Cadenced Runs of Impulse and Hybrid Control Systems

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Abstract

Impulse differential inclusions*, and in particular, hybrid control systems, are defined by a differential inclusion (or a control system) and a reset map. A* run *of an impulse differential inclusion is defined by a sequence of* cadences*, of* reinitialized states *and of* motives *describing the evolution along a given cadence between two distinct consecutive impulse times, the value of a motive at the end of a cadence being reset as the next reinitialized state of the next cadence.*

A cadenced run *is then defined by* constant cadence, initial state and motive*, where the value at the end of the cadence is reset at the same reinitialized state. It plays the role of a "discontinuous" periodic solution of a differential inclusion.*

We prove that if the sequence of reinitialized states of a run converges to some state, then the run converges to a cadenced run starting from this state, and that, under convexity assumptions, that a cadenced run does exist.

Keywords: hybrid control, impulse control, differential inclusion, viability, run, execution, periodic, cadenced run.

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Courses cadences de systmes contrl impulsionnels et hybrides

Rsum

Les inclusions diffrentielles impulsionnelles*, et en particulier, les systmes contrls hybrides, sont dfinis par une inclusion diffrentielle (ou un systme contrl) et une correspondance de rinitialisation. Une* course *d'une inclusion diffrentielle impulsionnelle est dfinie par une suite de* cadences*, d'*rinitialisations *et de* motifs *dcrivant l'volution dcrivant durant une cadence, la valeur du motif en fin de cadence tant rinitialise comme l'tat initial du motif suivant.*

Une course cadence *est alors dfinie par* une cadence constante, une tat initial et un motif *constants, et joue le rle d'une solution priodique "discontinue" d'une inclusion diffrentielle.*

Mots Cls: contrle hybride, contrle impulsionnel, inclusion diffrentielle impulsionnelle, solution priodique, course cadence

AMS Classification: 90

Introduction

Cadenced runs of impulse differential inclusions of hybrid control systems play the role of "discontinuous" periodic solutions of differential equations and inclusions, and therefore, share their importance.

Just to mention one motivation, "Integrate and Fire" models in neurobiology[3] provide a simple example of impulse differential equation extensively studied. Roughly speaking, investigators admit that the propagation of the nervous influx along the axon of a neuron is governed by a differential equation, and that a neuron "fires" by sending neurotransmitters through a synapse when a threshold is reached. Periodic firing times are naturally of interest, signing so to speak a biological characteristic of the neuron. Above all, as we shall see in this paper, they appear as soon as the sequence of reinitialized states converges to a limit, that is then the initial state of a cadenced run, playing the role of a limit cycle.

Impulse differential inclusions, and in particular, hybrid control systems, are defined by a differential inclusion (or a control system) and a reset map. A run of an impulse differential inclusion is defined by a sequence of cadences, of reinitialized states and of motives describing the evolution along a given cadence between two distinct consecutive impulse times, the value of a motive at the end of a cadence being reset as the next reinitialized state of the next cadence.

A cadenced run is then defined by constant cadence, initial state and motive, where the value at the end of the cadence is reset at the same reinitialized state.

A first advantage of introducing impulse differential inclusions is to summarize the usually protracted description of an hybrid system[4] by only two set-valued maps F — the right-hand side of the differential inclusion governing the continuous evolution of a hybrid system — and R, describing the reset map reinitializing the system when required and a constrained set K inside which the evolution of the "run" or "execution" must remain. Hence, for instance, the existence of a run of an hybrid system for every initial set becomes a viability problem of an adequate auxiliary subset under an impulse differential inclusion, that can be characterized elegantly end efficiently. A key towards success requires simplification of the problem to use only the relevant properties of the problem, the price to pay is abstraction. The abstraction process amounts here to

1. regard control systems as differential inclusions, i.e., differential equations with set-valued right hand sides,

2. regard hybrid control systems as "impulse differential inclusions" defined below,

[3] See [23, Bressloff & Coombes], [24, Brette], [25, Burgmann], [28, Destexhe], [47, 48, Shimokawa, Pakdaman & Sato] and [49, Shimokawa, Pakdaman, Takahata, Tanabe & Sato] for instance.

[4] See for instance among many papers and books [22, Branicky, Borkar & Mitter], [20, Bensoussan & Menaldi], [41, 42, Matveev & Savkin] and [46, Shaft & Schumacher].

And, on top of it, it is so much simpler and handier.

Outline: We begin by giving our definition of hybrid systems, that can be embedded in the framework of impulse differential inclusions. We then recall the characterization of viable subsets under an impulse differential inclusion and derive from it a necessary and sufficient condition for the existence of solutions to hybrid differential inclusions. Then, we devote the last section to the existence theorems of cadenced runs.

1 Hybrid Differential Inclusions

"Hybrid control systems", as they are called by engineers, or "multiple-phase dynamical economies", as they are called by economists (see for instance [27, Day]), or "Integrate and Fire" models in neurobiology — may be regarded as hybrid differential inclusions.

Here, $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ denote finite dimensional vector spaces. Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $P : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\begin{cases} i) & x'(t) \; = \; f(x(t), u(t)) \\ ii) & u(t) \; \in \; P(x(t)) \end{cases}$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, P(x)) := \{f(x,u)\}_{u \in P(x)}$.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.

