Conditional Viability for Impulse Differential Games

Jean-Pierre Aubin and Nicolas Seube [1]

Université de Paris-Dauphine and ENSIETA[2]

Abstract

We introduce in this paper the concept of "impulse evolutionary game".

Examples of evolutionary games are usual differential games, differentiable games with history (path-dependent differential games), mutational differential games, etc.

Impulse evolutionary systems and games cover in particular "hybrid systems" as well as "qualitative systems".

The conditional viability kernel of a constrained set (with a target) is the set of initial states such that for all strategies (regarded as continuous feedbacks, or, more generally chosen in a class of feedbacks for which solutions of the associated control system exist) played by the second player, there exists a strategy of the first player such that the associated run starting from this initial state satisfies the constraints until it hits the target.

This paper characterizes the concept of conditional viability kernel for "qualitative games" and of conditional valuation function for "qualitative games" maximinimizing an intertemporal criterion.

The theorems obtained so far about viability/capturability issues for evolutionary systems, conditional viability for differential games and about impulse and hybrid systems are used to provide characterizations of conditional viability under impulse evolutionary games.

---

[1] Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00281, [Evolution Equations]

[2] The authors thank Alexandre Bayen for his comments and help.

Keywords: impulse differential games, differential games, conditional viability, Hamilton-Jacobi-Bellman-Isaacs quasi-variational inequalities

AMS Classification: 93C10, 93C15, 93C55, 49J24, 49J40, 49J53

Introduction

The main theorems dealing with viability and capturability issues are valid for *evolutionary systems* introduced in [6, Aubin], encompassing systems associated with control systems and differential inclusions, history dependent (or path-dependent, memory-dependent) control systems and differential inclusions ([35, 36, 37, Haddad]), as well as other evolution systems such as parabolic type differential inclusions (see [46, 47, 48, Shi Shuzhong]) and mutational equations $\overset{\circ}{x} \ni f(x)$ on metric spaces (see [2, Aubin]). They are set-valued maps

$$x \in X \rightsquigarrow \mathcal{S}(x) \subset \mathcal{C}(0.\infty; X)$$

mapping a space X into the space of continuous functions $x(\cdot) : t \in \mathbf{R} \mapsto x(t) \in X$ starting at x at time 0 and satisfying few extra conditions.

The main example is provided by solution maps $\mathcal{S}_F$ associated with differential inclusions $x'(t) \in F(x(t))$.

On the other hand, the concept of evolutionary game has been introduced in [9, Aubin & Catté] : it is an evolutionary system parametrized by a given family $\widetilde{\mathcal{V}}$ of feedbacks $\widetilde{v} : x \in X \mapsto \widetilde{v}(x) \in \mathcal{V}$. An evolutionary game $\mathcal{G}$ is a set-valued map

$$\mathcal{G} : (x, \widetilde{v}) \in X \times \widetilde{\mathcal{V}} \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x) \in \mathcal{C}(0, \infty; X)$$

such that, for any $\widetilde{v} \in \widetilde{\mathcal{V}}$, $x \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ is an evolutionary system.

Usual *differential or dynamical games* provide examples of evolutionary games: A dynamical game (P, Q, F) is defined by

- a "set-valued feedback map" $P : X \rightsquigarrow \mathcal{P}$
- a "*tychastic*" (or perturbation, disturbance) set-valued map $Q : X \rightsquigarrow \mathcal{V}$
- a set-valued map $F : X \times \mathcal{P} \times \mathcal{V} \rightsquigarrow X$ describing the dynamics of the dynamical

game:

$$\left\{\begin{array}{lll} i) & x'(t) & \in \ F(x(t),u(t),v(t))) \\ ii) & u(t) & \in \ P(x(t)) \\ iii) & v(t) & \in \ Q(x(t)) \end{array}\right.$$

Naturally, a *dynamical game* generates the evolutionary game $(x,\widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ where $\widetilde{\mathcal{V}}$ is the set of continuous selections of the set-valued map Q (feedbacks) and where $\mathcal{S}_{\widetilde{v}}(x)$ is the set of solutions to the differential inclusion

$$x'(t) \ \in \ F(x(t),P(x(t)),\widetilde{v}(x(t)))$$

starting at x.

Among the issues covered by the theory of differential games we single-out the concepts of *conditional viability kernel* of a constrained set with target under an evolutionary game: It is the set of initial states such that for all strategies (regarded as continuous feedbacks, or, more generally chosen in a class of feedbacks for which solutions of the associated control system exist) played by the second player, there exists a strategy of the first player such that the associated run starting from this initial state satisfies the constraints until it hits the target (see the precise definition below).

Finally, *hybrid control systems* have been "embedded" in [19, Aubin, Lygeros, Quincampoix, Sastry & Seube]), [3, Aubin] in the framework of *impulse differential inclusions*, later generalized [16, 18, Aubin & Haddad] under the concept of *impulse evolutionary systems* $(\mathcal{S},\Phi)$ where $\Phi : X \rightsquigarrow X$ is a reset map. Evolutions governed by impulse hybrid games are called *runs or executions* depending upon the strategies played by the two players: They are governed by the evolutionary system $\mathcal{S}$ until they reach the domain of the reset map Φ. When this happens, the evolution is reinitialized with a new initial condition obtained through the reset map Φ (see the precise definition below).

In this paper, we shall introduce the concept of impulse evolutionary game and study the concept of conditional viability kernel of a subset K (with a (possibly empty) target $C \subset K$) under such an impulse evolutionary game, which has been already studied in [28, 29, Crück], [30, Crück & Saint-Pierre], [44, 45, 42, Saint-Pierre] and [31, Crück, Quincampoix & Saint-Pierre]. The point is not only to adapt to impulse evolutionary games the results known for usual evolutionary games, but to study the specific issues of impulse evolutionary games allowing to characterize the conditional viability kernel under an impulse evolutionary game in terms of conditional viability kernels under "continuous" evolutionary games through an adequate algorithm.

The theorems obtained so far about viability/capturability issues for evolutionary systems, conditional viability for differential games and about impulse and hybrid systems are used to provide characterizations of conditional viability under impulse evolutionary "qualitative games" and of conditional valuation function for "qualitative games" maximinimizing an intertemporal criterion. In this case, it is shown that the epigraph of valuation functions of a wide class of intertemporal optimization dynamical games are impulse conditional viability kernels, and thus, enjoy their properties. Among them, we found standard impulse optimal problems, impulse stopping time problems and impulse worst time problems, These three classes of problems are treated in a same general footing, the two last classes being treated independently by classical or viscosity methods. The characterization of a valuation function in terms of impulse conditional viability kernel allows us to adapt the Saint-Pierre viability kernel algorithm for computing impulse conditional viability kernel the regulation maps providing the runs. It bypass the characterization in terms of (generalized) solution of a quasi-variational Hamilton-Jacobi inequalities to which one can give a precise meaning by translating the tangential conditions provided by the viability theorem, as in [4, Aubin] for instance, avoiding the regularity conditions that classical methods or the concepts of viscosity require (Most often, the valuation functions are only lower semicontinuous). We do not reproduce these properties here (see [8, Aubin, Bayen, Bonneuil & Saint-Pierre]) for lack of space.

The paper is organized as follows. The first section introduces the evolutionary systems associated with control systems. The second section provides the main results on impulse control systems and differential inclusions: it introduces the concepts of runs, of impulse evolutionary systems and impulse viability kernels. The third section is devoted to conditional viability kernels of sets under impulse evolutionary games. These results are applied in the fourth section to impulse optimal control problems. We need to recall these recent results couched in terms of concepts of viability theory before tackling in the fifth section the maximinimization of an intertemporal criterion involving "discrete and continuous" cost functions, by relating the epigraph of the impulse valuation function to the concept of impulse viability kernel.

0.1 Control Systems and Differential Inclusions

Consider a system (f, U) where

1. $f : X \times \mathcal{U} \mapsto X$ is a map associating the *velocity* $f(x, u)$ of the state x with any state-parameter pair (x, u) ,

2. $U : X \rightsquigarrow \mathcal{U}$ is a set-valued map associating a set $U(x)$ of parameters feeding back on the state x with any state x.

The relevant properties of a set-valued map are portrayed in its graph:

Definition 0.1.1 *The graph* Graph(U) *of a set-valued map* $U : X \rightsquigarrow Y$ *is the set of pairs* $(x, y) \in X \times Y$ *satisfying* $y \in U(x)$.

The domain *of* U *is the subset of elements* $x \in X$ *such that* $U(x)$ *is not empty:*

$$\mathrm{Dom}(U) \ := \ \{x \in X \ | \ U(x) \neq \emptyset\}$$

The image *of* U *is the union of the images (or values)* $U(x)$*, when* x *ranges over* X*:*

$$\mathrm{Im}(U) \ := \ \bigcup_{x \in X} U(x)$$

The inverse U^{-1} *of* U *is the set-valued map from* Y *to* X *defined by*

$$x \in U^{-1}(y) \iff y \in U(x) \iff (x, y) \in \mathrm{Graph}(U)$$

If U is a single-valued map, it coincides with the usual concept of graph.

Definition 0.1.2 *A control system* (f, U) *defines the evolutionary system* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$*: for any* $x \in X$*,* $\mathcal{S}(x)$ *is the set of evolutions* $x(\cdot)$ *governed by*

$$\left\{ \begin{array}{lll} (i) & x'(t) & = \ f(x(t), u(t)) \\ (ii) & u(t) & \in \ U(x(t)) \end{array} \right. \tag{1}$$

starting from x.

Lemma 0.1.3 *Let* $F(x) := f(x, U(x))$ *denote the set of velocities of the parameterized system. The evolutions* $x(\cdot)$ *governed by the parameterized system*

$$\left\{ \begin{array}{lll} (i) & x'(t) & = \ f(x(t), u(t)) \\ (ii) & u(t) & \in \ U(x(t)) \end{array} \right. \tag{2}$$

are governed by the differential inclusion

$$x'(t) \ \in \ F(x(t)) \tag{3}$$

and conversely.

By taking $f(x,u) := u$ and $U(x) := F(x)$, any differential inclusion $x'(t) \in F(x(t))$ appears as a parameterized system (f,U) parameterized by its velocities. we consider mainly evolutions $x(\cdot)$ viable in a subset $K \subset X$ representing a constrained set (an environment) in which the trajectory of the evolution must remain forever:

$$\forall\, t \geq 0,\;\; x(t) \;\in\; K \tag{4}$$

Alternatively, a "target" $C \subset K$ being given, we distinguish evolutions $x(\cdot)$ capturing the target C in the sense that they are viable in K until they reach the target C in finite time:

$$\exists\, T \geq 0 \text{ such that} \begin{cases} x(T) \in C \\ \forall\, t \;\in\; [0,T],\; x(t) \;\in\; K \end{cases} \tag{5}$$

We say that an evolution is viable in K outside C if it is viable in K forever or until it reaches the target C in finite time.

