History (Path) Dependent Optimal Control and Portfolio Valuation and Management

Jean-Pierre Aubin[1] **and Georges Haddad**[2]

Abstract

Regarding the evolution of financial asset prices governed by an history dependent (path dependent) dynamical system as a prediction mechanism, we provide in this paper the dynamical valuation and management of a portfolio (replicating for instance European, American and other options) depending upon this prediction mechanism (instead of an uncertain evolution of prices, stochastic or tychastic).

The problem is actually set in the format of a viability/capturability theory for history dependent control systems and some of their results are then transferred to the specific examples arising in mathematical finance or optimal control. They allow us to provide an explicit formula of the valuation function and to show that it is the solution of a "Clio Hamilton-Jacobi-Bellman" equation. For that purpose, we introduce the concept of Clio derivatives of "history functionals" in such a way we can give a meaning to such an equation. We then obtain the regulation law *governing the evolution of optimal portfolios.*

Keywords: Hamilton-Jacobi-Bellman equations, history dependent control, path dependent control, functional differential inclusion, viability, capturability, portfolio valuation, portfolio management, Clio derivatives, chaining of functions.

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[1]14, rue Domat, F-75005 Paris cx (16), Tel/Fax. [33] (0)1-46-33-71-21
e-mail: J.P.Aubin@wanadoo.fr, WEB: http://viab.dauphine.fr

[2]Université Panthéon-Sorbonne, Président Honoraire, 12, Place du Panthéon, F-75005, Paris
Tel. [33] (0)1-40-46-28-02, Fax. [33] (0)1-46-34-20-56

Optimisation intertemporelle et valuation et gestion de portefeuilles dans le cadre de systèmes dépendant de l'histoire (ou des sentiers)

Résumé

En interprétant une évolution des prix d'actifs financiers gouvernée par un système dynamique dépendant de son histoire (history, memory ou path dependent systems) comme un mécanisme prédictif, *nous proposons dans cet article le calcul de la fonction de valuation et de la loi de régulation de portefeuilles (duplicant par exemple des options européennes, américaines ou autres) en fonction d'un tel mécanisme, au lieu de supposer que l'évolution des prix soit incertaine (stochastique ou tychastique).*

Le problème est formulé dans le cadre d'un problème de viabilité/capturabilité pour un système contrôlé dépendant de l'histoire et les résultats obtenus dans ce cadre général sont ensuite transférés à ces exemples issus de la finance mathématique et d'autres provenant de l'optimisation intertemporelle de critères portant sur les solutions de problèmes de contrôle dépendant de l'histoire.

Cela nous permet en particulier de donner une formule explicite de la fonctionnelle de valuation (qui dépend de l'histoire) et de montrer qu'elle est une solution d'un système d'équations aux dérivées partielles d'Hamilton-Jacobi-Bellman-Clio[3] *où l'on doit pour cela introduire le concept de "dérivées de Clio" de fonctionnelles définies sur les histoires (ou les sentiers). Cela nous permet d'adapter aux problèmes d'optimisation intertemporelle de solutions de systèmes contrôlés dépendant de l'histoire la théorie d'Hamilton-Jacobi-Bellman.*

[3] Clio, née comme les autres muses des amours de Zeus et de Mnénosyne, déesse de la mémoire, promettant de divulguer ce qui fut et de révéler ce qui sera, aura fourvoyé nombre de ceux qu'elle a séduit et qui ont cru à ses promesses. Les sectateurs de Clio tentent de donner un sens à la mémoire des actions des hommes collationnées par Mnénosyne de sorte que l'Histoire oscille du rassemblement des faits à leur interprétation. Dans cette rivalité permanente entre Mnénosyne et Clio, c'est cette dernière qui est souvent courtisée.

Introduction

Most models of dynamic valuation and management of portfolios made of shares of assets, including the replicating portfolios of financial options, assume that the future evolution of prices of the risky assets is uncertain, stochastic or tychastic[4].

In this paper, we shall assume instead that the future evolution of the asset prices can be predicted[5] or forecasted from its history through a convenient prediction mechanism, and based on such a prediction mechanism, valuate the portfolio and find the regulation law allowing the manager to modify his/her portfolio at each instant.

The class of prediction operators we study here is provided by history dependent (or path dependent, memory dependent, functional) control systems. A each instant, they associate with the history of the evolution up to this time and a control at this time (here, the portfolio) the velocity of the price. Therefore, given the initial history, they provide the future evolutions according to each open loop control.

In the financial example, the evolution of prices is just dictated in terms of the history of its evolution up to the initial time when the decision must be made.

We use here viability/capturability analysis, regarding inequality constraints at the exercise time (or the horizon) as targets to be reached at that time, and regarding inequality constraints that must be satisfied at each instant as viability constraints.

Such an approach has been pioneered in [42, 43, Zabczyk] in the case when the evolution of the prices is stochastic and discrete, in [36, 37, 38, Soner & Touzi] in the case when this evolution is stochastic and continuous, in [33, Pujal], [34, Pujal & Saint-Pierre], [11, Aubin, Pujal & Saint-Pierre]), [17, 18, 19, Bernhard] when the evolution is tychastic, discrete and/or continuous, and [10, Aubin & Doss] for relations between the tychastic and stochastic approaches).

On the other hand, this viability approach, combined with the epigraphical approached, has been used by Frankowska[6] to study the Hamilton-Jacobi-Bellman theory

[4]The theory of tychastic control (or "robust control") can be studied in the framework of dynamical games, when one player plays the role of Nature that chooses — plays — perturbations. These perturbations, disturbances, parameters that are not under the control of the controller or the decision-maker, could be called "random variables" if this vocabulary was not already confiscated by probabilists. We suggest to borrow to Charles Peirce the concept of *tyche*, one of the three words of classical Greek meaning "chance", and to call in this case the control system as a tychastic system.

[5]despite Paul Valéry's warning: *La prévision est un rêve duquel l'événement nous tire.* (Prevision is a dream from which reality takes us out).

[6]The epigraph of a real-valued function is what it is above its graph. The discovery that the main properties of a function involved and used in optimization problems deal with the epigraph of a function took its roots in the sixties in convex analysis after the pioneering works of Moreau and Rockafellar. The viability of the epigraph of a function was used in [1, 2, Aubin] in the context of

for optimal control problems.

However, for usual control problems, differentiable games and dynamical economic models, it has been shown that the viability approach for studying optimal intertemporal control problems and the viability/capturability approach to study the dynamic valuation of portfolios in mathematical finance are the same, and that it actually allows us to study other intertemporal extremality problems, such as stopping time problems (or obstacle problems) in control, "American options" in mathematical finance.

Finally, twenty years ago, viability theorems have been proved in [29, 30, 31, Haddad] for history dependent control systems and differential inclusions (then, called functional differential inclusions).

Therefore, all these tools can be gathered to provide an Hamilton-Jacobi-Bellman approach to history dependent control problems and portfolio management when the evolution of the prices is predicted from the past history. This is the very purpose of this paper.

The Hamilton-Jacobi-Bellman approach is based on the fact that the valuation function in mathematical finance or the value function of optimal control problems in control theory (and economic theory in the extent it uses intertemporal optimization) is a solution, taken in an adequate generalized sense, of a system of partial differential equations, the celebrated Hamilton-Jacobi-Bellman equations (or variational inequalities) in control, among which we find the Black-Scholes equations in finance when the evolution of the price is governed by a stochastic differential equation.

When the evolution depends upon its history defined as a continuous function $\varphi \in \mathcal{H}(\mathbf{R}^n) := \mathcal{C}(-\infty, 0, \mathbf{R}^n)$ defined for negative times, so does the valuation function $\mathbf{v} : \mathcal{H}(\mathbf{R}^n) \mapsto \mathbf{R}$. Therefore, we bump right away to the fact that the derivatives involved in what we suggest to call the Clio[7] Hamilton-Jacobi-Bellman equations are not the usual ones, defined through differential quotients

$$\nabla_h \mathbf{v}(\varphi)(\psi) \ := \ \frac{\mathbf{v}(\varphi + h\psi) - \mathbf{v}(\varphi)}{h}$$

Lyapunov functions. Their more and more frequent use in optimization and control since the sixties is handy because it allows us to replace inequalities constraints by membership relations, and becomes more and more familiar. It was taken up by Hélène Frankowska for characterizing value functions of optimal control problems in a long series of papers (see [23, 24, 25, 26, 27, 28, Frankowska]) to quote a few. This epigraphical approach in the field of Hamilton-Jacobi equations has since been taken up by other authors.

[7]Clio, muse of history, was born as the other muses out of the love between Zeus and Mnenosyne, Goddess of memory. If one chooses the name of path dependent dynamics instead of history dependent ones, we suggest to attribute the name of Mercury to these derivatives.