We shall say that K *is locally viable under* F if from every $x \in K$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on the nonempty interval $[0, T_x[$ in the sense

$$\forall \, t \in [0, T_x[, \;\; x(t) \; \in K$$

and that K is viable if we can take $T_x = +\infty$. It is *locally backward invariant under* F if for every $t_0 \in]0, +\infty[$, $x \in K$, for all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ arriving at x at time t_0, there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, and backward invariant if we can take $s = 0$.

We denote by

$$\text{Graph}(F) := \{(x, y) \in X \in Y \mid y \in F(x)\}$$

the graph of a set-valued map $F : X \rightsquigarrow Y$ and $\text{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its domain.

Let us set $x(^-t) := \lim_{\tau \mapsto t-} x(\tau)$ when $x(\cdot)$ is defined on some interval $[t - \eta, t[$ where $\eta > 0$, and, for consistency purposes, $x(s) = x(^-t)$ if $s = t$.

Definition 1.1 *An hybrid differential inclusion* (K, F, R) *is defined by*

1. *a finite dimensional vector space* E *of states* e *called* locations*,*
2. *a set-valued map* $K : E \rightsquigarrow X$ *associating with any location* e *a (possibly empty) subset* $K(e) \subset X$
3. *a set-valued map* $F : \text{Graph}(K) \rightsquigarrow X$ *with which we associate the differential inclusion* $x'(t) \in F(e, x(t))$*,*
4. *a set-valued map*[5] *(reset map)* $R : \text{Graph}(K) \rightsquigarrow E \times X$*.*

A run *of the impulse differential inclusion is a map* $(e(\cdot), x(\cdot))$ *from* $[0, T[$ *to* $X \times E$ *which is associated with a non decreasing sequence* $\mathcal{T}(e(\cdot), x(\cdot)) := \{t_n\}_{n \geq 0}$ *of* impulse or switching times $t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots < T$ *such that*

1. *either* $t_{n+1} = t_n$*,* $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(t_n))$ *and* $x(t_{n+1}) \in K(e(t_{n+1}))$*,*
2. *or* $t_{n+1} > t_n$*, and, for all* $t \in [t_n, t_{n+1}[$*,* $x(\cdot)$ *is a solution to the differential inclusion* $x'(t) \in F(e(t_n), x(t))$ *viable in* $K(e(t_n))$ *and we take* $(e(t_{n+1}), x(t_{n+1})) \in R(e(t_n), x(^-t_{n+1}))$ *and* $x(t_{n+1}) \in K(e(t_{n+1}))$*.*

When the maps F and R are single-valued, such hybrid systems are closely related to the differential automata introduced in [54, Taverni].

We observe right away that a map $(e(\cdot), x(\cdot))$ is a run of the hybrid differential inclusions if and only if $(e(\cdot), x(\cdot))$ is a run of

$$\begin{cases} i) & e'(t) \; = \; 0 \\ ii) & x'(t) \; \in \; F(e(t), x(t)) \end{cases}$$

"viable" in $\text{Graph}(K)$ until it reaches the domain of the reset map R.

Indeed the locations remain constant in the intervals $[t_n, t_{n+1}[$ since their velocities are equal to 0.

This leads us to regard $(e(\cdot), x(\cdot))$ as a run of an auxiliary system of impulse differential inclusions that we are about to define. We shall replace $E \times X$ by X, F and R by set-valued maps $F : X \rightsquigarrow X$ and $R : X \rightsquigarrow X$ and the graph $\text{Graph}(K) \in E \times X$ of the set-valued map K by a subset $K \subset X$.

[5] Usually, R is a single-valued defined on a subset $C \subset X$. We extend it to a set-valued map defined on the whole space X by setting $R(x) := \emptyset$ for any $x \notin C$, so that its extension is a set-valued map.

2 Impulse Differential Inclusions

Given a control system or a differential game described under the form of a differential inclusion $x' \in F(x)$ and constraints on the states represented by a closed subset K, there are no reasons why an arbitrary subset K should be viable under the differential inclusion $x' \in F(x)$.

Hence, the problem of reestablishing viability arises. One can imagine several mechanisms for this purpose:

1. Change either the dynamics or the set of constraints
   (a) either by changing the controls according to feedbacks or dynamic feedbacks that can be constructed (see for instance [1, 2, Aubin]),
   (b) or by changing the dynamics by, for instance, projecting the velocities onto the contingent cones and introducing viability multipliers (see for instance [1, 2, Aubin]),
   (c) or by restricting the constrained set to its viability kernel, which is by definition the largest subset viable under the dynamics,
   (d) or by letting the set of constraints evolve according to mutational equations, as in [4, Aubin].
2. or change the initial conditions by introducing a reset map R mapping any state of K to a (possibly empty) set $R(x) \subset X$ of new "initialized states".

This is the latter strategy we choose to use here: An impulse differential inclusion (and in particular, an impulse control system) is described by a pair (F, R), where the set-valued map $F : X \rightsquigarrow X$ mapping the state space $X := \mathbf{R}^n$ to itself governs the continuous evolution of the system in K and where R, the reset map, governs the discrete switches to new "initial conditions" when the continuous evolution is doomed to leave K.