Definition 0.1.4 *Let $K \subset X$ be a constrained set and $C \subset K$ be a target.*

1. *The subset $\mathrm{Viab}(K,C) := \mathrm{Viab}_{\mathcal{S}}(K,C)$ of initial states $x_0 \in K$ such that **at least one evolution** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

   When the target $C = \emptyset$ is the empty set, we say that $\mathrm{Viab}(K) := \mathrm{Viab}(K,\emptyset)$ is the viability kernel *of K.*

2. *The subset $\mathrm{Capt}(K,C) := \mathrm{Capt}_{\mathcal{S}}(K,C)$ of initial states $x_0 \in K$ such that **at least one evolution** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the* capture basin of C viable in K *under $\mathcal{S}$. When $K = X$ is the whole space, we say that $\mathrm{Capt}(C) := \mathrm{Capt}(X,C)$ is the* capture basin *of C.*

We say that

1. *a subset K is viable under $\mathcal{S}$ if $K = \mathrm{Viab}(K)$,*
2. *K is* viable outside the target $C \subset K$ *under the evolutionary system $\mathcal{S}$ if $K = \mathrm{Viab}(K,C)$,*
3. *that C is* isolated *in K if $C = \mathrm{Viab}(K,C)$,*

4. that K is a repeller *if* $\text{Viab}(K) = \emptyset$*, i.e., if the empty set is isolated in* K*.*

We start with a simple and important algebraic property of [9, Aubin & Catte]:

Theorem 0.1.5 *Let* $\mathcal{S}$ *be an evolutionary system,* $K \subset X$ *be a constrained set and* $C \subset K$ *be a target.*

The viability kernel $\text{Viab}(K, C)$ ***of*** K ***outside the target*** C ***is the unique subset between*** C ***and*** K ***that is both***

1. ***viable outside*** C *(and is the largest subset* $D \subset K$ *viable outside* C*),*
2. ***isolated in*** K *(and is the smallest subset* $D \supset C$ *isolated in* K*):*

$$\text{Viab}(K, \text{Viab}(K, C)) \;=\; \text{Viab}(K, C) \;=\; \text{Viab}(\text{Viab}(K, C), C) \qquad (6)$$

The viability kernel satisfies the properties of both the subsets viable outside a target and of isolated subsets in a constrained set, and is the unique one to do so.

This statement is at the root of uniqueness properties of solutions to some Hamilton-Jacobi-Bellman partial differential equations whenever the epigraph of a solution is a viability kernel of the epigraph of a function outside the epigraph of another function.

The uniqueness follows from

Lemma 0.1.6 *Let us consider a map* $(K, C) \mapsto \mathcal{A}(K, C)$ *satisfying*

$$\left\{ \begin{array}{ll} (i) & C \;\subset\; \mathcal{A}(K, C) \;\subset K \\ (ii) & (K, C) \mapsto \mathcal{A}(K, C) \text{ is increasing} \end{array} \right. \qquad (7)$$

1. *If* $\mathcal{A}(K, C) \;=\; \mathcal{A}(\mathcal{A}(K, C), C)$*, it is the largest fixed point of the map* $D \mapsto \mathcal{A}(D, C)$ *between* C *and* K*,*
2. *If* $\mathcal{A}(K, C) \;=\; \mathcal{A}(K, \mathcal{A}(K, C))$*, it is the smallest fixed point of the map* $E \mapsto \mathcal{A}(K, E)$ *between* C *and* K*.*

Then, any subset D *between* C *and* K *satisfying*

$$D \;\subset\; \mathcal{A}(D, C) \text{ and } \mathcal{A}(K, D) \;=\; D$$

is the unique *bilateral fixed point* D *between* C *and* K *of the map* $\mathcal{A}$ *in the sense that:*

$$\mathcal{A}(K, D) \;=\; D \;=\; \mathcal{A}(D, C)$$

and is equal to $\mathcal{A}(K, C)$*.*

See [9, Aubin & Catte] for a systematic study.

0.2 Conditional Viability Kernels of Sets under Impulse Evolutionary Systems

0.2.1 Runs

We refer to [15, 16, 17, 18, Aubin & Haddad] and [19, Aubin, Lygeros, Quincampoix, Sastry & Seube] for more details on the viability approach to impulse and hybrid control systems.

We identify $\mathcal{C}(0,0;X)$ with X and we define "runs" in the following way:

Definition 0.2.1 . *We say that a (finite or infinite) sequence*

$$\overrightarrow{x}(\cdot) \ := \ (\tau_n, x_n(\cdot))_{n\geq 0} \ \in \ \prod_{n\geq 0} \mathbf{R} \times \mathcal{C}(0,\tau_n;X)$$

is a run *(or an* execution *or a* punctuated evolution*)* $\overrightarrow{x}(\cdot)$ *made of*

1. *a finite or infinite sequence* $\tau(\overrightarrow{x}(\cdot)) \ := \ \{\tau_n\}_n$ *of nonnegative* cadences $\tau_n \in [0,+\infty[$,
2. *a sequence of* motives $x_n(\cdot) \in \mathcal{C}(0,\tau_n;X)$.

We associate with a run $\overrightarrow{x}(\cdot)$

1. *its* impulse set $\mathcal{T}(\overrightarrow{x}(\cdot)) := \{t_n\}_{n\geq 0}$ *of sequences of* impulse times $t_{n+1} := t_n + \tau_n$, $t_0 = 0$
2. *its development defined by*

$$\forall\, n \geq 0, \ \overrightarrow{x}(t_n) := x_n(0) \ \ \& \ \forall\, t \in [t_n, t_{n+1}[, \ \ \overrightarrow{x}(t) \ := \ x_n(t-t_n)$$

We say that the sequence of $x_n := x_n(0) \in X$ *is the sequence of* reinitialized states *of the run* $\overrightarrow{x}(\cdot)$.

A run $\overrightarrow{x}(\cdot)$ *is said to be* viable *in* K *on an interval* $I \subset \mathbb{R}_+$ *if for any* $t \in I$, $\overrightarrow{x}(t) \in K$.

Naturally, if $\tau_n = 0$, i.e., if $t_{n+1} = t_n$, we identify the motive $x_n(\cdot)$ with the reinitialized state $x_n(\cdot) \equiv x_n \in \mathcal{C}(0,0;X) \equiv X$, so that runs can be continuous-time evolutions, sequences (discrete time evolutions), or hybrids of them.

Definition 0.2.2 *A run* $\overrightarrow{x}(\cdot) := (\tau_n, x_n(\cdot))_{n \geq 0}$ *is said to be*

1. discrete *if for some* p *and for all* $n \geq p$, $\tau_n = 0$ *(the run ends with a sequence),*
2. *infinite* *if the sequence of cadences is infinite and finite otherwise. In this case, it stops at some* τ_N*, and we agree to say that the* N*th motive* $x_N(\cdot)$ *is taken on* $[0, +\infty[$ *and to set* $\tau_{N+1} = +\infty$ *(the run ends with a continuous-time evolution).*

If $t \notin \mathcal{T}(\overrightarrow{x}(\cdot))$ is not an impulse time of the run $\overrightarrow{x}(\cdot)$, then we can set $\overrightarrow{x}(t) = x(t)$ without ambiguity. At impulse time $t_n \in \mathcal{T}(\overrightarrow{x}(\cdot))$, we defined $\overrightarrow{x}(t_n) := x_n(0)$. But we also need to define the value of the run just before it "jumps" at impulse time:

Definition 0.2.3 *If* $t_n \in \mathcal{T}(\overrightarrow{x}(\cdot))$ *is an impulse time of the run* $\overrightarrow{x}(\cdot)$*, we set*

$$\overrightarrow{x}(^{-}t_n) := \begin{cases} \lim_{\tau \mapsto t_n -} x(\tau) & \text{if} \quad t_n > t_{n-1} \\ x_{n-1}(0) := \overrightarrow{x}(^{-}t_{n-1}) & \text{if} \quad t_n = t_{n-1} \end{cases}$$

We associate with a run $\vec{x}(\cdot)$ *its sequence of* jumps $\mathbf{s}(\vec{x}(\cdot)) := (\mathbf{s}_n(\vec{x}(\cdot)))_{n \geq 1}$ *defined by*

$$\mathbf{s}_n(\vec{x}(\cdot)) := x_n(0) - x_{n-1}(\tau_{n-1}) = \overrightarrow{x}(t_n) - \overrightarrow{x}(^{-}t_n)$$

We now adapt to runs the integral representation $x(t) = x_0 + \int_0^t x'(\tau)d\tau$ of an evolution:

Proposition 0.2.4 *Assume that the motives* $x_n(\cdot)$ *of a run* $\overrightarrow{x}(\cdot)$ *are differentiable on the intervals* $]0, \tau_n[$ *for all* n *such that the cadence* $\tau_n > 0$ *is positive. Therefore, for all* j*,*

$$\begin{cases} (i) \quad \forall\, t \in [t_{j-1}, t_j[, \quad \overrightarrow{x}(t) = x_0 + \sum_{k=1}^{j-1} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^t x'(\tau)d\tau \quad \text{when } t_{j-1} < t_j \\ (ii) \quad x_j := \overrightarrow{x}(t_j) = x_0 + \sum_{k=1}^{j} \mathbf{s}_k(\vec{x}(\cdot)) + \int_0^{t_j} x'(\tau)d\tau \end{cases}$$

0.2.2 Impulse Evolutionary Systems

Consider control system (Equation 1):

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases}$$

with state-dependent constraints on the controls, generating the *evolutionary system* denoted by $\mathcal{S}$ associating with any initial state x the subset $\mathcal{S}(x)$ of solutions $x(\cdot)$ to (Equation 1) starting at x.