If the limit

$$D\mathbf{v}(\varphi)(\psi) \ := \ \lim_{h\to 0+} \nabla_h \mathbf{v}(\varphi)(\psi)$$

exists and is linear and continuous on $\mathcal{H}(\mathbf{R}^n)$, then the gradient of $\mathbf{v}$ at φ is an element of the $\mathcal{H}(\mathbf{R}^n)^\star$ of $\mathcal{H}(\mathbf{R}^n)$, i.e., a vector measure on $]-\infty,0]$.

For Clio Hamilton-Jacobi-Bellman equations popping up in optimal control of history dependent control problems, the derivatives are taken in the "Clio sense" through differential quotients formed in another way, and measuring other kinds of variations. The addition operator $\varphi \mapsto \varphi + h\psi$ is replaced by the chaining operator $\diamondsuit_h u$ associating with each history $\varphi \in \mathcal{C}(-\infty,0;\mathbf{R}^n)$ the function $\varphi\diamondsuit_h u \in \mathcal{C}(-\infty,0;\mathbf{R}^n)$ defined by

$$(\varphi\diamondsuit_h u)(\tau) \ = \ \begin{cases} \varphi(\tau+h) & \text{if} \quad \tau \in]-\infty,-h] \\ \varphi(0)+(\tau+h)u & \text{if} \quad \tau \in [-h,0] \end{cases}$$

We no longer measure variations along the line $h \mapsto \varphi + h\psi$ passing at φ in the direction $\psi \in \mathcal{H}(\mathbf{R}^n)$, but along the chaining $h \mapsto \varphi\diamondsuit_h v \in \mathcal{H}(\mathbf{R}^n)$ of the history φ and a direction $u \in \mathbf{R}^n$ (instead of a function $\psi \in \mathcal{H}(\mathbf{R}^n)$).

The idea, made precise later in the paper[8], is to define a "Clio differential quotient"

$$\Delta_h \mathbf{v}(\varphi)(\psi) \ := \ \frac{\mathbf{v}(\varphi\diamondsuit_h u) - \mathbf{v}(\varphi)}{h}$$

If the limit

$$\mathbf{D}\mathbf{v}(\varphi)(u) \ := \ \lim_{h\to 0+} \Delta_h \mathbf{v}(\varphi)(u)$$

exists and is linear with respect to $u \in \mathbf{R}^n := \mathbf{R}^{n+1}$, then we say that $\mathbf{D}\mathbf{v}(\varphi)(u)$ *is the Clio derivative of* $\mathbf{v}$ *at* φ *in the direction* u and we set

$$\frac{\partial \mathbf{v}(\varphi)}{\partial x_i} := \mathbf{D}\mathbf{v}(\varphi)(0,\ldots,1,0,\ldots)$$

and

$$\mathbf{D}\mathbf{v}(\varphi)(u) \ := \ \sum_{i=1}^{n} \frac{\partial \mathbf{v}(\varphi)}{\partial x_i} u_i$$

Taking limits in weaker senses provide generalized derivatives, among which we shall select the one that sill allow us to state that a lower semicontinuous valuation function is a solution to a "Clio" Hamilton-Jacobi-Bellman equation of the form

[8] The rigorous definition is somewhat more involved.

$$-\frac{\partial \mathbf{v}(t,\varphi)}{\partial t} + \inf_{u \in U(\varphi)} \left(\sum_{i=1}^{n} \frac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} f_i(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi) + \mathbf{l}(\varphi,u) \right) = 0$$

Outline

The first section defines history dependent control systems and prediction operators.

Section 2 motivates our study by presenting the questions and their answers in the framework of portfolio valuation and management.

This is a specific problem that is generalized in the third section for any history dependent control system.

Section 4 is devoted to the proofs of the theorem by transforming the history dependent intertemporal optimization problems in the problem of controlling the evolution of the state constrained to belong to a given set of histories (history dependent viability constraints) until a finite time when it captures a given subset of histories, the target.

We then transfer the known results about this general viability/capturability problem to the specific examples that this paper proposes.

1 History or Path Dependent Control Systems

Let $X := \mathbf{R}^n$ be a finite dimensional vector space. We denote by

$$\mathcal{H}(X) := \mathcal{C}(-\infty, 0; X)$$

the history (or memory, path) space, supplied with the compact convergence topology, for which it is a Fréchet space and thus, metrizable.

We associate with any continuous function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ its *history (or path)*[9] $\kappa(t)x$ *up to time* t defined by:

$$\forall\, \tau \in]-\infty, 0], \quad (\kappa(t)x)(\tau) := x(t+\tau)$$

Then $\kappa(t)$ maps $\mathcal{C}(-\infty, +\infty; X)$ to $\mathcal{H}(X)$ and we observe that for any function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$, we have $x(t) = (\kappa(t)x)(0)$.

We introduce a map $f : \mathcal{H}(X) \mapsto X$ governing the evolution of a solution $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ to the history dependent (or path dependent, functional) differential equation

$$x'(t) = f(\kappa(t)x)$$

[9] often denoted by $x_t := \kappa(t)x$.

that associates with the history $\kappa(t)$ of the function $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ up to time $t \geq 0$ the velocity $x'(t)$ of the state at time t. We denote by $\mathcal{C}(\varphi)$ the set of solutions $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ of this history dependent differential equation starting at time 0 at the given initial history $\varphi \in \mathcal{H}(X)$.

This set-valued map $\mathcal{C} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ can be regarded as a prediction or an extrapolation system:

Definition 1 *A general* prediction process *is a set-valued map* $\mathcal{C} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ *associating with any* $\varphi \in \mathcal{H}(X)$ *a set* $\mathcal{C}(\varphi)$ *of future evolutions* $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ *satisfying* $\kappa(x) = \varphi$ *that is required to satisfy the two following properties*

1. *Let* $x(\cdot) \in \mathcal{C}(\varphi)$. *Then for all* $T \geq 0$, *the function* $y(\cdot)$ *defined by* $y(t) := x(t+T)$ *is a prediction* $y(\cdot) \in \mathcal{C}(\kappa(T)x)$ *associated with* $\kappa(T)x$,

2. *Let* $x(\cdot) \in \mathcal{C}(\varphi)$ *and* $T \geq 0$. *Then for every* $y(\cdot) \in \mathcal{C}(\kappa(T)x)$, *the function* $z(\cdot)$ *defined by*

$$z(t) \; := \left\{ \begin{array}{lll} x(t) & \text{if} & t \in]-\infty, T] \\ y(t-T) & \text{if} & t \geq T \end{array} \right.$$

   belongs to $\mathcal{C}(\varphi)$.

We denote from now on by φ an element of $\mathcal{H}(X)$ and by $\varphi(\cdot) : \mathbf{R}_+ \mapsto \mathcal{H}(X)$ a function associating with every nonnegative time $t \geq 0$ an history $\varphi(t) \in \mathcal{H}(X)$. Hence $\varphi(\cdot)$ can also be regarded as a function

$$(t, \tau) \in [0, +\infty[\times] - \infty, 0] \mapsto \varphi(t)(\tau)$$

This is the case of histories $\varphi(\cdot) := \kappa(\cdot)x$ of functions $x(\cdot) \in \mathcal{C}(0, +\infty; X)$.

It is more convenient to associate with a prediction process $\mathcal{C}$ mapping $\mathcal{H}(X)$ to $\mathcal{C}(-\infty, +\infty; X)$ an evolutionary system on the history space $\mathcal{H}(X)$:

Definition 2 *An evolutionary system on* $\mathcal{H}(X)$ *is a set-valued map* $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; \mathcal{H}(X))$ *satisfying*

1. *the* translation property*: Let* $\varphi(\cdot) \in \mathcal{B}(\varphi)$. *Then for all* $T \geq 0$, *the function* $\psi(\cdot)$ *defined by* $\psi(t) := \varphi(t+T)$ *belongs to* $\mathcal{B}(\varphi(T))$,

2. *the* concatenation property*: Let* $\varphi(\cdot) \in \mathcal{B}(\varphi)$ *and* $T \geq 0$. *Then for every history* $\psi(\cdot) \in \mathcal{B}(\varphi(T))$, *the history* $\xi(\cdot)$ *defined by*

$$\xi(t) \; := \left\{ \begin{array}{lll} \varphi(t) & \text{if} & t \in [0, T] \\ \psi(t-T) & \text{if} & t \geq T \end{array} \right.$$

   belongs to $\mathcal{B}(\varphi)$.