Such a hybrid evolution, mixing continuous evolution "punctuated" by discontinuous impulses at impulse times is called in the "hybrid system" literature a "run" or an "execution".

Definition 2.1 *Let us consider a finite dimensional vector space X, a closed subset $K \subset X$, a set-valued map $F : X \rightsquigarrow X$ and a set-valued map $R : X \rightsquigarrow X$, regarded as a* reset map*.*

We regard the pair (F, R) as the dynamics of an impulse differential inclusion*.*

A run *of the impulse differential inclusion is a map $x(\cdot)$ from $[0, T]$ to X if $T < +\infty$ or from $[0, +\infty[$ to X if $T = +\infty$ which is associated with a non decreasing sequence $\mathcal{T}(x(\cdot)) := \{t_n\}_{n \geq 0}$ of* impulse or switching times *$t_0 := 0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq T$ such that*

1. *$x(t_{n+1}) \in R(x(t_n))$ if $t_{n+1} = t_n$,*

2. *or else, on the interval $[t_n, t_{n+1}[$, $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ at time t_n until time t_{n+1} at which we take $x(t_{n+1}) \in R(x(^- t_{n+1}))$.*

We denote by $\tau_n := t_n - t_{n-1}$ the nth cadence *of the run and by $x_n(\cdot) := x(\cdot + t_n)$ the nth* motive *of the run, a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ on the interval $[0, \tau_n]$. The sequence of states $x(t_n)$ is called the sequence of* initialized states*.*

We say that a run $x(\cdot)$ is viable in K if for any $t \geq 0$, $x(t) \in K$.

At this stage, a run $x(\cdot)$ can just be a (discrete) sequence of states $x_{n+1} \in R(x_n)$ at a fixed time, or just a (continuous) solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$, or an hybrid of these two modes, the discrete and the continuous.

3 The Characterization Theorem

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 3.1 (Marchaud Map) *We shall say that F is a* Marchaud map *if*

$$\begin{cases} i) & \text{the graph of } F \text{ is closed} \\ ii) & \text{the values } F(x) \text{ of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear: } \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(x, u) \mapsto f(x, u)$ is continuous, affine with respect to the controls u and with linear growth and when P is Marchaud.

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$ the set of absolutely continuous functions $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t))$$

starting at time 0 at x: $x(0) = x$. The set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the solution map (or the set-valued flow) associated with F.

We recall the following version of the important Theorem 3.5.2 of **Viability Theory**, [1, Aubin]:

Theorem 3.2 *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}$ is upper semicompact*[6] *with nonempty values: This means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.*

Our purpose is to characterize the viability of a subset K under an impulse differential inclusion:

Definition 3.3 *We shall say that a subset K is viable*[7] *under an impulse differential inclusion (F, R) if from any initial state x of K starts at least one run viable in K.*

The Viability Theorem[8] and its consequences imply the following

Theorem 3.4 *Let (F, R) be an impulse differential inclusion and $K \subset X$ be a closed subset. Assume that F is Marchaud and that R is upper semicontinuous with compact images*[9]*. Then the following statements are equivalent*

1. *K is viable under (F, R),*
2. *The subset*[10] *$K \backslash R^{-1}(K)$ is locally viable under F,*
3. *K, F and R are linked through the tangential condition*[11]

$$\forall\, x \in K \backslash R^{-1}(K), \quad F(x) \cap T_K(x) \neq \emptyset$$

[6] Actually, a sizable part of the following results depend upon few properties: translation and concatenation and upper semicompactness properties of the set-valued map $x \rightsquigarrow \mathcal{S}_F(x)$. Therefore these results are common to other control problems, such as

1. control problems with memory (see the contributions [30, 31, 32, Haddad] of G. Haddad, some of them being presented in [1, Aubin]) — before known under the name of functional control problems, the new fashion calling them as "path dependent control systems"
2. parabolic type partial differential inclusions (see the contributions [50, 52, 53, Shi Shuzhong] of Shi Shuzhong, some of them being presented in [1, Aubin]) — also known as distributed control systems
3. "mutational equations" governing the evolution in metric spaces, including "morphological equations" governing the evolution of sets (see [4, Aubin] for instance).

[7] Viability issues ("positively invariance") for hybrid systems has been studied in [29, Hespanha & Morse] (Lemma 3).

[8] See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [1, Aubin].

[9] This assumption implies that $R^{-1}(K)$ is closed, which is the property we really need. It remains true when we assume only that the subsets $K \cap (R(x) + B)$ are compact, where B denotes the unit ball.

[10] The subset $K \backslash C$ denotes the intersection of K and the complement of C, i.e., is the set of elements of K which do not belong to C.

[11] The contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [14, Aubin & Frankowska]) or [43, Rockafellar & Wets] for more details).

(see [18, Aubin, Lygeros, Quincampoix, Sastry & Seube] or [5, Aubin] for a proof.)

Since the existence of solutions to hybrid differential inclusions amounts to the viability of the graph of the set-valued map K under an associated auxiliary impulse differential inclusion, we obtain a necessary and condition for the existence of solutions to hybrid differential inclusions thanks to Theorem 3.4.