Definition 0.2.5 *We introduce a set-valued map $\Phi : X \rightsquigarrow X$, regarded as a* reset map *governing the resetting of the initial conditions.*

The subset

$$Egress_{\Phi}(K) \;:=\; K \backslash \Phi^{-1}(K) \;=\; K \cap \Phi^{\ominus 1}(\complement K) \;=\; K \backslash Viab^{1}_{\Phi}(K) \tag{8}$$

is called the egress set *of K under Φ.*

The graphical restriction $\Phi|_{(K \to K)} : K \rightsquigarrow K$ *of Φ to $K \times K$ defined by*

$$\Phi|_{(K \rightsquigarrow K)}(x) \;:=\; \begin{cases} \Phi(x) \cap K & \text{if} \quad x \in Viab^{1}_{\Phi}(K) := K \cap \Phi^{-1}(K) \\ \emptyset & \text{if} \quad x \in Egress_{\Phi}(K) \;:=\; K \backslash \Phi^{-1}(K) \end{cases}$$

the graph of which being equal

$$\mathrm{Graph}(\Phi_{(K \rightsquigarrow K)}) \;=\; \mathrm{Graph}(\Phi) \cap (K \times K)$$

The domain $Viab^{1}_{\Phi}(K) := K \cap \Phi^{-1}(K)$ of the graphical restriction to K of the reset map Φ is known under various names, such as the transition set*,* stopping set *or* guard *in the control literature. The* discrete egress set *$Egress_{\Phi}(K) := K \backslash \Phi^{-1}(K)$ of K under Φ is called the* continuation set *in optimal impulse control theory. The image of the graphical restriction is equal to $K \cap \Phi(K)$.*

Remark: Extension of Reset Maps — In some problems, the reset map $\Phi : G \rightsquigarrow K$ is given as a set-valued map from a subset $G \subset K$ of K to K. This particular situation fits the general framework since Φ coincides with the graphical restriction

$$\Phi \;=\; \Phi|^{K}_{(K \rightsquigarrow K)}$$

of its extension $\Phi^{K} : X \rightsquigarrow X$ defined by

$$\Phi|^{K}(x) \;:=\; \begin{cases} \Phi(x) & \text{if} \quad x \in G \\ \emptyset & \text{if} \quad x \notin G \end{cases}$$

and thus, $\mathrm{Viab}^{1}_{\Phi^{K}}(K) = G$ and $\mathrm{Egress}_{\Phi^{K}}(K) = K \backslash G$. ■

An impulse evolutionary system is associated with an evolutionary system $\mathcal{S}$ governing the continuous-time components of a run and a reset map Φ governing its discrete components in the following way:

Definition 0.2.6 *Let $\Phi : X \rightsquigarrow X$ be the reset map and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ the evolutionary system associated with a control system. We associate with any initial state $x \in X$ the subset $\mathcal{R}(x) := \mathcal{R}_{(\mathcal{S}, \Phi)}(x)$ of runs $\overrightarrow{x}(\cdot)$ satisfying*

$$\forall\, n \geq 0, \begin{cases} (i) & x_n(\cdot) \in \mathcal{S}(x_n(0)) & \text{or} & (x_n(0), x_n(\cdot)) \in \mathrm{Graph}(\mathcal{S}) \\ (ii) & x_{n+1}(0) \in \Phi(x_n(\tau_n)) & \text{or} & (x_n(\tau_n), x_{n+1}(0)) \in \mathrm{Graph}(\Phi) \end{cases} \tag{9}$$

Such a run is called a solution to the impulse evolutionary system starting at x.

The set-valued map $\mathcal{R} : X \rightsquigarrow \prod_{n\geq 0} \mathbf{R}\times \mathcal{C}(0,\tau_n;X)$ is called the impulse evolutionary system *associated with the pair $(\mathcal{S},\Phi)$.*

A subset K is said to be viable outside a target $C \subset K$ under the impulse evolutionary system $\mathcal{R}$ is from every $x\in K$ starts at least one run $\overrightarrow{x}(\cdot)\in\mathcal{R}(\cdot)$ viable in K forever or until it reaches the target C in finite time.

We provide other formulations of solutions to impulse differential inclusions:

Lemma 0.2.7 *Let $\mathcal{R} := \mathcal{R}_{(\mathcal{S},\Phi)}$ be an impulse evolutionary system and consider a subset $K\subset X$. A run $\overrightarrow{x}(\cdot)\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)$ is a solution to the impulse evolutionary system if and only if for every $n\geq 0$, the impulse times are given by $t_{n+1} := t_n+\tau_n$, $t_0=0$,*

$$\left\{\begin{array}{l} \overrightarrow{x}(t_n) \ \in \ \Phi(\overrightarrow{x}(^-t_{n-1})) \\ \forall\, t\in[t_n,t_{n+1}[,\ \ \overrightarrow{x}(t) \ := \ x(\cdot) \text{ is an evolution stating at impulse time } t_n \text{ at } \overrightarrow{x}(t_n) \end{array}\right.$$

Such a solution can also be written in the form

$$\left\{\begin{array}{l} (i) \quad \forall\, t\in[t_{n-1},t_n[,\ \ \overrightarrow{x}(t) \ = \ x_0+\displaystyle\sum_{k=0}^{n-1}(\Phi-\mathbf{1})(\overrightarrow{x}(^-t_k))+\int_0^t F(x(\tau))d\tau \ \text{ when } t_{n-1}<t_n \\ (ii) \quad \overrightarrow{x}(t_n) \ = \ x_0+\displaystyle\sum_{k=0}^{n}(\Phi-\mathbf{1})(\overrightarrow{x}(^-t_k))+\int_0^{t_n} F(x(\tau))d\tau \end{array}\right.$$

We begin our study by characterizing subsets viable under a impulse evolutionary system in terms of viability kernels under its continuous-time component $\mathcal{S}$ and its discrete-time component Φ:

Theorem 0.2.8 *Let $\mathcal{R} := \mathcal{R}_{(\mathcal{S},\Phi)}$ be an impulse evolutionary system and two subsets $K\subset X$ and $C\subset K$. We set*

$$\mathrm{Viab}^1_\Phi(K,C) \ := \ C\cup(K\cap\Phi^{-1}(K))$$

A subset $K\subset X$ is viable outside the target C under the impulse evolutionary system $\mathcal{R}$ if and only if K is viable outside $\mathrm{Viab}^1_\Phi(K,C)$ under the evolutionary system $\mathcal{S}$:

$$K \ = \ \mathrm{Viab}_{\mathcal{S}}(K,\mathrm{Viab}^1_\Phi(K,C))$$

If

$$K \ = \ \mathrm{Capt}_{\mathcal{S}}(K,\mathrm{Viab}^1_\Phi(K,C))$$

then from any initial state x starts at least one run $\overrightarrow{x}(\cdot)\in\mathcal{R}(x)$ with finite cadences viable in K forever or until it reaches the target C in finite time. This happens whenever $K\backslash\mathrm{Viab}^1_\Phi(K,C)$ is a repeller.

0.2.3 Impulse Viability Kernels

We associate now with an impulse evolutionary system viability kernels of targets in the following way:

Definition 0.2.9 *Let K and $C \subset K$ be two subsets. The* impulse viability kernel *of K*

$$\text{ImpViab}(K, C) \ := \ \text{ImpViab}_{(\mathcal{S},\Phi)}(K, C)$$

with target C under the impulse evolutionary system $(\mathcal{S}, \Phi)$ is the subset of initial states $x \in K$ from which starts at least *one run viable in K forever or until it reaches the target C in finite time.*

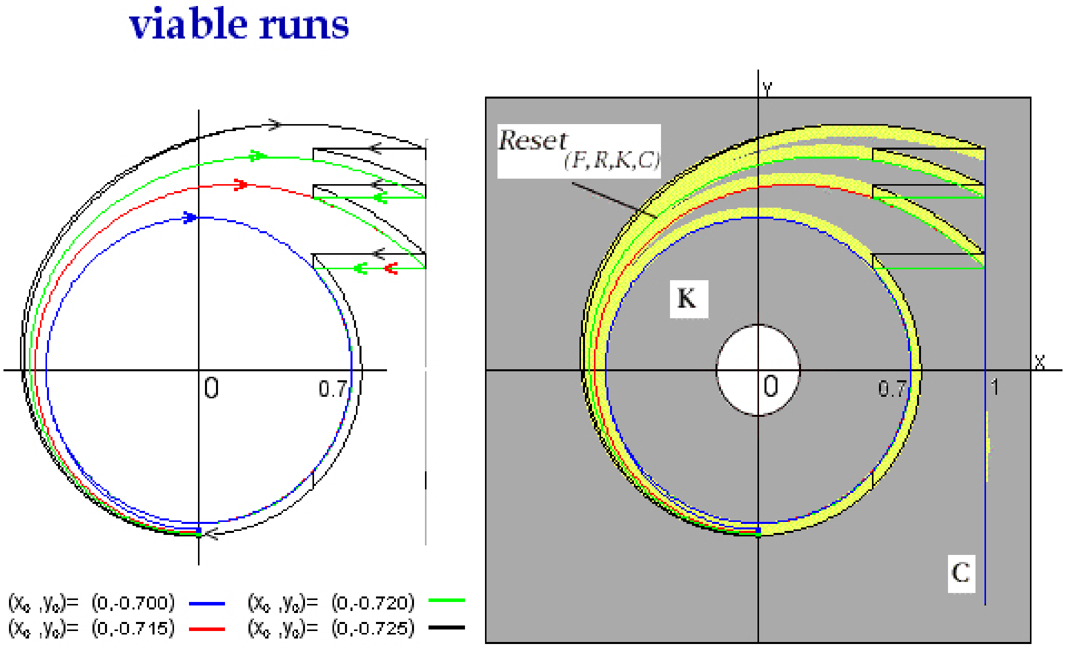

Figure 1: Example of an impulse viability kernel. The (single-valued) map F is defined by $F(x) := (x_2 + \frac{\|x\|-0.7}{\|x\|}x_1, -x_1 + \frac{\|x\|-0.7}{\|x\|}x_2)$. The (single-valued) map Φ maps point $(1, x_2)$ to points $(0.5, x_2)$. The subset K is the complement of a small ball around $(0, 0)$ in the square $[-1.2, +1.2]^2$. Source: Patrick Saint-Pierre.

We recall the fundamental characterization of impulse viability kernels:

Theorem 0.2.10 *The impulse viability kernel* $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *of* K *with a target* $C \subset K$ *is*

1. *the* largest *subset* D *satisfying* $C \subset D \subset K$ *and* $D \subset \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(D,C)$*,*
2. *the* smallest *subset* D *satisfying* $C \subset D \subset K$ *and* $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,D) \subset D$*,*
3. *the* unique *subset* D *satisfying* $C \subset D \subset K$ *and*

$$D \;=\; \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,D) \;=\; \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(D,C)$$

The map $(K,C) \mapsto \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ satisfies the property

$$C \;\subset\; \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C) \;\subset\; K$$

and is increasing in the sense that

$$\text{If } C_1 \subset C_2 \;\&\; K_1 \subset K_2, \;\; \text{then } \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K_1,C_1) \;\subset\; \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K_2,C_2)$$

Therefore Lemma 0.1.6 implies that the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ is the unique bilateral fixed point between C and K of the map $(K,C) \mapsto \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$.

Theorem 0.2.8 states that whenever K is viable outside C under an impulse evolutionary system, it is a fixed point of the map $D \mapsto \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C)$, that, as a matter of fact, plays an important role for characterizing impulse viability kernels.