Therefore, we associate with a prediction process $\mathcal{C}$ the evolutionary system $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0,\infty;\mathcal{H}(X))$ defined by

$$\mathcal{B}(\varphi) \;:=\; \{t \mapsto \varphi(t) := \kappa(t)x\}_{x(\cdot)\in\mathcal{C}(\varphi)}$$

associating with φ the histories of evolutions $t \mapsto \varphi(t) := \kappa(t)x$ associated with the predictions $x(\cdot) \in \mathcal{C}(\varphi)$ of prediction process.

Indeed, it satisfies:

1. the translation property: Let $\varphi(\cdot) \in \mathcal{B}(\varphi)$. Then for all $T \geq 0$, the function $\psi(\cdot)$ defined by $\psi(T) := \varphi(t+T)$ is the history $\psi(\cdot) := \kappa(\cdot)y \in \mathcal{B}(\kappa(s)x)$ of the prediction $y(\cdot)$ from $\kappa(s)x$,

2. the concatenation property: Let $\varphi(\cdot) \in \mathcal{B}(\varphi)$ be the history and $T \geq 0$. Then for every history $\psi(\cdot) \in \mathcal{B}(\kappa(s)x)$ of a prediction $y(\cdot)$ from $\kappa(s)x$, the function $\xi(\cdot)$ defined by
$$\xi(t) \;:=\; \begin{cases} \varphi(t) := \kappa(t)x & \text{if } \; t \in [0,T] \\ \psi(t-T) := \kappa(t-T)y & \text{if } \; t \geq T \end{cases}$$
is the history of the solution $z(\cdot)$ defined by
$$z(t) \;:=\; \begin{cases} x(t) & \text{if } \; t \in [0,s] \\ y(t-T) & \text{if } \; t \geq s \end{cases}$$
to the history dependent differential inclusion starting at the initial path φ, and thus, belongs to $\mathcal{B}(\varphi)$.

The solution map $\mathcal{B}$ has the advantage of mapping histories $\varphi \in \mathcal{H}(X)$ into time-dependent functions $t \mapsto \varphi(t) := \kappa(t)x$ taking their values in the same history space $\mathcal{H}(X)$, even though the intuitive view is to handle predictions that are functions $t \mapsto x(t)$ taking their values in the state space X. But the introduction of these definitions is justified by the fact that the main viability/capturability theorems hold true for evolutionary systems defined on any metric space, such as $\mathcal{H}(X)$ (see [6, 5, **?**, Aubin], [13, 14, Aubin-Haddad] and [7, Aubin & Catté] for recent results on that topic).).

Remark — This format allows us to associate a prediction process with and anticipation operator, that is a map $\phi : \mathcal{H}(X) \mapsto \mathcal{C}(0,\infty;X)$ from the history space to the future space $\mathcal{C}(0,\infty;X)$. Therefore, a differential equation with anticipation is defined from both an anticipation operator ϕ and a single-valued map $g : \mathcal{C}(0,\infty;X) \mapsto$

X by taking $f := g \circ \phi$ as the dynamics governing the evolution through the history dependent differential equation

$$x'(t) \;=\; g(\phi(\kappa(t)x))$$

Examples By the way, many examples of history dependent differential inclusion equations are governed by dynamics $f := g \circ \alpha$ where $\alpha : \mathcal{H}(X) \mapsto Y$ and $g : Y \mapsto X$ where Y is an intermediate space.

By taking $Y := X^{p+1}$, delays $\theta_0 := 0 > \theta_1 > \cdots > \theta_p$ and $\alpha(\varphi) := (\varphi(0), \varphi(\theta_1), \ldots, \varphi(\theta_p),)$, we obtain delay equations

$$x'(t) \;=\; g\left(\varphi(t), \varphi(t+\theta_1), \ldots, \varphi(t+\theta_p), \right)$$

governed by the dynamics f defined by

$$f(\varphi) \;:=\; g\left(\varphi(0), \varphi(\theta_1), \ldots, \varphi(\theta_p)\right)$$

The classical example is provided by Volterra dynamics defined through a "kernel" $k :]-\infty, 0] \times X \mapsto Y$, the associated integral operator $\alpha : \mathcal{H}(X) \mapsto Y$ defined by

$$\alpha(\varphi) \;:=\; \int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s)$$

and a single-valued map $g : Y \rightsquigarrow X$ by

$$f(\varphi) \;:=\; g\left(\int_{-\infty}^{0} k(-s, \varphi(s)) d\mu(s)\right)$$

where the measure $d\mu$ can be the Lebesgue measure dx or a discrete measure $\sum_{i=0}^{p} \delta_{\theta_i}$ where δ_{θ_i} are Dirac measures at times $\theta_0 := 0 > \theta_1 > \cdots > \theta_p$. $\square$

A history dependent (or path dependent) control system (U, f) is defined by

- a set-valued map $U : \mathcal{H}(X) \rightsquigarrow \mathcal{U}$ assigning to every history $\varphi \in \mathcal{H}(X)$ a subset $U(\varphi) \subset \mathcal{U}$ of controls,
- and a single-valued map $f : \text{Graph}(U) \mapsto X$ associating with every history-control pair (φ, u) the velocity $f(\varphi, u) \in X$ of the state.

It governs the evolution of the system according to the history dependent control system (or a path dependent control system, a functional control system)

$$\begin{cases} i) & x'(t) \ = \ f(\kappa(t)x, u(t)) \\ ii) & \text{where } u(t) \ \in \ U(\kappa(t)x) \end{cases}$$

The initial condition is an history $\varphi \in \mathcal{H}(X)$ and we require that at initial time 0,

$$\kappa(0)x \ := \ \varphi$$

is satisfied.

When we do not need the explicit use of the controls, they boil down to history dependent differential inclusions

$$x'(t) \ \in \ F(\kappa(t)x)$$

where $F : \mathcal{H}(X) \rightsquigarrow X$ is the set-valued map defined by

$$\forall \, \varphi \in \mathcal{H}(X), \ \ F(\varphi) \ := \ \{f(\varphi, u)\}_{u \in U(\varphi)}$$

We associate with it the prediction process $\mathcal{C} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$ associating with any $\varphi \in \mathcal{H}(X)$ the set of solutions $x(\cdot)$ to the history dependent differential equation starting at φ and the evolutionary system $\mathcal{B} : \mathcal{H}(X) \rightsquigarrow \mathcal{C}(0, \infty; \mathcal{H}(X))$ defined by $\mathcal{B}(\varphi) := \{\kappa(\cdot)x\}_{x(\cdot) \in \mathcal{C}(\varphi)}$.

2 Motivation: Dynamic Valuation of Portfolios under Price Predictive Models

Before proceeding further, we shall state in the viability/capturability framework a problem arising in dynamic valuation and management of portfolios. Here, we shall assume that a prediction process is used to forecast prices instead of assuming that we evaluate and manage portfolios under stochastic uncertainty (see [42, 43, Zabczyk] for the discrete case and [36, 37, 38, Soner & Touzi] for the continuous case) or tychastic uncertainty (see for instance [33, Pujal], [34, Pujal & Saint-Pierre] and [11, Aubin, Pujal & Saint-Pierre]), [17, 18, 19, Bernhard] for this approach and [10, Aubin & Doss] for relations between the tychastic and stochastic approach).

2.1 The Model

Let n financial assets $i = 1, \ldots, n$. The components of the state variable $x := (x_1, \ldots, x_n) \in X := \mathbf{R}^n$ are the prices of the n assets, usually denoted by $S := (S_1, \ldots, S_n)$ in the financial literature.

Instead of studying the evolution of prices under tychastic or stochastic uncertainty, we assume that we have a history dependent interest rate $\rho : \mathcal{H}(X) \mapsto X$ providing the interest rates $\rho_i(\varphi)$ in function of the history of the prices of the assets, using prediction or other extrapolation operators that are assumed to be given here[10].

The evolution of the prices of the assets is thus governed by an history-dependent differential equation [11]

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; x_i(t)\rho_i(\kappa(t)x)$$

We denote by $\mathcal{C}(\varphi)$ the set of solutions to the history dependent differential equation system satisfying $\kappa(0)x = \varphi$. *For simplicity of the exposition of this model, we assume (temporarily) that* $\mathcal{C} : \mathcal{H}(X) \mapsto \mathcal{C}(-\infty, +\infty; X)$ *is single-valued.*

A portfolio is an element $u := (u_1, \ldots, u_n) \in \mathbf{R}^n$ describing the number of shares of assets $i = 1, \ldots, n$. We may introduce constraints on the number of available shares of each asset. They range over a subset $U(\varphi) = U(\varphi_1, \ldots, \varphi_n)$ that can be a constant set $U \subset \mathbf{R}^n$ or can depend on the history of prices of the assets.