For that purpose, we need the definition of the contingent derivative $DK(e,x) : E \rightsquigarrow X$ of a set-valued map $K : E \rightsquigarrow X$ at a point (e,x) of its graph: It can be defined by

$$\text{Graph}(DK(e,x)) \ := \ T_{\text{Graph}(K)}(e,x)$$

We also need to introduce the set-valued map $K_1 : E \rightsquigarrow X$ defined by

$$\text{Graph}(K_1) \ := \ \text{Graph}(K)\backslash R^{-1}(\text{Graph}(K))$$

We observe that whenever $R(e,x) := R_E(e) \times R_X(x)$, the map K_1 can be defined through the formulas

$$\forall\, e \in E, \ \ K_1(e) \ = \ K(e) \cap R_X^{-1}(K(R_E(e))$$

Theorem 3.5 *Let (K,F,R) be a hybrid differential inclusion. Assume that F is Marchaud and that R is upper semicontinuous with compact images. Then the hybrid differential inclusion has a solution for every initial state if and only if*

$$\forall\, e \in E, \forall\, x \in K(e)\backslash K_1(e), \ \ F(e,x) \cap DK(e,x)(0) \neq \emptyset$$

We shall also need some other prerequisites from Viability Theory:

Definition 3.6 *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set.*

1. *The subset* $\text{Viab}(K,C)$ *of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under F.*

   A subset $C \subset K$ is said to be isolated in K by $\mathcal{S}$ *if it coincides with its viability kernel:*

   $$\text{Viab}(K,C) \ = \ C$$

2. *The subset* $\text{Capt}^K(C)$ *of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K and*

   $$\text{Capt}(C) \ := \ \text{Capt}^X(C)$$

   is said to be the capture basin *of C.*

3. *When the target $C = \emptyset$ is the empty set, we set*

$$\mathrm{Viab}(K) \;:=\; \mathrm{Viab}(K,\emptyset) \;\;\&\;\; \mathrm{Capt}^K(\emptyset) \;=\; \emptyset$$

and we say that $\mathrm{Viab}(K)$ *is the viability kernel of* K.

A subset K is a repeller *under F if its viability kernel is empty, or, equivalently, if the empty set is isolated in K.*

In other words, the viability kernel $\mathrm{Viab}(K)$ is the subset of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$. Furthermore, we observe that

$$\mathrm{Viab}(K,C) \;=\; \mathrm{Viab}(K\backslash C) \cup \mathrm{Capt}^K(\emptyset) \tag{1}$$

Consequently, the viability kernel $\mathrm{Viab}(K,C)$ of K with target C coincides with the capture basin $\mathrm{Capt}^K(C)$ of C viable in K whenever the viability kernel $\mathrm{Viab}(K\backslash C)$ is empty, i.e., whenever $K\backslash C$ is a repeller:

$$\mathrm{Viab}(K\backslash C) \;=\; \emptyset \;\Rightarrow\; \mathrm{Viab}(K,C) \;=\; \mathrm{Capt}^K(C) \tag{2}$$

This happens in particular when K is a repeller, or when the viability kernel $\mathrm{Viab}(K)$ of K is contained in the target C.

Hence, the concept of viability kernel with a target allows us to study both the viability kernel of a closed subset and the viable-capture basin of a target.

It will also be useful to handle hitting and exit functions and Theorem 3.8 below:

Definition 3.7 *Let $C \subset K \subset X$ be two subsets. The functional $\tau_K : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its* exit time $\tau_K(x(\cdot))$ *defined by*

$$\tau_K(x(\cdot)) \;:=\; \inf\{t \in [0,\infty[\;|\; x(t) \notin K\}$$

is called the exit functional.

The (constrained) hitting (or minimal time) functional $\varpi_{(K,C)}$ *defined by*

$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \;\&\; \forall s \in [0,t],\; x(s) \in K\;\}$$

has been introduced in [26, Cardaliaguet, Quincampoix & Saint-Pierre]). We set

$$\varpi_C(x(\cdot)) := \varpi_{(X,C)}(x(\cdot))$$

If $\mathcal{S}$ is the solution map associated with the map F, the function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\sharp(x) \;:=\; \sup_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot))$$

the upper exit function *and the function* $\varpi^{\flat}_{(K,C)} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *defined by*

$$\varpi^{\flat}_{(K,C)}(x) \; := \; \inf_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$

is called the lower (constrained) hitting function

We shall need the following:

Theorem 3.8 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and C and K be two closed subsets such that $C \subset K$. Then the hitting function $\varpi^{\flat}_{(K,C)}$ is lower semicontinuous and the exit function $\tau^{\sharp}_K$ is upper semicontinuous. Furthermore, for any $x \in \mathrm{Dom}(\varpi^{\flat}_{(K,C)})$, there exists one solution $x^{\flat}(\cdot) \in \mathcal{S}(x)$ which hits C as soon as possible before possibly leaving K*

$$\varpi^{\flat}_{(K,C)}(x) \; = \; \varpi_{(K,C)}(x^{\flat}(\cdot))$$

and for any $x \in \mathrm{Dom}(\tau^{\sharp}_K)$, there exists one solution $x^{\sharp}(\cdot) \in \mathcal{S}(x)$ which remains viable in K as long as possible:

$$\tau^{\sharp}_K(x) \; = \; \tau_K(x^{\sharp}(\cdot))$$

(see Proposition 4.2.4 of [1, 5, Aubin], for instance)

We deduce the following property of viable runs:

Proposition 3.9 *If*

$$\mathrm{Viab}(K) \cap R(K) \; = \; \emptyset \tag{3}$$

then the cadences τ_n of a run $x(\cdot)$ starting from $K\backslash\mathrm{Viab}(K)$ and viable in K are finite.