We begin by observing that

Lemma 0.2.11 *The map* $D \mapsto \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C))$ *is increasing and satisfies*

$$C \;\subset\; \mathrm{Viab}^1_{\Phi}(D,C) \;\subset\; \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C)) \;\subset\; D$$

We deduce another fixed point characterization of the impulse viability kernel:

Theorem 0.2.12 *The impulse viability kernel* $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *is the largest of the fixed points of the map* $D \mapsto \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C)$*:*

$$\mathrm{Viab}_{\mathcal{S}}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C), \mathrm{Viab}^1_{\Phi}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)), C) \;=\; \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$$

Proof — Theorem 0.2.8 implies that any subset D between C and $\mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C))$ is viable in $D \subset K$ outside C under the impulse evolutionary system $\mathcal{R}_{(\mathcal{S},\Phi)}$, so that it is contained in the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$.

Conversely, the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ is a fixed point of the map $D \mapsto \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}^1_{\Phi}(D,C)$. Indeed, by Theorem 0.1.5, the impulse viability kernel is itself viable in $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ outside C under the impulse evolutionary system, so that Theorem 0.2.8 implies that the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ is a fixed point. ■

Since the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ is equal to $\mathrm{Capt}(K, K \cap \Phi^{-1}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)))$ by Theorem 0.2.12, we infer how to regulate viable runs:

Theorem 0.2.13 *Starting from any initial state* $x_0 \in \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$*, a run* $\overrightarrow{x}(\cdot) = (\tau_n, x_n(\cdot))_{n\geq 0}$ *viable in* K *forever or until it reaches the target* C *in finite time is regulated in the following way: Assume that at stage* n*,* x_n *belong to the impulse viability kernel* $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *of* K *outside* C*.*

1. *if* $x_n \in C$*, then the run may stop,*
2. *if* $x_n \in \mathrm{Viab}^1_{\Phi}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)\backslash C$*, then we may take* $\tau_n = 0$ *and* $x_{n+1} \in \Phi(x_n) \cap \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *as the next reinitialization state,*
3. *if* $x_n \in \mathrm{Viab}^1_{\Phi}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)\backslash C$*, then we must take one motive motive* $x_n(\cdot) \in \mathcal{S}(x_n)$ *viable in* $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *forever (and the run may stop) or until the first time* $\tau_n > 0$ *when* $x_n(\tau_n) \in \mathrm{Viab}^1_{\Phi}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C),C)$*. In this case,*
    - *(a) either* $x_n(\tau_n) \in C$*, and the run may stop,*
    - *(b) or* $x_n(\tau_n) \in \Phi^{-1}(\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C))$*, and then, we take* $x_{n+1} \in \Phi(x_n(\tau_n) \cap \mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *as the next reinitialization state.*

We provide a way to build the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ is an intersection of viability kernels under the evolutionary system: This is the Impulse Viability Kernel Algorithm.

Theorem 0.2.14 *Assume that* K *is a compact repeller under* $\mathcal{S}$*, that* F *is Marchaud and that* $C \subset K$ *and the graph of* Φ *are closed.*

The sequence of subsets K_j *starting at* $K_0 := K$ *and defined recursively by*

$$K_{j+1} := \mathrm{Viab}_{\mathcal{S}}(K_j, \mathrm{Viab}^1_{\Phi}(K_j,C))$$

The impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$ *is obtained through the "Impulse Viability Kernel Algorithm":*

$$\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C) \;=\; \bigcap_{j}^{+\infty} K_j$$

Proof — We already know that

$$\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C) \;\subset\; \bigcap_{j=0}^{+\infty} K_j \;=:\; K_\infty$$

Conversely, take $x \in K_\infty$. Since for any $n \geq 0$, $x \in K_{n+1} := mbox\mathrm{Viab}_{\mathcal{S}}(K_j, \mathrm{Viab}_{\Phi}^{1}(K_j,C))$, there exists an evolution $x_n(\cdot) \in \mathcal{S}(x)$ and $\tau_n \leq \overline{\tau} < +\infty$ such that $x_n(\tau_n) \in \mathrm{Viab}_{\Phi}^{1}(K_j,C))$. Since the evolutionary system $\mathcal{S}$ is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges to some evolution $x_\infty(\cdot) \in \mathcal{S}(x)$ and a subsequence (again denoted by) τ_n converges to some $\tau_\infty \leq \overline{\tau}$, so that $x_n(\tau_n)$ converges to $x_\infty(\tau_\infty)$. We deduce that

$$x_\infty(\tau_\infty) \;\in\; \bigcap_{j=0}^{\infty} \mathrm{Viab}_{\Phi}^{1}(K_\infty,C)) \;\subset\; \mathrm{Viab}_{\Phi}^{1}(K_\infty,C))$$

Therefore, K_∞ is a fixed point of $D \mapsto \mathrm{Viab}_{\mathcal{S}}(D, \mathrm{Viab}_{\Phi}^{1}(D,C)$, and thus, contained in its largest fixed point, the impulse viability kernel $\mathrm{ImpViab}_{(\mathcal{S},\Phi)}(K,C)$. ■

0.3 Conditional Viability Kernels of Sets under Impulse Evolutionary Games

Definition 0.3.1 *A* dynamical game (P,Q,F) *is defined by*

- *a "set-valued feedback map"* $P : X \rightsquigarrow \mathcal{P}$
- *a "tychastic" (or perturbation, disturbance) set-valued map* $Q : X \rightsquigarrow \mathcal{V}$
- *a set-valued map* $F : X \times \mathcal{P} \times \mathcal{V} \rightsquigarrow X$ *describing the dynamics of the dynamical game:*

$$\left\{ \begin{array}{lll} (i) & x'(t) \;\in\; F(x(t),u(t),v(t))) \\ (ii) & u(t) \;\in\; P(x(t)) \\ (iii) & v(t) \;\in\; Q(x(t)) \end{array} \right.$$

Usual differential games provide examples of evolutionary games:

Definition 0.3.2 *A dynamical game generates the evolutionary game*

$$\mathcal{S}_{\widetilde{\mathcal{V}}}(x,\widetilde{v}) \in X \times \widetilde{\mathcal{V}} \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x) \in \mathcal{C}(0,\infty;X)$$

where $\widetilde{\mathcal{V}}$ *is the set of continuous selections of the set-valued map* Q *(feedbacks) and where* $\mathcal{S}_{\widetilde{v}}(x)$ *is the set of solutions to the differential inclusion*

$$x'(t) \;\in\; F(x(t), P(x(t)), \widetilde{v}(x(t)))$$

starting at x*.*

We introduce impulse evolutionary dynamical games:

Definition 0.3.3 *Consider a family* $\Phi_{\widetilde{\mathcal{V}}}$ *of set-valued maps* $\Phi_{\widetilde{v}} : X \rightsquigarrow X$*, regarded as* reset maps *indexed by* $\widetilde{v} \in \widetilde{\mathcal{V}}$ *and an evolutionary game*

$$\mathcal{S}_{\widetilde{\mathcal{V}}} : (x,\widetilde{v}) \in X \times \widetilde{\mathcal{V}} \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x) \in \mathcal{C}(0,\infty;X)$$

We denote by $\mathcal{R}_{\widetilde{v}} := \mathcal{R}_{(\mathcal{S}_{\widetilde{v}},\Phi_{\widetilde{v}})}$ *the impulse evolutionary system indexed by* $\widetilde{v} \in \widetilde{\mathcal{V}}$*. The associated* impulse evolutionary game *is the family* $\mathcal{R}_{\widetilde{\mathcal{V}}} := (\mathcal{R}_{\widetilde{v}})_{\widetilde{v}\in\widetilde{\mathcal{V}}}$ *of impulse evolutionary systems.*

Definition 0.3.4 *Consider an impulse evolutionary game* $\mathcal{R}_{\widetilde{\mathcal{V}}} := (\mathcal{S}_{\widetilde{v}}, \Phi_{\widetilde{v}})_{\widetilde{v}} \in \widetilde{\mathcal{V}}$ *and two subsets* K *and* $C \subset K$*. We set*

$$\mathrm{ImpViab}_{\widetilde{v}}(K,C) \;:=\; \mathrm{ImpViab}_{(\mathcal{S}_{\widetilde{v}},\Phi_{\widetilde{v}})}(K,C)$$

and say that the subset

$$\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C) \;:=\; \bigcap_{\widetilde{v}\in\widetilde{\mathcal{V}}} \mathrm{ImpViab}_{\widetilde{v}}(K,C)$$

is the conditional impulse viability kernel with a target under the impulse evolutionary game $(\mathcal{S},\Phi)$*.*

We say that K *is* conditionally viable outside C *if*

$$K \;\subset\; \mathrm{CondImpViab}_{\mathcal{G}}(K,C)$$

i.e., if for any state $x_0 \in K$*,* for all feedbacks $\widetilde{v}$, there exists at least *one run* $\overrightarrow{x}(\cdot) \in \mathcal{R}_{\widetilde{v}}(x_0)$ *viable in* K *forever or until it reaches* C*.*

Proposition 0.3.5 *If a subset D between C and K satisfies*

$$\left\{\begin{array}{ll} (i) & \forall\, \widetilde{v} \in \widetilde{\mathcal{V}},\;\; D \;\subset\; \mathrm{ImpViab}_{\widetilde{v}}(K,D) \\ (ii) & \exists\, \widetilde{v}_0 \in \widetilde{\mathcal{V}} \;\; \textit{such that}\, \mathrm{ImpViab}_{\widetilde{v}_0}(D,C) \;\subset\; D \end{array}\right. \tag{10}$$

then D is equal to $\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$ *and is the* unique bilateral fixed point *of* $\bigcap_{\widetilde{v}\in\widetilde{\mathcal{V}}} \mathrm{ImpViab}_{\widetilde{v}}(\cdot,\cdot)$*:*

$$\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}\Big(\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C),C\Big) \;=\; \mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$$

$$=\; \mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}\Big(K,\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)\Big)$$

We deduce the following fixed-point characterization of conditional impulse viability kernels:

Theorem 0.3.6 *The conditional viability kernel* $\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$ *of a subset K with target $C \subset K$ is the* smallest *subset D between C and K satisfying*

$$\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,D) \;\subset\; D$$

and actually is a fixed point

$$\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$$

$$=\; \bigcap_{\widetilde{v}\in\widetilde{\mathcal{V}}} \mathrm{ImpViab}_{\widetilde{v}}\Big(K,\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)\Big)$$

Unfortunately, the conditional impulse viability kernel $\mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$ of K outside the target $C \subset K$ is not itself necessarily conditionally viable outside the target C, i.e., it is not a fixed point of the map $K \mapsto \mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$.