The associated capital (or the value of the portfolio) y (usually denoted by W in the financial literature) can be written

$$y \; := \; \langle u, x \rangle \; = \; \sum_{i=1}^{n} u_i x_i$$

The dynamics of the portfolio is assumed to be constrained[12] by the dynamical

[10] It is a routine task to derive from this paper and [33, Pujal] the case when we still introduce tychastic uncertainty in the predictive model, the evolution of the prices being governed by the history dependent tychastic system

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \; = \; x_i(t)\rho_i(\kappa(t)x, v(t)) \text{ where } v(t) \in Q(\kappa(t)x)$$

This framework is also the right one to study the valuation of portfolios when the evolution of prices is governed by a thychastic system and when the contingent claim $\mathbf{u}$ introduced later is history (or path) dependent.

[11] or any prediction process $\mathcal{C} : \mathcal{H}(\mathbf{R}^n) \mapsto \mathcal{C}(-\infty, +\infty; \mathbf{R}^n)$. We need only the fact that the prediction process derives from an history dependent differential equation only for deriving the Hamilton-Jacobi-Bellman equations.

[12] but no constraints are required on the size of the exchanges $u'(t)$, that can be infinite.

inequalities

$$\langle u'(t), x(t)\rangle \ = \ -\mathbf{m}(\kappa(t)x, u(t))\langle u(t), x(t)\rangle$$

It states that the cost $\langle u'(t), x(t)\rangle$ of an instantaneous exchange $u'(t)$ of the portfolio $u(t)$ at price $x(t)$ is proportional to the value (or the capital) $\langle u(t), x(t)\rangle$ of the portfolio. Usually, the cost $\mathbf{m}$ is assumed to be equal to 0 (self-financing assumption). Here, the function $\mathbf{m}$ could be regarded as a very specific example of "transaction costs". However, real transaction costs depend upon the absolute value of the instantaneous exchange u'_i of shares, and in this case, these instantaneous exchanges u'_i have to be used as components of the state of an augmented system[13], as in [11, Aubin, Pujal & Saint-Pierre].

We deduce that the evolution of price-capital pair $(x(t), y(t))$ is governed by the history dependent control system

$$\begin{cases} i) \quad \forall\, i = 1, \ldots, n, \ \ x'_i(t) \ = \ x_i(t)\rho_i(\kappa(t)x) \\ ii) \quad y'(t) \ = \ \displaystyle\sum_{i=0}^{n} u_i(t)x_i(t)\rho_i(\kappa(t)x) - \mathbf{m}(\kappa(t)x, u(t))y(t) \\ \quad\quad \text{where } u(t) \ \in \ U(\kappa(t)x) \end{cases} \quad (1)$$

controlled by the portfolio.

Let us also consider a given history-dependent (but time-independent) function $\mathbf{u} : \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$, known under the name of contingent claim in the financial literature.

For instance, we can single out history dependent contingent claims of the form

$$\mathbf{u}(\varphi) \ := \ \mathbf{w}\left(\int_{-\infty}^{0} k(-s, \varphi(s))d\mu(s)\right)$$

or

$$\mathbf{u}(\varphi) \ := \ \mathbf{w}\left(\varphi(0), \varphi(\theta_1), \ldots, \varphi(\theta_p)\right)$$

Given an exercise time $T > 0$ and an initial history $\varphi \in \mathcal{H}(X)$ of prices, the general problem of dynamic portfolio valuation is to find an initial capital y such that there

[13] in this case, constraints are also set on the sizes of the exchanges u'_i, taking into account some inertia of the managers when they have to make transactions.

exists a portfolio $u(\cdot)$ satisfying one of the three following option rules[14] are satisfied:

$$\left\{\begin{array}{ll} i) & \sum_{i=0}^n u_i(T)x_i(T) \ \geq \ \mathbf{u}(\kappa(T)x) \\ & \text{(European Options)} \\ ii) & \forall\, t \in [0,T],\ \ \sum_{i=0}^n u_i(t)x_i(t) \ \geq \ \mathbf{u}(\kappa(t)x) \\ & \text{(American Options)} \\ iii) & \exists\, t^* \in [0,T] \ \text{ such that } \sum_{i=0}^n u_i(t^*)x_i(t^*) \ \geq \ \mathbf{u}(\kappa(t^*)x) \\ & \text{(First Time Options)} \end{array}\right. \tag{2}$$

Actually, in order to treat the three option rules (2) as particular cases of a more general framework, we introduce two functions $\mathbf{b} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ (constraint function) and $\mathbf{c} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ (objective function) satisfying

$$\forall\, (t,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X),\ \ 0 \ \leq \ \mathbf{b}(t,\varphi) \ \leq \ \mathbf{c}(t,\varphi) \ \leq \ +\infty$$

with which we require that there exist a portfolio $u(\cdot)$ and a time $t^* \in [0,T]$ satisfying conditions:

$$\left\{\begin{array}{ll} i) & \forall\, t \in [0,t^*],\ \ y(t) \geq \mathbf{b}(T-t,\kappa(t)x) \\ & \text{(dynamical constraint)} \\ ii) & y(t^*) \geq \mathbf{c}(T-t^*,\kappa(t^*)x) \\ & \text{(final objective)} \end{array}\right. \tag{3}$$

are satisfied.

We can recover in particular *the three option rules (2) associated with a same contingent function* $\mathbf{u} : \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$. *They can be written in the form (3) by adequate choices of pairs* $(\mathbf{b},\mathbf{c})$ *of functions associated with* $\mathbf{u}$: Indeed, denoting by $\mathbf{u}_\infty$ the function defined by

$$\mathbf{u}_\infty(t,\varphi) \ := \left\{\begin{array}{ll} \mathbf{u}(\varphi) & \text{if} \quad t=0 \\ +\infty & \text{if not} \end{array}\right. \tag{4}$$

and by $\mathbf{0}$ the function defined by

$$\mathbf{0}(t,\varphi) \ = \left\{\begin{array}{ll} 0 & \text{if} \quad t \geq 0, \\ +\infty & \text{if not} \end{array}\right.$$

1. by taking $\mathbf{b}(t,\varphi) := \mathbf{0}(t,\varphi)$ and $\mathbf{c}(t,\varphi) = \mathbf{u}_\infty(t,\varphi)$, we obtain the rule for the European option (2)i),

[14] These rules, among many other ones, are applied in finance theory for determining the portfolios replicating options when the function $\mathbf{u}$ is regarded as a claim function.

2. by taking $\mathbf{b}(t,\varphi) := \mathbf{u}(\varphi)$ and $\mathbf{b}(t,\varphi) := \mathbf{u}_\infty(t,\varphi)$, we obtain the rule for the American option (2)ii),

3. by taking $\mathbf{b}(t,\varphi) := \mathbf{0}(t,\varphi)$ and $\mathbf{c}(t,\varphi) = \mathbf{u}(\varphi)$, we obtain the rule for the first time option(2)iii).

2.2 The Questions Raised

The problems are to

1. find the valuation subset $\mathcal{V}_{(\mathbf{b},\mathbf{c})} \subset \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+$ of triples (T,φ,y) made of the exercise time T, the initial price history φ and the initial capital y such that there exist at least a portfolio $u(\cdot)$ and a time $t^* \in [0,T]$ for which conditions (3) are satisfied,

2. associate with any exercise time T and initial price history φ the smallest capital $V(T,\varphi)$:
$$V_{(\mathbf{b},\mathbf{c})}(T,\varphi) \ := \inf_{(T,\varphi,y)\in\mathcal{V}_{(\mathbf{b},\mathbf{c})}} y$$
The function $(T,\varphi) \mapsto V_{(\mathbf{b},\mathbf{c})}(T,\varphi)$ — called the southern border of $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ — is said to be the valuation function, i.e., the minimal initial capital y satisfying the two constraints (3).

3. find the regulation map $\Gamma_{(\mathbf{b},\mathbf{c})}$ associating with any $(T,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X)$ a subset $\Gamma_{(\mathbf{b},\mathbf{c})}(T,\varphi)$ of portfolios $u \in U(\varphi)$ that are optimal in the sense that the optimal capital is governed by the history dependent control system
$$\left\{\begin{array}{ll} i) & \forall\, i = 1,\ldots,n, \ \ x_i'(t) \ = \ x_i(t)\rho_i(\kappa(t)x) \\ ii) & y'(t) \ = \ \displaystyle\sum_{i=0}^{n} u_i(t)x_i(t)\rho_i(\kappa(t)x) - \mathbf{m}(\kappa(t)x, u(t))y(t) \\ iii) & \text{where } u(t) \ \in \ \Gamma_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x) \\ & \text{(regulation law)} \end{array}\right. \tag{5}$$

This regulation law is the main answer the theory brings: it tells the manager at each instant how to change his portfolio in terms to time $T-t$ left to exercise time and the prices of the assets.