If we assume furthermore that $R(K) \cap K$ is compact and that F is Marchaud, then there exist a finite scalar $\overline{T} < +\infty$ such that the cadences τ_n of any run $x(\cdot)$ starting from $K\backslash R^{-1}(K)$ and viable in K belong to the interval $[0, \overline{T}]$.

Proof — Let us consider a run $x(\cdot)$ associated with sequences $\{\tau_n\}_n$ of cadences, $\{x_n\}_n \in K$ of reinitialized conditions and $\{x_n(\cdot)\}_n \in \mathcal{S}(x_n)$ of motives viable in K on the intervals $[0, \tau_{n+1}[$. Since any reinitialized state x_n belongs to $K \cap R(K)$, and thus, disjoint from $\mathrm{Viab}(K)$, all solutions $y(\cdot) \in \mathcal{S}(x_n)$ starting from x_n leave K in finite time. Therefore, since the run $x(\cdot)$ is viable in K, each motive $x_n(\cdot) \in \mathcal{S}(x_n)$ reaches $K \cap R^{-1}(K)$ in finite time τ_n before leaving K.

Now, if we assume that F is Marchaud, we introduce

$$\overline{T} \; := \; \sup_{x\in K\cap R(K)} \tau^{\sharp}_K(x)$$

By Theorem 3.8, the subset $K \cap R(K)$ being compact, the exit function being finite and upper semicontinuous, we infer that $\overline{T}$ is finite. □

4 Cadenced Runs

4.1 Definitions

Definition 4.1 *A run $x(\cdot)$ of a impulse differential inclusion (F, R) is said to be* cadenced *with rhythm $\overline{\tau}$ if it is associated with the sequence of impulse times $\{n\overline{\tau}\}_{n\geq 0}$ and with a* stationary motive *$\overline{x}(\cdot) \in \mathcal{S}(\overline{x})$, solution to the differential inclusion $x' \in F(x)$ starting at $\overline{x}$ and satisfying $\overline{x} \in R(\overline{x}(^{-}\overline{\tau}))$.*

The behavior of the run is "summarized" by the "initialization map" $U := U^K_{(F,R)}$ associating with each initial condition $x_0 \in K$ the (possibly empty) set of new initialized conditions $x_1 \in R(x(^{-}t_1)) \cap K$ when $x(\cdot)$ ranges over the set of solutions to the differential inclusion $x' \in F(x)$ viable in K until they reach $R^{-1}(K)$ at time $t_1 \geq 0$ at $x(^{-}t_1) \in R^{-1}(K)$.

Indeed, the sequence of successive initial conditions x_n of a viable run $x(\cdot)$ of the impulse differential inclusion (F, R) — constituting the "discrete component of the run" — is governed by the discrete system $x_n \in U^K_{(F,R)}(x_{n-1})$ starting at x_0. The knowledge of the sequence of initialized states x_n allows us to reconstitute the "continuous component" of the run by solving the differential inclusion $x' \in F(x)$ starting at each reinitialized state x_n and satisfying the end-point condition $x_{n+1} \in R(x(^{-}t_{n+1}))$, which exists thanks to the definition of the map $U^K_{(F,R)}$.

The initial states of cadenced runs are obviously the fixed points $\overline{x} \in U^K_{(F,R)}(\overline{x})$ of the initialization map $U^K_{(F,R)} : K \rightsquigarrow K$[12].

We need to introduce the reachable map $\vartheta(\cdot, x) := \vartheta_{\mathcal{S}}(\cdot, x)$ defined by

$$\forall\, x \in X, \;\; \vartheta(t, x) \;:=\; \{x(t)\}_{x(\cdot)\in\mathcal{S}(x)}$$

We associate with it the reachable tube $t \rightsquigarrow \vartheta(t, C)$ defined by

$$\vartheta(t, C) \;:=\; \{x(t)\}_{x(\cdot)\in\mathcal{S}(C)}$$

Remark: Poincaré Return Map — Cadenced runs play the role of "discontinuous periodic solutions" of a differential inclusion. The link between cadenced runs of an impulse differential inclusion and periodic solutions of a differential inclusion is provided

[12] The values of the initialization map $U^K_{(F,R)}$ are not necessarily convex, so that the Kakutani type Fixed-Point Theorem — that states that whenever K is convex and compact and that $S : K \rightsquigarrow K$ is an upper semicontinuous with nonempty convex compact values map, then there exists a fixed point $\overline{x} \in S(\overline{x})$ of S — cannot be applied. See for instance [3, Aubin]. We shall overcome this difficulty in the last part of the paper.

by Poincaré maps, that can be regarded as reinitialized map associated with the special reset map $\mathbf{I}_C$ defined by

$$\mathbf{I}_C(x) := \begin{cases} x & \text{if} \quad x \in C \\ \emptyset & \text{if} \quad x \notin C \end{cases}$$

where $C \subset X$ is a hyperplane. Indeed, the Poincaré return map of a differential inclusion $x' \in F(x)$ is the initialization map of the special impulse differential inclusion $(F, \mathbf{I}_C)$ because in this case $U_{(F,\mathbf{I}_C)}(x) = \vartheta(t,x) \cap C$. The value $U_{(F,\mathbf{I}_C)}(x)$ provides all the points of the trajectories of the solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ crossing the hyperplane C.