As in the case of dynamical games, we follow the idea of Pierre Cardaliaguet by introducing the impulse discriminating kernel $\mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$ of K with target C:

Definition 0.3.7 *Let us consider an impulse evolutionary game $(\mathcal{S},\Phi)$ and two subsets K and $C \subset K$.*

We say that the largest subset D between C and K conditionally viable in K outside C under the impulse evolutionary game is the impulse discriminating kernel of K with target C under the impulse evolutionary game $\mathcal{R}_{\widetilde{\mathcal{V}}}$*, denoted by* $\mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$.

Matheron's Theorem (see [9, Aubin & Catté] for instance) implies that the map $K \mapsto \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$ is the unique map onto the family of subsets D between C and K conditionally viable in K outside C associated with the map $K \mapsto \text{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$ and that $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$ is the largest fixed point of this map:

$$\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C), C) \ = \ \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$$

We state the following properties of impulse discriminating kernels:

Proposition 0.3.8 *Let us consider an evolutionary game $(x, \widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$. Then*

$$\left\{ \begin{array}{ll} (i) & C \mapsto \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C) \text{ is increasing} \\ (ii) & \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)) \\ & = \ \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C) \end{array} \right. \quad (11)$$

Proof — Let $C_1 \subset C_2 \subset K$ and $x \in \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C_1)$. Hence, for any feedback $\widetilde{v} \in \widetilde{\mathcal{V}}$, there exists at least one run $\overrightarrow{x}(\cdot)$ belongs to $\mathcal{R}_{\widetilde{v}}(x)$ viable in $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C_1)$ forever or until it reaches C_1 — and thus, C_2 — in finite time. Hence $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C_1)$ is conditionally viable outside C_2, so that $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C_1) \subset \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C_2)$.

For proving that $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)) = \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$, take any

$$x_0 \ \in \ \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C))$$

For any $\widetilde{v}$, there exists a run $\overrightarrow{x}(\cdot) \in \mathcal{S}_{\widetilde{v}}(x_0)$ viable in $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C))$ forever or else, until it possibly reaches the subset $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$ of $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C))$ at some *finite time* $T > 0$ at $x(T) \in \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$. In this case, we concatenate $\overrightarrow{x}(\cdot)$ with one run $\overrightarrow{y}(\cdot) \in \mathcal{R}_{\widetilde{v}}(x(T))$ starting from $x(T)$ viable in $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$, and thus, in $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C))$, until it possibly reaches C. Hence, for any $\widetilde{v}$, $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C))$ captures C, i.e., $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)) = \text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$. ■

We thus deduce that the impulse discriminating kernel is the unique bilateral fixed point:

Theorem 0.3.9 *The impulse discriminating kernel* $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, C)$ *of a subset K with target $C \subset K$ is*

1. *the* largest *subset D between C and K conditionally viable outside the target C,*
2. *the* smallest *subset D between C and K satisfying* $\text{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K, D) = D$,

3. the unique *minimax D between C and K in the sense that*

$$D \;=\; \mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,D) \;=\; \mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(D,C)$$

As Pierre Cardaliaguet did with differential games, we relate the concepts of impulse discriminating kernels and conditional impulse viability kernels through the Discriminating Impulse Viability Kernel Algorithm:

Theorem 0.3.10 *We set $K_0 := K$ and*

$$\forall\, i \geq 0, \;\; K_i \;:=\; \mathrm{CondImpViab}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K_{i-1},C) = \bigcap_{\widetilde{v}\in\widetilde{\mathcal{V}}} \mathrm{ImpViab}_{\widetilde{v}}(K_{i-1},C)$$

Let us assume that the evolutionary game $(x,\widetilde{v}) \rightsquigarrow \mathcal{S}_{\widetilde{v}}(x)$ is upper semicompact. Then

$$\mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C) \;=\; \bigcap_{i=1}^{\infty} K_i$$

Proof — We already know that

$$\mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C) \;\subset\; \bigcap_{i=1}^{\infty} K_i$$

We have to check that the intersection $K_\infty := \bigcap_{i\geq 0} K_i$ of the closed subsets K_i is conditionally viable outside C, i.e., that for each $\widetilde{v}$, K_∞ captures C under $\mathcal{R}_{\widetilde{v}}$. Indeed, for a fixed $\widetilde{v}$,

$$C \;\subset\; K_{i+1} \;\subset\; \mathrm{ImpViab}_{\widetilde{v}}(K_i,C) \;\subset\; K_i$$

So, K_∞ is the intersection of the decreasing sequence of the viability kernels $\mathrm{ImpViab}_{\widetilde{v}}(K_i,C)$ viable under the set-valued map $\mathcal{S}_{\widetilde{v}}$. Since $\mathcal{S}_{\widetilde{v}}$ is assumed to be upper semicompact, the stability theorem of impulse viability kernels implies that K_∞ itself is viable outside C under this map $\mathcal{S}_{\widetilde{v}}$. Hence, K_∞ is viable outside C under every set-valued maps $\mathcal{S}_{\widetilde{v}}$, and thus, is contained in the impulse discriminating kernel $\mathrm{ImpDisc}_{\mathcal{R}_{\widetilde{\mathcal{V}}}}(K,C)$. ■

0.4 Impulse Optimal Control

We recall here the results of [12, Aubin & Dordan] on the links between impulse viability kernels and impulse optimal control problems that we shall use for intertemporal impulse differential games.

Let us introduce a discount factor m and a "Lagrangian"

$$\mathbf{l} : (x,u) \in X \times \mathcal{U} \mapsto \mathbf{l}(x,u) \in \mathbb{R}$$

used to measure a cumulated cost over time, that depends upon the state and the controls. Let us introduce also

1. a reset map $\Phi : \mathbb{R}_+ \times X \rightsquigarrow X$,
2. an extended function $\mathbf{w} : \mathbb{R}_+ \times X \times X \mapsto \mathbb{R} \cup \{+\infty\}$ satisfying

$$\mathbf{w}(t,x,x^*) \;=\; +\infty \text{ whenever } x^* \notin \Phi(t,x) - x$$

with which we associate the set-valued map $\Psi : \mathbb{R}_+ \times X \times Y \rightsquigarrow \mathbb{R}_+ \times X \times \mathbb{R}$ defined by

$$\Psi(t,x,y) \;:=\; (t, x + x^*, y - y^*) \text{ where } \; y^* \;\geq\; \mathbf{w}(t,x,x^*)$$

Note that this condition implicitly implies that $x^* \in \Phi(t,x) - x$.

We denote by $\mathcal{T}(\overrightarrow{x}(\cdot);t)$ the set of impulse times $t_n \leq t$ less than or equal to t.

We introduce the impulse Cost Functionals:

Definition 0.4.1 *We associate with any extended function* $\mathbf{v} : \mathbb{R}_+ \times X \mapsto \mathbb{R}_+ \cup \{+\infty\}$, *any* $t \geq 0$ *and any* $(\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)) \in \mathcal{R}_{(\mathcal{C},\Phi)}(x)$ *the functional*

$$\left\{ \begin{array}{l} \mathbf{Imp(J)_v}(t; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x) \\ \\ :=\; e^{-mt}\mathbf{v}(T-t, x(t)) + \displaystyle\int_0^t e^{-m\tau}\mathbf{l}(x(\tau), u(\tau))d\tau \\ \\ + \displaystyle\sum_{t_j \in \mathcal{T}(\overrightarrow{x}(\cdot);t)} e^{-mt_j}\mathbf{w}(T - t_j, \overrightarrow{x}(^-t_j), \mathbf{s}_j(\overrightarrow{x}(\cdot))) \end{array} \right.$$

We now introduce a function $\mathbf{c} : \mathbb{R}_+ \times X \mapsto \mathbb{R}_+ \cup \{+\infty\}$ used as an objective and a function $\mathbf{b} \geq \mathbf{c}$ defining dynamical constraints: It is used as a lower bound in the sense that we take into account at each instant the maximal cumulated cost up to the current time t:

$$\mathbf{Imp(I)_b}(t; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x) \;:=\; \sup_{s \in [0,t]} \mathbf{Imp(J)_b}(s; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x)$$

We next integrate this cumulated cost together with the cost $\mathbf{Imp(J)_c}(t;(x(s),u(s)))(T,x)$ associated with the objective function $\mathbf{c}$ by introducing the new cost functions

$$\left\{\begin{array}{l}\mathbf{Imp(L)_{(b,c)}}(t;(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(T,x)\\ :=\ \max(\mathbf{Imp(I)_b}(t;x(\cdot),\overrightarrow{u}(\cdot))(T,x),\mathbf{Imp(J)_c}(t;(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(T,x))\end{array}\right.$$

and the functional

$$\mathbf{Imp(V)_{(b,c)}}(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))(T,x)\ :=\ \inf_{t\in[0,T]}\ \inf_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);T)}\mathbf{Imp(L)_{(b,c)}}(t;(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(T,x)$$

Definition 0.4.2 *The impulse valuation function* $\mathbf{Imp(V)}^{\inf}_{\mathbf{(b,c)}}$ *is defined by*

$$\mathbf{Imp(V)}^{\inf}_{\mathbf{(b,c)}}(T,x)\ :=\ \inf_{(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{C},\Phi)}(x)}\mathbf{Imp(V)_{(b,c)}}(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))(T,x)$$

Examples: — Consider a given time-independent function $\mathbf{u}: X\mapsto\mathbb{R}\cup\{+\infty\}$, regarded as a "spot" cost function in economics or as an obstacle in problems of unilateral mechanics, for instance. State constraints are involved in the domain

$$\mathrm{Dom}(\mathbf{u})\ :=\ \{x\in X\mid \mathbf{u}(x)\ <\ +\infty\}$$

We set

$$\left\{\begin{array}{l}\mathbf{Imp(J)_u}(t,\mathcal{T}(x(\cdot));(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(x)\\ \\ :=\ e^{-mt}\mathbf{u}(x(t))+\displaystyle\int_0^t e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau\\ \\ +\displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t)}e^{-mt_j}\mathbf{w}(\mathbf{s}_j(\vec{x}(\cdot))))\end{array}\right.$$

We associate with it the function $\mathbf{u}_\infty$ defined by

$$\mathbf{u}_\infty(t,x,u)\ :=\left\{\begin{array}{ccc}\mathbf{u}(x,u) & \text{if} & t=0\\ +\infty & \text{if not} & \end{array}\right.\tag{12}$$

and by $\mathbf{0}$ the function defined by

$$\mathbf{0}(t,x,u)\ =\left\{\begin{array}{ccc}0 & \text{if} & t\geq 0,\\ +\infty & \text{if not} & \end{array}\right.$$

We obtain three valuation functions $\mathbf{Imp(V)}^{\inf}_{(\mathbf{0},\mathbf{u}_\infty)}$, $\mathbf{Imp(V)}^{\inf}_{(\mathbf{0},\mathbf{u})}$ and $\mathbf{Imp(V)}^{\inf}_{(\mathbf{u},\mathbf{u}_\infty)}$:

1. **Impulse Optimal Control Problems**: They are defined on a prescribed time horizon by

$$\mathbf{Imp(V)}^{\inf}_{(\mathbf{0},\mathbf{u}_\infty)}(T,x)\ :=\ \inf_{(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)}\mathbf{Imp(J)_u}(T,\mathcal{T}(x(\cdot));(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(x)\tag{13}$$

When $m = 0$, the above problem boils down to the impulse Bolza problem

$$\left\{ \begin{array}{l} \mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{0},\mathbf{u}_\infty)}(T,x) \\ := \inf_{((\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \left(\mathbf{u}(x(T)) + \int_0^T \mathbf{l}(x(\tau),u(\tau))d\tau + \sum_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);T)} \mathbf{w}(\mathbf{s}_j(\overrightarrow{x}(\cdot))) \right) \end{array} \right.$$

and when furthermore $\mathbf{l} = 0$, the impulse Mayer problem

$$\mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{0},\mathbf{u}_\infty)}(T,x) \;=\; \inf_{((\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \left(\mathbf{u}(x(T)) + \sum_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);T)} \mathbf{w}(\mathbf{s}_j(\overrightarrow{x}(\cdot))) \right)$$

2. **Impulse Stopping Time Problems**: They are defined by

$$\mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{0},\mathbf{u})}(T,x) \;:=\; \inf_{(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \; \inf_{t\in[0,T]} \mathbf{Imp}(\mathbf{J})_{\mathbf{u}}(t,\mathcal{T}(x(\cdot));(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(x) \quad (14)$$

When $m = 0$, the above problem boils down to the classical impulse stopping time problem

$$\left\{ \begin{array}{l} \mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{0},\mathbf{u})}(T,x) \\ := \inf_{((\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \; \inf_{t\in[0,T]} \left(\mathbf{u}(x(t)) + \int_0^t \mathbf{l}(x(\tau),u(\tau))d\tau + \sum_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);t)} \mathbf{w}(\mathbf{s}_j(\overrightarrow{x}(\cdot))) \right) \end{array} \right.$$

where we single-out the time (an no longer the prescribed horizon) where the cost is minimized.

3. **Impulse Worst Time Problems**: They are defined by

$$\mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{u},\mathbf{u}_\infty)}(T,x) \;:=\; \inf_{(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \; \sup_{t\in[0,T]} \mathbf{Imp}(\mathbf{J})_{\mathbf{u}}(t,\mathcal{T}(x(\cdot));(\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot)))(x) \quad (15)$$

where we want to minimize the worst value of the functional on the time interval and not at the prescribed time only.

When $m = 0$, the above problem boils down to

$$\left\{ \begin{array}{l} \mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{u},\mathbf{u}_\infty)}(T,x) \\ := \inf_{((\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \; \sup_{t\in[0,T]} \left(\mathbf{u}(x(t)) + \int_0^t \mathbf{l}(x(\tau),u(\tau))d\tau + \sum_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);t)} \mathbf{w}(\mathbf{s}_j(\overrightarrow{x}(\cdot))) \right) \end{array} \right.$$

and when $\mathbf{l} = 0$, to

$$\mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{u},\mathbf{u}_\infty)}(T,x) \;:=\; \inf_{((\overrightarrow{x}(\cdot),\overrightarrow{u}(\cdot))\in\mathcal{R}_{(\mathcal{S},\Phi)}(x)} \; \sup_{t\in[0,T]} \left(\mathbf{u}(x(t)) + \sum_{t_j\in\mathcal{T}(\overrightarrow{x}(\cdot);t)} \mathbf{w}(\mathbf{s}_j(\overrightarrow{x}(\cdot))) \right)$$

One can devise many other examples. ■

0.4.1 Characterization of the Impulse Valuation Function

We need to introduce the auxiliary evolutionary control system

$$\left\{\begin{array}{llll} (i) & \tau'(t) & = & -1 \\ (ii) & x'(t) & = & f(x(t),u(t)) \text{ where } u(t) \ \in \ U(x(t)) \\ (iii) & y'(t) & = & my(t) - \mathbf{l}(x(t),u(t)) \end{array}\right. \tag{16}$$

where

$$y(t) \ = \ e^{-\int_0^t m(x(s),u(s))ds}\left(y - \int_0^t e^{\int_0^\tau m(x(s),u(s))ds}\mathbf{l}(x(\tau),u(\tau))d\tau\right) \tag{17}$$

with which we associate the auxiliary reset map $\Psi : \mathbb{R}_+ \times X \times Y \rightsquigarrow \mathbb{R}_+ \times X \times Y$ defined by

$$\Psi(t,x,y) \ := \ (t, x+x^*, y-y^*) \text{ where } \ y^* \ \geq \ \mathbf{w}(t,x,x^*)$$

The "continuous component" $y(\cdot)$ of a run $(\overrightarrow{x}(\cdot), \overrightarrow{y}(\cdot)))$ of this auxiliary impulse control evolutionary system is given on each interval $[t_n, t_{n+1}[$ by

$$\left\{\begin{array}{l} y(t) \ \leq \ e^{-mt}\left[y_0 - \displaystyle\sum_{t_j \leq t} e^{-mt_j}y_j^* \right. \\ \\ \left. -\displaystyle\int_0^t e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau\right] \end{array}\right.$$

We prove it by induction. Assume that it is proved for t_n. Let us take $t_n :=^- t_n$, $x_n := x(^-t_n) + x_n^*$ and $y_n := y(^-t_n) - y_n^*$ where $x_n^* \in \Phi(T-t_n, x(^-t_n))$ and $y_n^* \geq \mathbf{w}(T-t_n, x(^-t_n), x_n^*)$. Since on the interval $[t_n, t_{n+1}]$ we have

$$y(t) \ = \ e^{m(t_n-t)}\left[y_n - \int_{t_n}^t e^{m(t_n-\tau)}\mathbf{l}(x(\tau),u(\tau))d\tau\right]$$

we infer that for any $t \in [t_n, t_{n+1}[$

$$\begin{cases} y(t) \;\le\; e^{m(t_n-t)}\Bigg[y(^-t_n) - y_n^* - \displaystyle\int_{t_n}^{t} e^{m(t_n-\tau)}\mathbf{l}(x(\tau),u(\tau))d\tau\Bigg] \\ \le\; e^{m(t_n-t)}\Bigg[e^{mt}\Bigg[y_0 - \displaystyle\sum_{t_j\le t_{n-1}} e^{-mt_j}y_j^* \\ -\displaystyle\int_0^{t_n} e^{-mt_n}\mathbf{l}(x(\tau),u(\tau))d\tau\Bigg] - y_n^* - \displaystyle\int_{t_n}^{t} e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau\Bigg] \\ =\; e^{mt}\Bigg[y_0 - \displaystyle\sum_{t_j\le t} e^{-mt_j}y_j^* - \displaystyle\int_0^{t} e^{-m\tau}(x(\tau),u(\tau))d\tau\Bigg] \end{cases}$$

and that for $t := t_{n+1}$, we obtain

$$\begin{cases} y(t_{n+1}) \;:=\; y(^-t_{n+1}) - y_{n+1}^* \;\le\; e^{mt_{n+1}}\Bigg[y_0 - \displaystyle\sum_{t_j\le t_{n+1}} e^{mt_j}y_j^* \\ -\displaystyle\int_0^{t_{n+1}} e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau\Bigg] \end{cases}$$

because

$$y_{n+1}^* \;=\; e^{mt_{n+1}}\left(e^{-mt_{n+1}}y_{n+1}^*\right)$$

The same remark implies that when $t_{n+1} = t_n$, the "discrete component" is obtained by the same formula.

We now provide a viability characterization of the impulse valuation function:

Theorem 0.4.3 *Assume that the extended functions* $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial and non negative. Then the impulse valuation function* $\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf}$ *is given by the formula*

$$\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf}(T,x) := \inf_{(T,x,y)\in \mathrm{ImpViab}_{((Equation\ 16),\Psi)}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))} Y$$

Proof — To say that the run $(T-\cdot, \overrightarrow{x}(\cdot), \overrightarrow{y}(\cdot))$ is viable in $\mathcal{E}p(\mathbf{b})$ on the interval $[0,t^*]$ means that for every $s \in [0,t^*]$,

$$\begin{cases} (i) & \overrightarrow{y}(s) \;\ge\; \mathbf{b}(T-s, \overrightarrow{x}(s)) \\ (ii) & \overrightarrow{y}(t^*) \;\ge\; \mathbf{c}(T-t^*, \overrightarrow{x}(t^*)) \end{cases}$$

The first inequality means that for every $s \in [0,t^*]$,

$$y_0 \;\ge\; \sum_{t_j\le s} e^{-mt_j}y_j^* + \int_0^s e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau + e^{-ms}\mathbf{b}(T-s,x(s))$$

and the second inequality that

$$y_0 \;\geq\; \sum_{t_j \leq t^*} e^{-mt_j} y_j^* + \int_0^{t^*} e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau + e^{-mt^*}\mathbf{c}(T-t^*, \overrightarrow{x}(t^*))$$

By definition,

$$y_j^* \;\geq\; \mathbf{w}(T-t_j, \overrightarrow{x}(^-t_j), x_j^*) \;=\; \mathbf{w}(T-t_j, \overrightarrow{x}(^-t_j), \mathbf{s}_j(\overrightarrow{x}(\cdot)))$$

The above inequalities amount to writing that

$$y_0 \;\geq\; \mathbf{Imp(I)_b}(t^*; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x)$$

and that

$$y_0 \;\geq\; \mathbf{Imp(J)_c}(t^*; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x))$$

i.e., that

$$y_0 \;\geq\; \mathbf{Imp(L)_{(b,c)}}(t^*; (\overrightarrow{x}(\cdot), \overrightarrow{u}(\cdot)))(T,x)$$

Therefore, the valuation function inherits the properties of the viability kernels. ■

0.4.2 Fixed-Point Characterization of the Impulse Valuation Function

We associate with any extended function $\mathbf{v} : \mathbb{R}_+ \times X \mapsto \mathbb{R}_+ \cup \{+\infty\}$, any $t \geq 0$ and any $(x(\cdot),u(\cdot)) \in \mathcal{C}(x)$ the functionals

$$\left\{ \begin{array}{l} \mathbf{J_v}(t; (x(\cdot),u(\cdot)))(T,x) \\ \\ := \; e^{-mt}\mathbf{v}(T-t, x(t)) + \displaystyle\int_0^t e^{-m\tau}\mathbf{l}(x(\tau),u(\tau))d\tau \end{array} \right.$$

and

$$\mathbf{I_b}(t; (x(\cdot),u(\cdot)))(T,x) \;:=\; \sup_{s\in[0,t]} \mathbf{J_b}(s; (x(\cdot),u(\cdot)))(T,x)$$