We shall provide a formula of the valuation function in the cases of the European, American and first-time options in this section and in the general case later on. For that purpose, we associate with the contingent claim function **u** the functional

$$\left\{ \begin{array}{l} \mathbf{J_u}(t; u(\cdot))(\varphi) \ := \ e^{\int_0^t \mathbf{m}(\kappa(s)x, u(s))ds} \mathbf{u}(\kappa(t)x) \\ - \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x, u(s))ds} \sum_{i=0}^n u_i(\tau)\kappa(\tau)x_i\rho_i(\kappa(\tau)x_i)d\tau \end{array} \right.$$

where $x(\cdot) := \mathcal{C}(\varphi)$ is the evolution of the price starting from the initial history φ. We set

$$\mathbf{I_0}(t; u(\cdot))(\varphi) \ := \ \inf_{s\in[0,t]} \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x, u(s))ds} \sum_{i=0}^n u_i(\tau)\kappa(\tau)x_i\rho_i(\kappa(\tau)x_i)d\tau$$

We shall prove that for each of the three option rules, the three corresponding valuation functions are given by the following formulas:

1. **European Options**:

$$V_{(\mathbf{0},\mathbf{u}_\infty)}(T, \varphi) \ = \ \inf_{u(\cdot)\in U(\kappa(\cdot)x)} \max\left[\mathbf{J_u}(T; u(\cdot))(\varphi), -\mathbf{I_0}(T; u(\cdot))(\varphi)\right] \qquad (6)$$

2. **American Options**:

$$V_{(\mathbf{u},\mathbf{u}_\infty)}(T, \varphi) \ = \ \inf_{u(\cdot)\in U(\kappa(\cdot)x)} \sup_{t\in[0,T]} \mathbf{J_u}(t; u(\cdot))(\varphi) \qquad (7)$$

3. **First Time options**:

$$V_{(\mathbf{0},\mathbf{u})}(T, \varphi) \ = \ \inf_{u(\cdot)\in U(\kappa(\cdot)x)} \inf_{t\in[0,T]} \max\left[\mathbf{J_u}(t; u(\cdot))(\varphi), -\mathbf{I_0}(t; u(\cdot))(\varphi)\right] \qquad (8)$$

Hence the valuation function defined naturally — however, implicitly — in terms of a financial story as the southern border of the valuation set is actually the valuation function of intertemporal minimization history dependent control problem: a Bolza type minimal control problem in the case of European options, a minimal control problem of a non classical functional in the case of American options and an "obstacle" or "stopping time" problem in the case of First-Time options.

One could stop here and conclude that we can use known results of optimal control theory. This could be true in the case of European options, that are Bolza type optimal control problems. But it is well known that even in the history independent framework, stopping time problems are harder to study than Bolza problems, and American options are more difficult to study than European options.

Actually, we propose just the opposite: the results we shall expose in full generality in the viability/capturability format imply corresponding results enjoyed by a whole class of intertemporal minimization (and maximization) of history-dependent functionals that could be much more difficult to obtain directly.

In particular, we shall relate these history-dependent valuation functions to solutions of adequate generalizations of Hamilton-Jacobi-Bellman partial differential equations and variational inequalities.

2.3 Clio Derivatives

For that purpose, we have first to define "Clio partial derivatives" of history dependent functions:

Definition 3 *We set*

$$\mathcal{A}(X) \;:=\; \{x(\cdot) \in \mathcal{C}(0,+\infty;X) \quad \textit{such that } x(0) \;=\; 0\}$$

*Let $h > 0$ be given. The h-*chaining *(or* chaining *when there are no ambiguities) $\varphi\diamondsuit_h\psi$ is the bilinear form from $\mathcal{H}(X) \times \mathcal{A}(X) \mapsto \mathcal{H}(X)$ defined by*

$$(\varphi\diamondsuit_h\psi)(\tau) \;:=\; \begin{cases} \varphi(\tau+h) & \text{if } \;\tau \in]-\infty,-h] \\ \varphi(0)+\psi(\tau+h) & \text{if } \;\tau \in [-h,0] \end{cases}$$

We see that $(\varphi\diamondsuit_h\psi)(0) = \varphi(0) + h\frac{\psi(h)}{h}$.

As an example, the chaining of $\varphi \in \mathcal{H}(X)$ and $u \in X$ is defined by

$$(\varphi\diamondsuit_h u)(\tau) \;=\; \begin{cases} \varphi(\tau+h) & \text{if } \;\tau \in]-\infty,-h] \\ \varphi(0)+(\tau+h)u & \text{if } \;\tau \in [-h,0] \end{cases}$$

This chaining operation $\diamondsuit_h u : \varphi \in \mathcal{H}(X) \mapsto \varphi\diamondsuit_h u \in \mathcal{H}(X)$ by a vector u replaces the addition $x \in X \mapsto x + hu \in X$ for defining difference quotients and derivatives of a function $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$. Actually, for any function $\varphi \in \mathcal{H}(X)$ such that $x = \varphi(0)$, we see that $(\varphi\diamondsuit_h u)(0) = x + hu$.

Definition 4 *Let $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ be a history dependent function. We shall that*

$$\mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(\lambda,u) \;:=\; \liminf_{h\to 0+,\; \frac{\psi(h)}{h}\to u,\; \psi_h\in\mathcal{A}(X)} \frac{\mathbf{v}(t+h\lambda,\varphi\diamondsuit_h\psi_h) - \mathbf{v}(t,\varphi)}{h}$$

is the Clio epiderivative *of the function* $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ *at* (t, φ) *in the direction* (λ, u).

In other words, there exist a sequence $h_n > 0$ converging to 0, a sequence $\psi_n \in \mathcal{A}(X)$ such that $\frac{\psi_n(h_n)}{h_n}$ converges to u and a sequence v_n converging to $\mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(\lambda,u)$ such that

$$\forall\, n \geq 0, \;\; v_n \;\geq\; \frac{\mathbf{v}(t + h_n\lambda, \varphi\diamondsuit_{h_n}\psi_n) - \mathbf{v}(t,\varphi)}{h_n}$$

Naturally, if the map $(\lambda, u) \mapsto \mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(\lambda,u)$ is linear (and continuous when X is a topological vector space), then we can write

$$\mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(\lambda,u) \;=\; \frac{\partial\mathbf{v}(t,\varphi)}{\partial t}\lambda + \sum_{i=1}^{n}\frac{\partial\mathbf{v}(t,\varphi)}{\partial x_i}u_i$$

where we set

$$\begin{cases} \dfrac{\partial\mathbf{v}(t,\varphi)}{\partial t} & := \quad \mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(1,0,\ldots,0) \\ \dfrac{\partial\mathbf{v}(t,\varphi)}{\partial x_i} & := \quad \mathbf{D}_\uparrow\mathbf{v}(t,\varphi)(0,0,\ldots,1,\ldots,0) \end{cases}$$

and regard $\dfrac{\partial\mathbf{v}(t,\varphi)}{\partial x_i}$ as the Clio partial derivative of $\mathbf{v}$ at (t,φ) with respect to x_i.

2.4 History Hamilton-Jacobi-Bellman Equations

We denote by $(\mathbf{b},\mathbf{c})$ one of the three pairs $(\mathbf{0},\mathbf{u}_\infty)$, $(\mathbf{u},\mathbf{u}\infty)$ and $(\mathbf{0},\mathbf{u})$ involved in the three option rules studied in this section. When the valuation function $V_{(\mathbf{b},\mathbf{c})}$ is Clio differentiable, it is actually the largest solution $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ between the functions $\mathbf{b}$ and $\mathbf{c}$ to the nonlinear Hamilton-Jacobi-Isaacs partial differential inequalities (that play the role of Black-Scholes partial differential equations when the evolution of prices is governed by a stochastic differential equation):

$$-\frac{\partial\mathbf{v}(t,\varphi)}{\partial t} + \inf_{u\in U(\varphi)}\left(\sum_{i=0}^{n}\left(\frac{\partial\mathbf{v}(t,\varphi)}{\partial x_i} - u_i\right)\varphi_i(0)\rho_i(\varphi) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi)\right) \;\leq\; 0$$

satisfying the initial condition

$$\mathbf{v}(0,\varphi) \;=\; \mathbf{u}(\varphi)$$

on each of the subsets

1. **European case**:

$$\Omega_{(\mathbf{0},\mathbf{u}_\infty)}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t>0 \;\&\; \mathbf{v}(t,\varphi) \;\geq\; 0\}$$

2. **American Case**

$$\Omega_{(\mathbf{u},\mathbf{u}\infty)}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t>0 \;\&\; \mathbf{v}(t,\varphi) \;\geq\; \mathbf{u}(\varphi)\}$$

3. **First time case**

$$\Omega_{(\mathbf{0},\mathbf{u})}(\mathbf{v}) \;:=\; \{(t,\varphi) \mid t>0 \;\&\; \mathbf{u}(\varphi) \;>\; \mathbf{v}(t,\varphi) \;\geq\; 0\}$$

Actually, this is true even when the valuation function is only lower semicontinuous: it is enough to replace the Clio derivatives of the valuation function by its Clio epiderivative: The valuation function $V_{(\mathbf{b},\mathbf{c})}$ is actually the largest solution $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ to

$$\begin{cases} \mathbf{D}_\uparrow \mathbf{v}(t,\varphi)(-1,\varphi_1(0)\rho_1(\varphi),\ldots,\varphi_n(0)\rho_n(\varphi)) \\ +\inf_{u\in U(\varphi)} \left(\mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi) - \sum_{i=1}^n u_i\varphi_i(0)\rho_i(\varphi)\right) \;\leq\; 0 \end{cases}$$

on the corresponding set $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$.