The cadenced runs $\overline{x}(\cdot)$ of rhythm $\overline{\tau}$ are periodic solutions of period $\overline{\tau}$ crossing the hyperplane C at $\overline{x} \in C$. □

Remark: Periodic Runs — We can define also the concept of *periodic runs* of (discrete) period N that start from initial states that are N-periodic discrete solutions of the initialization map $U^K_{(F,R)}$. A N-periodic run $x(\cdot)$ is then associated with a N-periodic sequence of impulse times $t_n = t_{n+N}$, a N-periodic sequence of reinitialized states $x(t_n) = x(t_{n+N})$ and a N-periodic sequence of motives $x_n(\cdot) = x_{n+N}(\cdot)$. Such a periodic run is thus known whenever are known the N first cadences and the N reinitialized conditions such that $x_N = x_0$. The existence of such periodic runs is not studied in this paper (see [17, Aubin & Haddad]). In the case of single-valued hybrid systems, the existence of periodic solutions to differential automata and an adaptation of the Poincaré-Bendixon Theorem to differential automata can be found in [40, 41, 42, Matveev & Savkin].

4.2 Existence of Cadenced Runs

We begin with the following asymptotic property of a run that implies the existence of a cadenced run:

Theorem 4.2 *Let us assume that F is Marchaud, that R is upper semicontinuous, that $R(K)$ is compact and that $R(K) \cap \text{Viab}(K) = \emptyset$. Let $x(\cdot)$ be a run associated with a sequence $\mathcal{T}(x(\cdot))$ of impulse times t_n viable in K.*

If the sequence of reinitialized states $x(t_n)$ of the run $x(\cdot)$ converges to some $\overline{x}$, then a subsequence of the motives $x(\cdot + t_n)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ and viable in K.

Proof — We shall set from now on $C := K \cap R^{-1}(K)$, that plays the role of a target.

Let us assume that the sequence $x(t_n)$ converges to some $\overline{x} \in R(K)$. If we denote by $x_n(\cdot) := x(\cdot + t_n)$ the nth motive of the run, we see that $x_n(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$ starting at $x(t_n)$ and satisfying $x_n(\tau_{n+1}) \in R^{-1}(x(t_{n+1}))$.

Since $\overline{x}$ does not belong to the viability kernel $\text{Viab}(K)$ of K under F by assumption, Proposition 3.9 implies that the cadences $\tau_n := t_n - t_{n-1}$ are bounded by a finite time $\overline{T}$.

By Theorem 3.2, a subsequence $x_{n_p}(\cdot)$ converges uniformly on the compact interval $[0, \overline{T}]$ to some solution $\overline{x}(\cdot) \in \mathcal{S}(\overline{x})$ to the differential inclusion $x' \in F(x)$ starting at $\overline{x}$. Another subsequence of cadences $\tau_{n_{p_q}+1}$ converges to some $\overline{\tau} \in [0, \overline{T}]$. Hence $x_{n_{p_q}}(\tau_{n_{p_q}+1})$ converges to $\overline{x}(\overline{\tau})$. Since $x_{n_{p_q}}(\tau_{n_{p_q}+1})$ belongs to $R^{-1}(x(t_{n_{p_q}+1}))$, since $x(t_{n_{p_q}+1})$ converges also to $\overline{x}$ by assumption and since the graph of the reset map R is closed, we infer that $\overline{x}(\overline{\tau})$ belongs to $R^{-1}(\overline{x})$. Hence a subsequence of the motives $x_n(\cdot) := x(\cdot + t_n)$ of the run $x(\cdot)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ of rhythm $\overline{\tau}$. □

We next provide a sufficient condition for the existence of a cadenced run when the dynamics f governing the continuous evolution is single-valued and Lipschitz.

Proposition 4.3 *Let $f : X \mapsto X$ be a Lipschitz single-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : C \rightsquigarrow K$ be an upper semicontinuous set-valued map with nonempty compact convex images.*

Let us assume that

$$\forall\, x \in K \backslash C, \quad f(x) \;\in\; T_K(x) \tag{4}$$

that $\text{Viab}(K \backslash R^{-1}(K)) = \emptyset$ *and that*

$$\text{the hitting function } \varpi^{\flat}_{(K,R^{-1}(K))} \text{ is continuous} \tag{5}$$

Then there exists a cadenced run of the impulse differential equation (f, R) viable in K.

Proof — The Viability Theorem implies that from every $x \in K$ starts a solution $x(\cdot) =: \vartheta(\cdot, x)$ to the differential equation $x' = f(x)$, that is unique because f is assumed to be Lipschitz.