We next integrate them by introducing the new cost functions

$$\mathbf{L_{(b,c)}}(t; (x(\cdot),u(\cdot)))(T,x) \;:=\; \max(\mathbf{I_b}(t; x(\cdot),u(\cdot))(T,x), \mathbf{J_c}(t; (x(\cdot),u(\cdot)))(T,x))$$

and introduce the functional

$$\mathbf{V_{(b,c)}}(x(\cdot),u(\cdot))(T,x) \;=\; \inf_{t\in[0,T]} \mathbf{L_{(b,c)}}(t; (x(\cdot),u(\cdot)))(T,x)$$

The valuation function of the (non-impulsive) intertemporal minimization problem is defined by

$$\mathbf{V}_{(\mathbf{b},\mathbf{c})}^{\inf}(T,x) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(x)} \mathbf{V}_{(\mathbf{b},\mathbf{c})}(x(\cdot),u(\cdot))(T,x)$$

Theorem 0.4.4 *Let us set*

$$\mathbf{v} * \mathbf{w}(t,x) \;:=\; \inf_{x^*} \left(\mathbf{v}(t,x+x^*) + \mathbf{w}(t,x,x^*)\right)$$

Then the impulse valuation function $\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf}$ *is the unique fixed point*

$$\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf} \;=\; \mathbf{V}_{(\mathbf{b},\max(\mathbf{c},\min(\mathbf{b},(\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf} * \mathbf{w}))))}$$

of the map

$$\mathbf{v} \mapsto \mathbf{V}_{(\mathbf{b},\max(\mathbf{c},\min(\mathbf{b},(\mathbf{v}*\mathbf{w}))))}$$

Furthermore,

$$\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\inf} \;=\; \inf_{j\geq 0} \mathbf{v}_j$$

where $\mathbf{v}_0 := \mathbf{c}$ *and*

$$\mathbf{v}_{j+1} \;:=\; \mathbf{V}_{(\mathbf{b},\max(\mathbf{c},\min(\mathbf{b},(\mathbf{v}_j*\mathbf{w}))))}$$

Proof — The impulse capture basin is the unique fixed point of the map $\mathcal{E}p(\mathbf{v}) \subset \mathbb{R}_+ \times X \times \mathbb{R}_+ \mapsto \text{Capt}(\mathcal{E}p(\mathbf{b}), \mathcal{E}p(\mathbf{c}) \cup (\mathcal{E}p(\mathbf{b}) \cap \Psi^{-1}\mathcal{E}p(\mathbf{b})))$

We first observe that

$$(\mathbf{v} * \mathbf{w})(t,x) \;=\; \inf_{(t,x,y)\in\Phi^{-1}(\mathcal{E}p(\mathbf{v}))} y$$

Furthermore, under adequate conditions, if $\mathbf{v}$ and $\mathbf{w}$ are lower semicontinuous, so is $\mathbf{v} * \mathbf{w}$ and in this case,

$$\Phi^{-1}(\mathcal{E}p(\mathbf{v})) \;=\; \mathcal{E}p(\mathbf{v} * \mathbf{w})$$

Indeed, to say that (t,x,y) belongs to $\Phi^{-1}(\mathcal{E}p(\mathbf{v}))$ amounts to saying that there exist $y^* \geq \mathbf{w}(t,x,x^*)$ such that $(t,x+x^*,y-y^*) \in \mathcal{E}p(\mathbf{v})$, i.e., such that

$$(\mathbf{v} * \mathbf{w})(t,x) \;\leq\; \mathbf{v}(t,x+x^*) + \mathbf{w}(t,x,x^*) \;\leq\; y - y^* + y^* \;=\; y$$

Therefore, the epigraph of the function

$$(t,x) \mapsto \max(\mathbf{c}(t,x), \min(\mathbf{b}(t,x), (\mathbf{v} * \mathbf{w})(t,x)))$$

is equal to

$$\mathcal{E}p(\mathbf{c}) \cup (\mathcal{E}p(\mathbf{b}) \cap \Psi^{-1}\mathcal{E}p(\mathbf{b})))$$

We then know that the valuation function $\mathbf{V}_{(\mathbf{b},\mathbf{c})}^{\inf}$ is the southern border of the viable-capture basin $\text{Capt}_{((Equation\ 16),\Psi)}(\mathcal{E}p(\mathbf{b}), \mathcal{E}p(\mathbf{c}))$ of $\mathcal{E}p(\mathbf{c})$ viable in $\mathcal{E}p(\mathbf{b})$ under $((Equation\ 16), \Psi)$.

The algorithm follows from Theorem 0.2.14. ■

0.5 Impulse Intertemporal Dynamical Games

We consider an evolutionary game and the functional

$$\left\{\begin{array}{l}\mathbf{Imp(J)_v}(t;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)\\[2ex] :=\ e^{\int_0^t \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{v}(T-t,x(t))+\displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{l}(x(\tau),u(\tau),\widetilde{v}(x(\tau)))d\tau\\[2ex] +\displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t)} e^{\int_0^{t_j} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{w}(T-t_j,x(^-t_j),\mathbf{s}_j(\vec{x}(\cdot)))\end{array}\right.$$

The function $\mathbf{c}$ is used as an objective. The function $\mathbf{b}$ defines dynamical constraints: It is used as a lower bound in the sense that we take into account at each instant the maximal cumulated cost up to the current time t:

$$\mathbf{Imp(I)_b}(t;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)\ :=\ \sup_{s\in[0,t]}\mathbf{Imp(J)_b}(s;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)$$

We next integrate this cumulated cost together with the cost $\mathbf{Imp(J)_c}(t;(x(s),u(s));\widetilde{v})(T,x)$ associated with the objective function $\mathbf{c}$ by introducing the new cost functions

$$\left\{\begin{array}{l}\mathbf{Imp(L)_{(b,c)}}(t;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)\\ :=\ \max(\mathbf{Imp(I)_b}(t;x(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x),\mathbf{Imp(J)_c}(t;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x))\end{array}\right.$$

and the functional

$$\mathbf{Imp(V)_{(b,c)}}(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)\ :=\ \inf_{t\in[0,T]}\ \inf_{t_j\in\mathcal{T}(\vec{x}(\cdot);T)}\mathbf{Imp(L)_{(b,c)}}(t;(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)$$

Then the impulse valuation function $\mathbf{Imp(V)}^{\sup\inf}_{\mathbf{(b,c)}}$ is defined by

$$\mathbf{Imp(V)}^{\sup\inf}_{\mathbf{(b,c)}}(T,x)\ :=\ \sup_{\widetilde{v}}\ \inf_{(\vec{x}(\cdot),\vec{u}(\cdot))\in\mathcal{R}_{\widetilde{v}}(x)}\mathbf{Imp(V)_{(b,c)}}(\vec{x}(\cdot),\vec{u}(\cdot));\widetilde{v})(T,x)$$

Theorem 0.5.1 *Let us assume that the extended functions* $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial and non negative. Then the conditional impulse valuation function* $\mathbf{Imp(V)}^{\sup\inf}_{\mathbf{(b,c)}}$ *is the southern border of the conditional impulse viability kernel basin* $\mathcal{V}_{\mathbf{(b,c)}} := \mathrm{CondImpViab}_{\mathcal{B}}(\mathcal{E}p(\mathbf{b}),\mathcal{E}p(\mathbf{c}))$ *of* $\mathcal{E}p(\mathbf{b})$ *with target* $\mathcal{E}p(\mathbf{c})$ *under the auxiliary impulse evolutionary game* $(\mathcal{B}_{\widetilde{v}},\Psi)$.

Then for any feedback $\widetilde{v}$, *there exists a run* $(\vec{x}(\cdot),\vec{u}(\cdot)) \in \mathcal{R}_{\widetilde{v}}(x)$ *such that*

$$\begin{cases} \mathbf{Imp}(\mathbf{V})^{\sup\inf}_{(\mathbf{b},\mathbf{c})}(T,x) \\ \geq\ e^{\int_0^t \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{b}(T-t,x(t)) + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{l}(x(\tau),u(\tau),\widetilde{v}(x(\tau)))d\tau \\ + \displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t)} e^{\int_0^{t_j} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{w}(T-t_j,x(^-t_j),\mathbf{s}_j(\vec{x}(\cdot))) \end{cases} \tag{18}$$

until the first time $t^* \in [0,T]$ *when*

$$\begin{cases} \mathbf{Imp}(\mathbf{V})^{\sup\inf}_{(\mathbf{b},\mathbf{c})}(T,x) \\ \geq\ e^{\int_0^{t^*} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{c}(T-t^*,x(t^*)) + \displaystyle\int_0^{t^*} e^{\int_0^\tau \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{l}(x(\tau),u(\tau),\widetilde{v}(x(\tau)))d\tau \\ + \displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t^*)} e^{\int_0^{t_j} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{w}(T-t_j,x(^-t_j),\mathbf{s}_j(\vec{x}(\cdot))) \end{cases} \tag{19}$$

Finally, the valuation function is the unique *solution* $\mathbf{v}$ *to functional equation stating that the functions* $\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{v})}$ *and* $\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}$ *have the same* $\sup\inf$*:*

$$\begin{cases} \mathbf{v}(T,x) \ = \ \mathbf{Imp}(\mathbf{V})^{\sup\inf}_{(\mathbf{b},\mathbf{v})}(T,x) \\ := \ \sup_{\widetilde{v}} \inf_{((\vec{x}(\cdot),\vec{u}(\cdot))\in\mathcal{R}_{\mathcal{C},\Phi}(x)} \mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{v})}((x(\cdot),u(\cdot)))(T,x) \end{cases} \tag{20}$$

Unfortunately, the conditional impulse valuation function does not satisfy the property

$$\begin{cases} \mathbf{Imp}(\mathbf{V})^{\sup\inf}_{(\mathbf{b},\mathbf{c})}(T,x) \\ \geq\ e^{\int_0^t \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{Imp}(\mathbf{V})^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,x(t)) + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{l}(x(\tau),u(\tau),\widetilde{v}(x(\tau)))d\tau \\ + \displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t)} e^{\int_0^{t_j} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{w}(T-t_j,x(^-t_j),\mathbf{s}_j(\vec{x}(\cdot))) \end{cases} \tag{21}$$

However, there exists a largest function $\mathbf{Imp}(\mathbf{V})^{\text{disc}}_{(\mathbf{b},\mathbf{c})}$ among the functions $\mathbf{d}$ between