2.5 The Regulation Map

We shall prove that the regulation map $\Gamma_{(\mathbf{b},\mathbf{c})}$ is defined on the corresponding set $\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v})$ by

$$\begin{cases} \Gamma_{(\mathbf{b},\mathbf{c})}(t,\varphi) \;:=\; \Big\{ u\in U(\varphi) \;\text{ such that } \mathbf{D}_\uparrow \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi)(-1,\varphi_1(0)\rho_1(\varphi),\ldots,\varphi_n(0)\rho_n(\varphi)) \\ +\mathbf{m}(\varphi,u)\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi) - \displaystyle\sum_{i=1}^n u_i\varphi_i(0)\rho_i(\varphi) \leq 0 \Big\} \end{cases}$$

hence the regulation law providing the portfolios is

$$u(t) \;\in\; \Gamma_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x)$$

3 Intertemporal Optimization of History Dependent Functionals

The history dependent control system (1) is actually a particular case of a system of the form[15]

$$\begin{cases} i) & x'(t) = f(\kappa(t)x, u(t)) \\ ii) & y'(t) = -\mathbf{m}(\kappa(t)x, u(t))y(t) - \mathbf{l}(\kappa(t)x, u(t)) \\ iii) & \text{where } u(t) \in U(\kappa(t)x) \end{cases} \quad (9)$$

We denote now by $\mathcal{C}(\varphi) \subset \mathcal{C}(0,\infty;X)$ the set of solutions[16] $x(\cdot)$ to the history dependent control system

$$\begin{cases} i) & x'(t) = f(\kappa(t)x, u(t)) \\ ii) & \text{where } u(t) \in U(\kappa(t)x) \end{cases} \quad (10)$$

satisfying $\kappa(0)x = \varphi$ and by $\mathcal{B}(\varphi) \subset \mathcal{C}(0,\infty;\mathcal{H}(X))$ the subset of associated evolutions $\kappa(\cdot)x$.

Definition 5 *Given the history dependent control system (1) and two functions* $\mathbf{b} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ *(constraint function) and* $\mathbf{c} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ *(objective function) satisfying*

$$\forall\ (t,\varphi) \in \mathbf{R}_+ \times \mathcal{H}(X),\ \ 0 \ \le\ \mathbf{b}(t,\varphi)\ \le\ \mathbf{c}(t,\varphi)\ \le\ +\infty$$

the valuation *subset* $\mathcal{V}_{(\mathbf{b},\mathbf{c})} \subset \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+$ *is the subset of triples* (T,φ,y) *such that there exist* at least *one evolution* $(x(\cdot),u(\cdot)) \in \mathcal{C}(\varphi)$ *starting from* φ *and a time* $t^\star \in [0,T]$ *such that conditions (3):*

$$\begin{cases} i) & \forall\ t \in [0,t^\star],\ \ y(t) \ge \mathbf{b}(T-t, \kappa(t)x, u(t)) \\ & \text{(dynamical constraint)} \\ ii) & y(t^\star) \ge \mathbf{c}(T-t^\star, \kappa(t^\star)x, u(t^\star)) \\ & \text{(final objective)} \end{cases}$$

are satisfied.

In order to compute the southern border of the valuation set subset $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$, we associate with any history dependent function $\mathbf{v} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ the

[15]In the financial example, we took $\mathbf{l}(\varphi,u) := \sum_{i=1}^{n} p_i\varphi_i(0)\rho_i(\varphi)$.

[16]for simplicity, we do not make explicit the control $u(\cdot) \in L^1(0,\infty;\mathcal{U})$ involved in the definition of the solution.

functional

$$\left\{ \begin{array}{l} \mathbf{J_v}(t;(x(\cdot),u(\cdot)))(T,\varphi) \ := \ e^{\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{v}(T-t,\kappa(t)x) \\ + \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{l}(\kappa(\tau)x,u(\tau))d\tau \end{array} \right.$$

where $\varphi := \kappa(0)x$. We next associate with $\mathbf{b}$ the functional

$$\mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi) \ := \ \sup_{s\in[0,t]} \mathbf{J_b}(s;(x(\cdot),u(\cdot)))(T,\varphi)$$

We integrate this cumulated cost together with the cost $\mathbf{J_c}(t;(x(s),u(s)))(T,\varphi)$ associated with the function $\mathbf{c}$ by introducing the new cost functions

$$\mathbf{L_{(b,c)}}(t;(x(\cdot),u(\cdot)))(T,\varphi) \ := \ \max(\mathbf{I_b}(x(\cdot),u(\cdot))(T,\varphi),\mathbf{J_c}(t;(x(\cdot),u(\cdot)))(T,\varphi))$$

and define the history dependent functional

$$\mathbf{V_{(b,c)}}(x(\cdot),u(\cdot))(T,\varphi) \ = \ \inf_{t\in[0,T]} \mathbf{L_{(b,c)}}(t;(x(\cdot),u(\cdot)))(T,\varphi)$$

We shall prove that the southern border

$$V_{(\mathbf{b},\mathbf{c})}(T,\varphi) \ := \ \inf_{(T,\varphi,y)\in\mathcal{V}_{(\mathbf{b},\mathbf{c})}} y \tag{11}$$

of the valuation set $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ is equal to the valuation of the history dependent minimization problem

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \ := \ \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V_{(b,c)}}(x(\cdot),u(\cdot))(T,\varphi)$$

and gather other characterizations:

Theorem 6 *Let us assume that the extended functions* $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial and non negative.*

Then the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ *satisfies*

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \ := \ \inf_{(T,\varphi,y)\in\mathcal{V}_{(\mathbf{b},\mathbf{c})}} y$$

When $(T,\varphi,\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi))$ *belongs to the valuation set* $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$*, there exists at least one solution* $(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)$ *starting from* φ *satisfying the inequality:*

$$\left\{ \begin{array}{l} \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \\ \geq \ e^{\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x) + \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{l}(\kappa(\kappa)x,u(\tau))d\tau \end{array} \right. \tag{12}$$

until the first time $t^* \in [0,T]$ *when*

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t^*,\kappa(t^*)x) = \mathbf{c}(T-t^*,\kappa(t^*)x)$$

and any such evolution is an optimal evolution for the optimal time t^*.

Furthermore, if $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) > \mathbf{b}(T,\varphi)$*, any such evolution actually satisfies*

$$\begin{cases} \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \\ = \; e^{\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x) + \displaystyle\int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds}\mathbf{l}(\kappa(\kappa)x,u(\tau))d\tau \end{cases} \tag{13}$$

until the first time $t^{**} \in [0,t^*]$ *when*

$$V^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \;=\; \mathbf{I_b}(t^{**};(x(\cdot),u(\cdot))))(T,\varphi)$$

Finally, the valuation function is the unique *solution* $\mathbf{v}$ *to the system of two following functional equations stating that the functions* $\mathbf{V}_{(\mathbf{b},\mathbf{v})}$ *and* $\mathbf{V}_{(\mathbf{v},\mathbf{c})}$ *have the same infimum than* $\mathbf{V}_{(\mathbf{b},\mathbf{c})}$*:*

$$\begin{cases} \mathbf{V}^{\inf}_{(\mathbf{v},\mathbf{c})}(T,\varphi) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V}_{(\mathbf{v},\mathbf{c})}((x(\cdot),u(\cdot)))(T,\varphi) \\ = \; \mathbf{v}(T,\varphi) \\ = \; \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{v})}(T,\varphi) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V}_{(\mathbf{b},\mathbf{v})}((x(\cdot),u(\cdot)))(T,\varphi) \end{cases} \tag{14}$$