Since $\text{Viab}(K \backslash R^{-1}(K))$ is empty, the solution $x(\cdot)$ leaves K in finite time whenever $x \in K \backslash C$, and leaves it through C by Theorem 3.4 since $K \backslash C$ is locally viable thanks to assumption (4).

We thus infer that the hitting function $\varpi^{\flat}_{(K,R^{-1}(K))}$ is finite on K. Since it is assumed to be continuous, the map $x \rightsquigarrow \vartheta(\varpi^{\flat}_{(K,R^{-1}(K))}(x), x)$ is also continuous. Hence the set-valued map $S : x \in K \rightsquigarrow R(\vartheta_f(\varpi^{\flat}_{(K,R^{-1}(K))}(x), x)) \subset K$ is upper semicontinuous with closed convex images since R enjoys these properties.

The Kakutani Fixed-Point Theorem implies that the set-valued map $S : K \rightsquigarrow X$ has a fixed point $\overline{x}$, from which starts a solution $\overline{x}(\cdot)$ to the differential equation $x' = f(x)$ satisfying $\overline{x} \in R(\overline{x}(\overline{\tau}))$, which is then the motive $\overline{x}(\cdot)$ of a cadenced run of rhythm $\overline{\tau} := \varpi^{\flat}_{(K,R^{-1}(K))}(\overline{x})$. □

The problem now is to provide sufficient conditions for the hitting function to be continuous instead of being only lower semicontinuous.

When L is a closed subset with nonempty interior such that $C := K \cap L$, a sufficient condition is that it coincides with the hitting function $\varpi^\flat_{\mathrm{Int}(L)}$, which is upper semicontinuous. For instance, when $C := \partial K$ and when K is the closure of its interior — which is the case when K is a closed convex subset with nonempty interior — one can take L to be the closure of the complement of K.

For this purpose, we need to introduce the Dubovitsky-Miliutin and hypertangent cones:

Definition 4.4 *The* Dubovitsky-Miliutin tangent cone $D_L(x)$ *to* L *is defined by:*

$$\begin{cases} v \in D_L(x) \text{ if} \\ \exists\ \varepsilon > 0,\ \exists\, \alpha > 0 \text{ such that } x+]0,\alpha](v+\varepsilon B) \subset L \end{cases}$$

The hypertangent cone $H_L(x)$ *to* L *at* $x \in \partial L$ *is the set of elements* $v \in X$ *such that there exist* $\varepsilon > 0$, $\delta > 0$ *and* $\eta > 0$ *for which*

$$B(x,\eta) \cap L+]0,\delta](v+\varepsilon B) \ \subset\ L$$

We recall that for any x in the boundary of L, the *Dubovitsky-Miliutin cone* $D_L(x)$ to L at x is the complement of the contingent cone $T_{X\backslash L}(x)$ to the complement $X\backslash L$ of L at $x \in \partial L$:

$$\forall\, x \in \partial L,\ \ D_L(x) \ = \ X \setminus T_{X\backslash L}(x)$$

and that the graph of the hypertangent cone is open in $L \times X$ (see Chapter 4 of [14, Aubin & Frankowska] for more details).

Proposition 4.3.5 of [1, Aubin] states that if the set-valued map F is Marchaud and if $F(x) \subset D_L(x)$ at $x \in \partial L$, then there exist $\rho_x > 0$ and $T_x > 0$ such that, for all solutions to the differential inclusion $x' \in F(x)$,

$$\forall\, t \in [0,T_x],\ \ d(x(t),\partial L) \ \geq\ \rho_x t$$

This implies that whenever $f(x) \in D_L(x)$, then $\varpi^\flat_{\mathrm{Int}(L)}(x) = 0$. We thus deduce from Theorem 3.8 the following Lemma:

Lemma 4.5 *Assume that* f *is a Lipschitz single-valued map, that* L *is a closed subset with a nonempty interior and that for every* $x \in \partial L$, $f(x) \in D_L(x)$. *Then the hitting functions* $\varpi^\flat_L$ *and* $\varpi^\flat_{\mathrm{Int}(L)}$ *coincide and, consequently, are continuous on the capture basin* $\mathrm{Capt}(L)$ *of* L *under* f.

We deduce from Proposition 4.3 and the above Lemma that

Proposition 4.6 *Let $f : X \mapsto X$ be a Lipschitz single-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : K \rightsquigarrow K$ be an upper semicontinuous set-valued map with closed convex images and that $R(K)$ is compact. Let us assume that L is a closed subset with nonempty interior such that $C := K \cap R^{-1}(K) \cap L$ is not empty and different from K, that*

$$\begin{cases} i) & \forall\, x \in K \backslash C, \;\; f(x) \;\in\; T_K(x) \\ ii) & \forall\, x \in K \cap \partial L, \;\; f(x) \;\in\; D_L(x) \end{cases} \tag{6}$$

and that $\mathrm{Viab}(K \backslash R^{-1}(K))$ *is empty.*

Then there exists a cadenced run to the impulse differential equation (f, R) viable in K.

