P) \rightsquigarrow C$ to the Hamilton-Jacobi system of first-order differential inclusions (12) satisfying condition*

$$(V \circ \Phi)(t,x) \;\subset\; V(t,x) \tag{14}$$

Knowing the detector and its derivatives, we introduce the associated regulation map $\mathbf{R}_{(P,C)}$ defined by

$$\mathbf{R}(P,C)(t,x,y) \;:=\; \{u \in U(x)\;\; 0 \in D\mathbf{D}_{(P,C)}(t,x,y)(-1,-f(x,u))\}$$

that allows to regulate the detected runs:

Theorem 7.3 *Let us assume that F is Marchaud, that C is closed and the graph of the tube $P(\cdot)$ are closed and that for all $t \geq 0$, $P(t) \cap \Phi(P(t))$ is closed.*

Starting from $x_0 \in \mathbf{D}_{(P,C)}(T,x)$,

1. *either $x_0 \in \Phi^{-1}(C)$ and we take $\tau_0 = 0$ and a next reinitialization state $x_1 \in \Phi(x_0) \cap C$,*

2. *or $x_0 \notin \Phi^{-1}(C)$ and we take for first motive $x_0(\cdot)$ an evolution governed by*

$$\begin{cases} i) & x'(t) \;=\; f(x(t),u(t)) \\ ii) & u(t) \;\in\; \mathbf{R}_{(P,C)}(t,x(t),x_0) \end{cases}$$

   until the first time τ_0 (if any) when $x_0 \in \mathbf{D}_{(P,C)}(\tau_0,\Phi(x_0(\tau_0)))$ and we take $x_1 \in \Phi(x_0(\tau_0))$.

We then proceed recursively

Proof — Indeed, since (T, x, x_0) belongs to the impulse capture basin of $\{0\} \times \text{Graph}(\mathbf{I}|_C)$ viable in $\text{Graph}(P) \times C$ under the auxiliary impulse evolutionary system $(\{-1\} \times \mathcal{S}_- \times \{0\}, \mathbf{I} \times \Phi^{-1} \times \mathbf{I})$ by Lemma 2.3, then

1. either (T, x, x_0) belongs to $((\mathbf{I} \times \Phi \times \mathbf{I})(\text{Graph}(\mathbf{D}(P, C)))$, i.e.,, $x_0 \in \mathbf{D}(T, x_{-1})$ where $x_{-1} \in \Phi^{-1}(x)$, and thus, we take $(T_1, x_{-1}, y_1) \in (\mathbf{I} \times \Phi \times \mathbf{I})^{-1}(T, x, x_0)$, i.e., $T_1 = T$, $y_1 = x_0$ and $x_{-1} \in \Phi^{-1}(x)$ such that $x_0 \in \mathbf{D}_{(P,C)}(T, x_{-1})$

2. or we take a solution $t \mapsto (T - t, \widehat{x}(t), x_0)$ where $\widehat{x}(\cdot)$ is a solution to the differential inclusion
$$\widehat{x}'(t) \;=\; -f(\widehat{x}(t), \widehat{u}(t))$$
starting at x such that $(T - t, \widehat{x}(t), x_0)$ belongs to the graph of $\mathbf{D}_{(P,C)}$ until the first time $\tau_{-1} > 0$ (if any) when $(T - \tau_{-1}, \widehat{x}(T - \tau_{-1}), x_0)$ belongs to $(\mathbf{I} \times \Phi \times \mathbf{I})(\text{Graph}(\mathbf{D})_{(P,C)})$, i.e., such that $x_0 \in \mathbf{D}_{(P,C)}(T - \tau_{-1}, x_{-1})$ where $x^{-1} \in \Phi^{-1}(\widehat{x}(T - \tau_{-1}))$. It satisfies
$$\forall\, t \in [0, \tau_{-1}[, \;\; x_0 \;\in\; \mathbf{D}_{(P,C)}(T - t, \widehat{x}(t)) \;=\; \mathbf{D}_{(P,C)}(T - t, x(T - t))$$
This implies that
$$\text{for almost all } t \in [0, T - \tau_{-1}[, \;\; 0 \;\in\; D\mathbf{D}(T - t, \widehat{x}(t), x_0)(-1, -f(\widehat{x}(t), \widehat{u}(t)))$$
i.e., that
$$\text{for almost all } t \in [0, T - \tau_{-1}[, \;\; \vec{u}(t) \;\in\; \mathbf{R}_{(P,C)}(T - t, \vec{x}(t), x_0)$$
Let us set $x(t) := \widehat{x}(T - t)$ and $u(t) := \widehat{u}(T - t)$, so that $x(\cdot)$ is a solution to the control system
$$x'(t) \;=\; f(x(t), u(t))$$
starting at $\widehat{x}(T - \tau_1)$, satisfying $x(T - \tau_{-1}) = x$ and
$$\forall\, t \in]T - \tau_{-1}, T], \;\; x_0 \;\in\; \mathbf{D}_{(P,C)}(t, x(t))$$
and thus
$$\text{for almost all } t \in]T - \tau_{-1}, T], \;\; u(t) \;\in\; \mathbf{R}_{(P,C)}(t, x(t), x_0)$$

We proceed in this way until time T when $\vec{x}(0) = x_0$. □

Hence, feedbacks $\mathbf{r}$ regulating the motives of the runs starting from x_0, reaching x at time T and satisfying $\vec{x}(t) \in P(t)$ for all $t \in [0,T]$ are selections $\mathbf{r}(t,x,x_0) \in \mathbf{R}_{(P,C)}(t,x,x_0)$ with enough regularity for implying that to the differential equation

$$x'(t) \ = \ f(x(t), \mathbf{r}(t, x(t), x_0))$$

have solution starting at x_0 at time 0.

Naturally, continuous selections do satisfy this condition. But other less regular selections may have solutions : we can take, for instance, for $\mathbf{r}^0(t,x,x_0) \in \mathbf{R}_{(P,C)}(t,x,x_0)$ the element of $\mathbf{R}_{(P,C)}(t,x,x_0)$ with minimal norm, or use other selection mechanisms (see for instance Chapter 6 of [1, Aubin]). Observe also that we only need continuity with respect to x and measurability with respect to t (see [**?**, **?**, Frankowska, Plaskacz & Rzezuchowski]).

Once can also "differentiate" the regulation map for expressing the velocities $u'(t)$ of the control $u(\cdot)$ and thus, obtain dynamical feedbacks (see for instance Chapter 7 of [1, Aubin]) and heavy solutions that minimize the chattering.

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