The next theorem characterizes the valuation functions as a generalized solution $\mathbf{v}$ to the following Clio Hamilton-Jacobi-Bellman partial differential equation

$$-\frac{\partial \mathbf{v}(t,\varphi)}{\partial t} + \inf_{u\in U(\varphi)}\left(\sum_{i=1}^n \frac{\partial \mathbf{v}(t,\varphi)}{\partial x_i} f_i(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi) + \mathbf{l}(\varphi,u)\right) \;\le\; 0$$

on the subset

$$\Omega_{(\mathbf{b},\mathbf{c})}(\mathbf{v}) \;:=\; \{(t,\varphi)\in\mathbf{R}_+\times\mathcal{H}(X) \mid \mathbf{b}(t,\varphi) \;\le\; \mathbf{v}(t,\varphi) \;<\; \mathbf{c}(t,\varphi)\}$$

More precisely, since the valuation function is only lower semicontinuous and may take infinite values, the derivatives are taken in the "Clio sense" defined earlier:

Theorem 7 *Let us assume that the control system* $(U,f,\mathbf{l},\mathbf{m})$ *is Marchaud in the sense that*

$$\begin{cases} i) & f \text{ and } U \text{ are Marchaud} \\ ii) & \sup_\varphi \|U(\varphi)\| \;<\; +\infty \\ iii) & \mathbf{m} \text{ and } \mathbf{l} \text{ are continuous with linear growth and convex with respect to } u \end{cases} \tag{15}$$

and that the functions $\mathbf{b}$ *and* $\mathbf{c}$ *are nontrivial, non negative and lower semicontinuous.*

Then the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ *is lower semicontinuous and is characterized as the smallest of the nonnegative lower semicontinuous functions* $\mathbf{v} : \mathbf{R}_+ \times X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *satisfying for every* $(t, \varphi) \in]0, \infty[\times X$

$$\left\{ \begin{array}{ll} i) & \mathbf{b}(t,\varphi) \ \leq \ \mathbf{v}(t,\varphi) \ \leq \ \mathbf{c}(t,\varphi) \\ ii) & \forall\ (t,\varphi) \in \Omega_{(\mathbf{b},\mathbf{c})}\mathbf{v}), \\ & \inf_{u\in U(\varphi)} \left(\mathbf{D}_{\uparrow}\mathbf{v}(t,\varphi)(-1, f(\varphi,u)) + \mathbf{l}(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{v}(t,\varphi)\right) \ \leq \ 0 \end{array} \right. \quad (16)$$

Let us set

$$\mathbf{R}(t,\varphi) \ := \ \{u \in U(\varphi) \mid \mathbf{D}_{\uparrow}\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi)(-1, f(\varphi,u)) + \mathbf{l}(\varphi,u) + \mathbf{m}(\varphi,u)\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(t,\varphi) \leq 0\}$$

Knowing the valuation function, an optimal solution is obtained in the following way: Starting from φ_0 *such that* $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi_0) < \mathbf{c}(T,\varphi_0)$*, any solution* $(x(\cdot), u(\cdot))$ *to the control system*

$$\left\{ \begin{array}{ll} i) & x'(t) \ = \ f(\kappa(t)x, u(t)) \\ ii) & u(t) \ \in \ \mathbf{R}(T-t, \kappa(t)x) \\ & \text{(regulation law)} \end{array} \right. \quad (17)$$

is an optimal solution, and the first time $t^* \geq 0$ *when*

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t^*, \kappa(t^*)x) \ = \ \mathbf{c}(T-t^*, \kappa(t^*)x)$$

is the optimal time.

4 The Viability/Capturability Strategy

We shall prove the above Theorems 6 and 7 from the fact that the valuation set is nothing other than a viable-capture basin of the epigraph of the function $\mathbf{c}$ under an auxiliary history dependent differential inclusion. This is the purpose of this section.

4.1 The Auxiliary Evolutionary System

Indeed, using the epigraphs of the functions $\mathbf{b}$ and $\mathbf{c}$, we can translate the viability/capturability conditions (3) in the following geometric form:

$$\left\{ \begin{array}{ll} i) & \forall\ t \in [0, t^*], \ \ (T-t, \kappa(t)x, y(t)) \in \mathcal{E}p(\mathbf{b}) \\ & \text{(viability constraint)} \\ ii) & (T-t^*, \kappa(t^*)x, y(t^*)) \in \mathcal{E}p(\mathbf{c}) \\ & \text{(capturability of a target)} \end{array} \right. \quad (18)$$

On the other hand, the function $t \mapsto (T-t, x(t), y(t))$ is actually governed by the history dependent control system

$$\left\{\begin{array}{lrcl} i) & \tau'(t) & = & -1 \\ ii) & x'(t) & = & f(\kappa(t)x, u(t)) \\ iii) & y'(t) & = & -\mathbf{m}(\kappa(t)x, u(t))y(t) - \mathbf{l}(\kappa(t)x, u(t)) \\ iv) & \text{where } u(t) & \in & U(\kappa(t)x) \end{array}\right. \tag{19}$$

Then it is easy to check that this is an history dependent control system associated with the set-valued map $G := (U, f, \mathbf{l}, \mathbf{m}) : \mathcal{H}(\mathbf{R} \times X \times \mathbf{R}) \rightsquigarrow \mathbf{R} \times X \times \mathbf{R}$ defined by

$$G(\omega, \varphi, \eta) \; := \; \{-1\} \times \{(f(\varphi, u), -\mathbf{m}(\varphi, u)\eta(0) - \mathbf{l}(\varphi, u))\}_{u \in U(\varphi)}$$

Actually, we introduce also the set-valued map $G_{\leq} := (U, f, \mathbf{l}, \mathbf{m})_{\leq}$ defined by

$$G_{\leq}(\omega, \varphi, \eta) \; := \; \{-1\} \times \bigcup_{u \in U(\varphi)} [-\lambda - \mu\eta(0), -\mathbf{m}(\varphi, u)\eta(0) - \mathbf{l}(\varphi, u)]$$

where

$$\lambda \; := \sup_{(\varphi,u) \in \mathrm{Graph}(U)} |\mathbf{l}(\varphi, u)| \;\&\; \mu \; := \sup_{(\varphi,u) \in \mathrm{Graph}(U)} |\mathbf{m}(\varphi, u)|$$

governing the solutions to the control system

$$\left\{\begin{array}{lrcl} i) & \tau' & = & -1 \\ ii) & x'(t) & = & f(\kappa(t)x, u(t)) \\ iii) & y'(t) & = & -\mathbf{m}(\kappa(t)x, u(t))y(t) - \mathbf{l}(\kappa(t)x, u(t)) - v(t) \\ & \multicolumn{3}{l}{\text{where } u(t) \in U(\kappa(t)x) \;\&\; v(t) \in [0, \lambda + \mu y(t)]} \end{array}\right. \tag{20}$$

We observe that G and $G_{\leq}$ are Lipschitz whenever the set-valued map U is Lipschitz and the maps f, $\mathbf{m}$ and $\mathbf{l}$ are Lipschitz and bounded on the graph of U and that $G_{\leq}$ is Marchaud and nontrivial whenever[17] whenever (15) holds true.

For simplicity, we shall denote by

$$\mathbf{X} \; := \; \mathcal{H}(\mathbf{R} \times X \times \mathbf{K})$$

the auxiliary state space. We associate with the epigraphs of the functions $\mathbf{b}$ and $\mathbf{c}$ the subsets

$$\left\{\begin{array}{lrcl} i) & \mathbf{K} & := & \{\mathbf{x} := (\omega, \varphi, \eta) \in \mathbf{X} \text{ such that } (\omega(0), \varphi, \eta(0)) \in \mathcal{E}p(\mathbf{b})\} \\ ii) & \mathbf{C} & := & \{\mathbf{x} := (\omega, \varphi, \eta) \in \mathbf{X} \text{ such that } (\omega(0), \varphi, \eta(0)) \in \mathcal{E}p(\mathbf{c})\} \end{array}\right.$$

[17] These assumptions can be relaxed to the price of more technical conditions.

We shall denote by $\mathcal{S} : \mathbf{X} \mapsto \mathcal{C}(0,\infty;\mathbf{X})$ the evolutionary system associated with G, mapping any $\mathbf{x} := (\omega, \varphi, \eta)$ to the set $\mathcal{S}(\mathbf{x}) := \mathcal{S}(\omega, \varphi, \eta)$ of evolutions $\mathbf{x}(t) := (\omega(t), \varphi(t), \eta(t))$ where $\varphi(t) \in \mathcal{B}(\varphi)$, where $\omega(t)(\tau) := \omega(0) - t - \tau$, $\eta(t)(\tau) := y(t+\tau)$ and where

$$y(t) \; := \; e^{-\int_0^t \mathbf{m}(\kappa(s)x, u(s))ds} \left(\eta(0) - \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds} \mathbf{l}(\kappa(\tau)x, u(\tau))d\tau \right)$$

and by $\mathcal{S}_{\leq} : \mathbf{X} \mapsto \mathcal{C}(0,\infty;\mathbf{X})$ the evolutionary system associated with $G_{\leq}$ mapping any $\mathbf{x} := (\omega, \varphi, \eta)$ to the set $\mathcal{S}_{\leq}(\mathbf{x}) := \mathcal{S}_{\leq}(\omega, \varphi, \eta)$ of evolutions $\mathbf{x}(t) := (\omega(t), \varphi(t), \eta(t))$ where

$$y(t) \; \leq \; e^{-\int_0^t \mathbf{m}(\kappa(s)x, u(s))ds} \left(\eta(0) - \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds} \mathbf{l}(\kappa(\tau)x, u(\tau))d\tau \right)$$

These solution maps define two evolutionary systems on the set $\mathbf{X}$ associated with the dynamics $(U, f, \mathbf{l}, \mathbf{m})$ defining the history dependent control system.