We now use the techniques of [33, Haddad & Lasry] for extending Proposition 4.6 to the case of impulse differential inclusions:

Theorem 4.7 *Let $F : X \mapsto X$ be a Marchaud set-valued map, $K \subset X$ be a convex compact subset, $C \subset K$ be closed and $R : C \rightsquigarrow K$ be an upper semicontinuous set-valued map with nonempty compact convex images. Let us assume that L is a closed subset with nonempty interior such that $C := K \cap R^{-1}(K) \cap L$ is not empty and different from K, that*

$$\begin{cases} i) & \forall\, x \in K \backslash C, \;\; F(x) \,\cap\, T_K(x) \,\neq\, \emptyset \\ ii) & \forall\, x \in K \cap \partial L, \;\; F(x) \,\subset\, H_L(x) \end{cases} \tag{7}$$

that $R(K) \cap \mathrm{Viab}(K)$ and $\mathrm{Viab}(K \backslash R^{-1}(K))$ are empty.

Then there exists a cadenced run of the impulse differential inclusion (F, R) viable in K.

Proof — Following [33, Haddad & Lasry], we use their basic Lemma (see also Theorem 1.6.1 of [11, Aubin & Cellina]) for approximating the Marchaud map F by Lipschitz Marchaud set-valued maps F_n defined by

$$\forall\, x \in K \backslash C, \;\; F_n(x) \;:=\; \sum_{\text{finite}} \psi_i^n(x) C_i^n \tag{8}$$

where ψ_i^n are Lipschitz and C_i^n are convex compact subsets contained in the image of F. They satisfy:

$$\begin{cases} i) & \forall\, n \geq 0, \;\; F(x) \subset \cdots \subset F_n(x) \\ ii) & \forall\, \varepsilon > 0, \; \exists\, N(\varepsilon, x) \;\mid\; F_n(x) \subset F(x) + \varepsilon B \end{cases} \tag{9}$$

Therefore, assumption (6) implies that

$$\forall\, n \geq 0,\ \forall\, x \in K\backslash C,\ \ F_n(x) \cap T_K(x) \neq \emptyset \tag{10}$$

We define now the set-valued map $G_n : K \rightsquigarrow X$ by

$$\forall\, n \geq 0,\ \forall\, x \in K\backslash C,\ \ G_n(x) \ := \ F_n(x) + \frac{1}{n}B \tag{11}$$

It is obvious that these set-valued maps G_n are Lipschitz with closed convex values. Moreover, since $\text{Int}(T_K(x))$ is nonempty, then $v + \frac{1}{n}B$ belongs to the interior of $T_K(x)$ for all $v \in T_K(x)$. Therefore (9) implies that

$$\forall\, n \geq 0,\ \forall\, x \in K\backslash C,\ \ G_n(x) \cap \text{Int}(T_K(x)) \neq \emptyset \tag{12}$$

We also deduce that

$$\forall\, n \geq 0,\ \forall\, x \in K \cap \partial L,\ \ G_n(x) \ \subset \ H_L(x) \tag{13}$$

Since G_n is Lipschitz and since the maps $x \rightsquigarrow \text{Int}(T_K(x))$ and $x \rightsquigarrow H_K(x)$ have an open graph, then there exists a Lipschitz selection g_n:

$$\begin{cases} i) & \forall\, x \in K\backslash C,\ \ g_n(x) \ \in \ G_n(x) \cap \text{Int}(T_K(x)) \\ ii) & \forall\, x \in K \cap \partial L,\ \ g_n(x) \ \in \ H_L(x) \end{cases} \tag{14}$$

Hence, by Proposition 4.6, there exist for each n an initial state $\overline{x}_n$ from which starts a solution $\overline{x}_n(\cdot)$ to the differential equation $x' = g_n(x)$ satisfying $\overline{x}_n \in R(\overline{x}_n(\overline{\tau}_n))$, which is the motive $x_n(\cdot)$ of a cadenced run of rhythm $\overline{\tau}_n := \varpi^{g_n}_{(K,R^{-1}(K))}(\overline{x}_n)$.

Since K is compact, a subsequence of such initial states (again denoted by) $\overline{x}_n$ converges to some $\overline{x} \in K \cap R(K)$. Since $\overline{x}$ does not belong to the viability kernel $\text{Viab}(K)$ of K under F, Proposition 3.9 implies that the cadences $\tau_n := t_n - t_{n-1}$ are bounded by a finite time $\overline{T}$.

By Theorem 3.2, a subsequence (again denoted by) $\overline{x}_n(\cdot)$ converges uniformly on the compact interval $[0, \overline{T}]$ to some solution $\overline{x}(\cdot) \in \mathcal{S}(\overline{x})$ to the differential inclusion $x' \in F(x)$ starting at $\overline{x}$. Another subsequence of cadences (again denoted by) $\overline{\tau}_n$ converges to some $\overline{\tau} \in [0, \overline{T}]$. Hence $\overline{x}_n(\overline{\tau}_n)$ converges to $\overline{x}(\overline{\tau})$. Since $\overline{x}_n(\overline{\tau}_n)$ belongs to $R^{-1}(\overline{x}_n)$ and since the graph of the reset map R is closed, we infer that $\overline{x}(\overline{\tau})$ belongs to $R^{-1}(\overline{x})$. Hence a subsequence of the motives $x_n(\cdot) := x(\cdot + t_n)$ of the run $x(\cdot)$ converges to the motive $\overline{x}(\cdot)$ of a cadenced run starting at $\overline{x}$ of rhythm $\overline{\tau}$. □

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