$\mathbf{b}$ and $\mathbf{c}$ satisfying

$$\left\{\begin{array}{l} \mathbf{d}(T,x) \\ \geq \ e^{\int_0^t \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{d}(T-t,x(t)) + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{l}(x(\tau),u(\tau),\widetilde{v}(x(\tau)))d\tau \\ + \displaystyle\sum_{t_j\in\mathcal{T}(\vec{x}(\cdot);t)} e^{\int_0^{t_j} \mathbf{m}(x(s),u(s),\widetilde{v}(x(s)))ds}\mathbf{w}(T-t_j,x(^-t_j),\mathbf{s}_j(\vec{x}(\cdot))) \end{array}\right. \tag{22}$$

This is the impulse discriminating valuation function denoted by $\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\text{disc}}$. It satisfies

$$0 \ \leq \ \mathbf{b}(t,x) \ \leq \ \mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\sup\inf}(t,x) \ \leq \ \mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\text{disc}}(t,x) \ \leq \ \mathbf{c}(t,x) \ \leq \ +\infty$$

The Cardaliaguet Algorithm implies that the impulse discriminating valuation function is obtained as

$$\mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c})}^{\text{disc}}(t,x) \ = \ \inf_{i\geq 0}\mathbf{c}_i(t,x)$$

where the functions are defined recursively by $\mathbf{c}_0 = \mathbf{c}$ and

$$\mathbf{c}_{i+1}(t,x) \ := \ \mathbf{Imp}(\mathbf{V})_{(\mathbf{b},\mathbf{c}_i)}^{\sup\inf}(t,x)$$

Bibliography

[1] AUBIN J.-P. (1991) **Viability Theory**

[2] AUBIN J.-P. (1999) **Mutational and morphological analysis: tools for shape regulation and morphogenesis**, Birkhäuser

[3] AUBIN J.-P. (1999) **Impulse Differential Inclusions and Hybrid Systems: A Viability Approach**, Lecture Notes, University of California at Berkeley

[4] AUBIN J.-P. (2000) *Optimal impulse control problems and quasi-variational inequalities thirty years later: a viability approach*, in **Optimal Control and Partial Differential Equations**, 311-324, IOS Press

[5] AUBIN J.-P. (2000) *Boundary-Value Problems for Systems of First-Order Partial Differential Inclusions*, NoDEA, 7, 67-90

[6] AUBIN J.-P. (2001) *Viability Kernels and Capture Basins of Sets under Differential Inclusions*, SIAM J. Control, 40, 853-881

[7] AUBIN J.-P. (2002) *Boundary-Value Problems for Systems of Hamilton-Jacobi-Bellman Inclusions with Constraints*, SIAM J. Control, 41, 425-456

[8] AUBIN J.-P., BAYEN A., BONNEUIL N. & SAINT-PIERRE P. (in preparation) **Viability, Control and Games: Regulation of Complex Evolutionary Systems Under Uncertainty and Viability Constraints**

[9] AUBIN J.-P. & CATTE F. (2001) *Fixed-Point and Algebraic Properties of Viability Kernels and Capture Basins of Sets*, Set-Valued Analysis, 10, 379-416

[10] AUBIN J.-P. & DORDAN O. (1996) *Fuzzy Systems, Viability Theory and Toll Sets*, In HANDBOOK OF FUZZY SYSTEMS, MODELING AND CONTROL, Hung Nguyen Ed.. Kluwer, 461-488

[11] AUBIN J.-P. & DORDAN O. (2001) *Dynamical Qualitative Analysis of Evolutionary Systems*, Proceedings of the ECC 2001 Conference

[12] AUBIN J.-P. & DORDAN O. (in preparation) *Impulsive Optimal Control and Stopping Time Problems in Finite Horizon*,

[13] AUBIN J.-P. & DORDAN O. (in preparation) *Dynamic Management of Portfolios with Impulse Transactions under Tychastic Uncertainty*,

[14] AUBIN J.-P. & FRANKOWSKA H. (1990) **Set-Valued Analysis**, Birkhäuser, Boston, Basel, Berlin

[15] AUBIN J.-P. & HADDAD G. (2001) *Cadenced Runs of Impulse and Hybrid Control Systems*, International Journal Robust and Nonlinear Control, 11, 401-415

[16] AUBIN J.-P. & HADDAD G. (2001) *Path-dependent impulse and hybrid control systems*, in HYBRID SYSTEMS: COMPUTATION AND CONTROL, 119-132, Di Benedetto & Sangiovanni-Vincentelli Eds, Proceedings of the HSCC 2001 Conference, LNCS 2034, Springer-Verlag

[17] AUBIN J.-P. & HADDAD G. (2002) *History (Path) Dependent Optimal Control and Portfolio Valuation and Management*, J. Positivity, 6, 331-358

[18] AUBIN J.-P. & HADDAD G. (2002) *Impulse Capture Basins of Sets under Impulse Control Systems, J. Mathematical Analysis and Applications*, 275, 676-692

[19] AUBIN J.-P., LYGEROS J., QUINCAMPOIX. M., SASTRY S. & SEUBE N. (2002) *Impulse Differential Inclusions: A Viability Approach to Hybrid Systems*, IEEE Transactions on Automatic Control, 47, 2-20

[20] BARLES G. (1985) *Deterministic impulse control problems* SIAM J. Control Optim., 23, 419-432

[21] BAYEN A.M. , CRUCK E. & TOMLIN C. (2002) *Guaranteed overapproximations of unsafe sets for continuous and hybrid systems : Solving the Hamilton-Jacobi equation using viability techniques* in Hybrid Systems : Computation and Control, TOMLIN C. & GREENSTREET M., Eds, LNCS 2289, 90-104, Springer-Verlag

[22] BAYEN A.M. , GRIEDER P. & TOMLIN C. (2002) *A control theoretic prdictive model for sector-based air traffic flow* Proceedings of the AIAA Conference on Guidance, Navigation and Control, paper 2002-5011

[23] BAYEN A.M. , GRIEDER P. & TOMLIN C. (to appear) *Lagrangian delay predictive model for sector based air trafic flow* AIAAA Journal on Guidance, Controls and Dynamics

[24] BAYEN A.M. , GRIEDER P., SIPMA H., MEYER G. & TOMLIN C. (2002) *Delay predictive models of the national airspace system using hybrid control theory* Proceedings of the American Control Conference

[25] BAYEN A.M. , MITCHELL I., OISHI M. & TOMLIN C. (to appear) *Lagrangian delay predictive model for sector based air trafic flow*

[26] CARDALIAGUET P. (1994) *Domaines dicriminants en jeux différentiels*, Thèse de l' Université de Paris-Dauphine

[27] CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. (1994) *Temps optimaux pour des problèmes avec contraintes et sans contrôlabilité locale* Comptes-Rendus de l'Académie des Sciences, Série 1, Paris, 318, 607-612

[28] CRUCK E. (2001) *Problèmes de cible sous contraintes d'état pour des systèes non linéaires avec sauts d'état*, Comptes-Rendus de l'Académie des Sciences, PARIS,

[29] CRUCK E. (2001) *Target problems under state constraints for nonlinear controlled impulsive systems*, Journal of Mathematical Analysis and Applications 270(2):636-656, 2002

[30] CRUCK E. & SAINT-PIERRE P. (to appear) *Nonlinear ImpulseTarget Problems under State Constraint: A Numerical Analysis based on Viability Theory*, Set-Valued Analysis

[31] CRUCK E., QUINCAMPOIX M. & SAINT-PIERRE P. (preprint) *Values for Pursuit-Evasion Games with Impulsive Dynamics*,

[32] DORDAN O. (1990) *Algorithme de simulation qualitative d'une équation différentielle sur le simplexe*, Comptes-Rendus de l'Académie des Sciences, Paris, 310, 479-482

[33] DORDAN O. (1992) *Mathematival problems arising in qualitative simulation of a differential equation*, Artificial Intelligence, 55, 61-86

[34] DORDAN O. (1995) ANALYSE QUALITATIVE, Masson

[35] HADDAD G. (1981) *Monotone trajectories of differential inclusions with memory*, Isr. J. Math., 39, 83-100

[36] HADDAD G. (1981) *Monotone viable trajectories for functional differential inclusions*, J. Diff. Eq., 42, 1-24

[37] HADDAD G. (1981) *Topological properties of the set of solutions for functional differential differential inclusions*, Nonlinear Anal. Theory, Meth. Appl., 5, 1349-1366

[38] LYGEROS J., TOMLIN C. & SASTRY S. (1997) *Multiobjective hybrid Controlled synthesis*, in HYBRID AND REAL-TIME SYSTEMS, Springer-Verlag, 109-123

[39] MITCHELL I. , BAYEN A. & TOMLIN C. (2002) *Computing reachable sets for continuous dynamic games using level-set methods* IEEE Transactions on Automatic Control

[40] MITCHELL I. , BAYEN A. & TOMLIN C. (2002) *Validating a Hamilton-Jacobi approximation to hybrid system reachable sets* in Hybrid Systems : Computation and Control, DI BENEDETTO M-D.& SANGIOVANI A., Eds, LNCS 2289, 90-104, Springer-Verlag

[41] ROCKAFELLAR R.T. & WETS R. (1997) **Variational Analysis**, Springer-Verlag

[42] SAINT-PIERRE P. (2003) *Extension of the viability kernel algorithm to discontinuous set-valued maps with adaptative step discretization*,

[43] SAINT-PIERRE P. (2003) *Viable Capture Basin for Studying Differential and Hybrid Games*, Applied Mathematics & Optimisation, 29, 187-209

[44] SAINT-PIERRE P. (2001) *Approximation of Viability Kernels and Capture Basin for Hybrid Systems*, Proceedings of European Control Conference ECC, J.L. Martins de Carvalho, Editor, 2776-2783

[45] SAINT-PIERRE P. (2002) *Hybrid Kernels and Capture Basins for Impulse Constrained Systems*, Proceedings of Hybrid Systems: Computation and Control, HSCC 2002,C. Tomlin, M. Greenstreet Editors, Lecture Notes in Computer Science, 2289, Springer

[46] SHI SHUZHONG (1986) *Théorèmes de viabilité pour les inclusions aux dérivées partielles*, C. R. Acad. Sc. Paris, 303, Série I, 11-14

[47] SHI SHUZHONG (1988) *Nagumo type condition for partial differential inclusions*, Nonlinear Analysis, T.M.A., 12, 951-967

[48] SHI SHUZHONG (1989) *Viability theorems for a class of differential-operator inclusions*, J. of Diff. Eq., 79, 232-257

[49] TOMLIN C. , MITCHELL I., BAYEN A. & OISHI M. (to appear) *Computational techniques for the verification and control of hybrid systems* Proceedings of the IEEE

[50] TOMLIN C. BOYD S., MITCHELL I., BAYEN A., JOHANNSON M. & XIAO L. (2003) *Computational tools for the verification of hybrid systems* in Software-Enabled Control, SAMAD T. & BALAS G., Eds, Wiley

Contents