4.2 Viable-Capture Basins

We recall the following definitions of viability theory:

Definition 8 *Let $\mathbf{C} \subset \mathbf{K} \subset \mathbf{X}$ be two subsets, $\mathbf{C}$ being regarded as a target, $\mathbf{K}$ as a constrained set and $\mathcal{S} : \mathbf{X} \rightsquigarrow \mathcal{C}(0,\infty;\mathbf{X})$ be an evolutionary system.*

1. *The subset* $\mathrm{Capt}(\mathbf{K}, \mathbf{C})$ *of initial states $\mathbf{x} \in \mathbf{K}$ such that $\mathbf{C}$ is reached in finite time before possibly leaving $\mathbf{K}$ by at least one evolution $\mathbf{x}(\cdot) \in \mathcal{S}(\mathbf{x})$ starting at $\mathbf{x}$ is called the* viable-capture basin *of $\mathbf{C}$ in $\mathbf{K}$ under the evolutionary system $\mathcal{S}$ and* $\mathrm{Capt}(\mathbf{C}) := \mathrm{Capt}(\mathbf{X}, \mathbf{C})$ *is said to be the* capture basin *of $\mathbf{C}$.*

2. *We say that the subset* $\mathrm{Viab}(\mathbf{K})$ *of elements $\mathbf{x} \in \mathbf{K}$ from which starts at least one evolution $\mathbf{x}(\cdot) \in \mathcal{S}(\mathbf{x})$ viable in $\mathbf{K}$ forever is the* viability kernel *of $\mathbf{K}$ under the evolutionary system $\mathcal{S}$. A subset $\mathbf{K}$ is a* repeller *under $\mathcal{S}$ if its viability kernel is empty.*

We observe that the subset $\mathbf{K}$ associated with the epigraph of the extended function $\mathbf{b} : \mathbf{R}_+ \times \mathcal{H}(X) \mapsto \mathbf{R} \cup \{+\infty\}$ is a repeller under the auxiliary control evolutionary system $\mathcal{S}$, since any evolution $t \mapsto (T - t, \kappa(\cdot)x, \kappa(\cdot)y)$ starting at $(\omega, \varphi, \eta) \in \mathbf{K}$ leaves $\mathbf{K}$ before time T. Also. it is obvious that

$$\mathrm{Capt}_{\mathcal{S}}(\mathbf{K}, \mathbf{C}) \; \subset \; \mathrm{Capt}_{\mathcal{S}_{\leq}}(\mathbf{K}, \mathbf{C})$$

Then we can check easily that

Lemma 9 *The valuation set* $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ *is the set of triples* $(\omega(0), \varphi, \eta(0))$ *when* (ω, φ, η) *ranges over the capture basin of the subset* $\mathbf{K}$ *associated with the epigraph of* $\mathbf{b}$ *viable in the subset* $\mathbf{C}$ *associated with epigraph of* $\mathbf{b}$ *under the evolutionary system* $\mathcal{S}$ *associated with* $(U, f, \mathbf{l}, \mathbf{m})$*:*

$$\mathcal{V}_{(\mathbf{b},\mathbf{c})} \;=\; \{(\omega(0), \varphi, \eta 0)\}_{(\omega,\varphi,\eta)\in \mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{C})}$$

Furthermore,

$$\mathrm{Capt}_{\mathcal{S}}(\mathbf{K}, \mathbf{C}) \;=\; \mathrm{Capt}_{\mathcal{S}_{\leq}}(\mathbf{K}, \mathbf{C})$$

and the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ *is the southern border of the valuation set* $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$*.*

Proof — To say that a pair (ω, φ, η) belongs to the viable-capture basin $\mathrm{Capt}_{\mathcal{S}_{\leq}}(\mathbf{K}, \mathbf{C}) \subset \mathbf{K}$ means that there exist a solution $(\omega(\cdot), \kappa(\cdot)x, \kappa(\cdot)y) \in \mathcal{S}(\omega, \varphi, \eta)$ to the auxiliary control evolutionary system and some $t^\star \geq 0$ such that

$$\begin{cases} i) & \forall\, t \in [0, t^\star], \;\; \mathbf{x}(t) \;\in\; \mathbf{K} \\ ii) & \mathbf{x}(t^\star) \;\in\; \mathbf{C} \end{cases}$$

The evolution $t \mapsto \mathbf{x}(t) := (\omega(t), \varphi(t), \eta(t))$ is actually associated with the solution $t \mapsto (T - t, x(t), y(t))$ to the history dependent control problem, where $\omega(t)(\tau) := \omega(0) - t - \tau$, $\eta(t)(\tau) := y(t + \tau)$ and $\varphi(t) := \kappa(t)x$ and where

$$y(t) \;\leq\; y_0(t) := e^{-\int_0^t \mathbf{m}(\kappa(s)x,u(s))ds} \left(\eta(0) - \int_0^t e^{\int_0^\tau \mathbf{m}(\kappa(s)x,u(s))ds} \mathbf{l}(\kappa(\tau)x, u(\tau))d\tau\right)$$

where $u(\cdot) \in U(x(\cdot))$ is a control associated with the solution $x(\cdot)$ to the history dependent control system (20).

We deduce from the definition of the subsets $\mathbf{K}$ and $\mathbf{C}$ that this condition amounts to saying that

$$\begin{cases} i) & \forall\, t \in [0, t^\star], \;\; (T - t, \kappa(t)x, y(t)) \;\in\; \mathcal{E}p(\mathbf{b}) \\ \\ ii) & (T - t^\star, \kappa(t^\star)x, y(t^\star)) \;\in\; \mathcal{E}p(\mathbf{c}) \end{cases}$$

or, equivalently, that

$$\begin{cases} i) & \forall\, t \in [0, t^\star], \;\; y_0(t) \;\geq\; y(t) \;\geq\; \mathbf{b}(T - t, \kappa(t)x) \\ \\ ii) & y_0(t^\star) \;\geq\; y(t^\star) \;\geq\; \mathbf{c}(T - t^\star, \kappa(t^\star)x \end{cases}$$

Since $(\omega(\cdot), \kappa(\cdot)x, \kappa(\cdot)y_0)$ obviously belongs to $\mathcal{S}(\mathbf{x})$, we infer first that

$$\mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{C}) \;=\; \mathrm{Capt}_{\mathcal{S}_{\leq}}(\mathbf{K},\mathbf{C})$$

This viability/capturability condition can also still be written

$$\begin{cases} i) \quad y \geq e^{\int_0^t M(\kappa(s)x,u(s))ds}\mathbf{c}(T-t,\kappa(t)x,u(t)) + \displaystyle\int_0^t e^{\int_0^\tau M(\kappa(s)x,u(s))ds}\mathbf{l}(\widetilde{x}(\tau),u(\tau))d\tau \\ ii) \quad y \geq e^{\int_0^{t^*} M(\kappa(s)x,u(s))ds}\mathbf{c}(T-t^*,\kappa(t^*)x) + \displaystyle\int_0^{t^*} e^{\int_0^\tau M(\kappa(s)x,u(s))ds}\mathbf{l}(\widetilde{x}(\tau),u(\tau))d\tau \end{cases} \tag{21}$$

Since we set

$$\mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi) \;:=\; \sup_{s\in[0,t]} \mathbf{J_b}(s;(x(\cdot),u(\cdot)))(T,\varphi)$$

and

$$\mathbf{L_{(b,c)}}(t;(x(\cdot),u(\cdot)))(T,\varphi) \;:=\; \max\left[\mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi),\mathbf{J_c}(t;(x(\cdot),u(\cdot))(T,\varphi)\right]$$

this is equivalent to state that (T,φ,y) belongs to the valuation set $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$.

4.3 The Double Fixed-Point Property

We deduce from [7, Aubin & Catté] that $\mathbf{D} := \mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{C})$ is the unique subset $\mathbf{D}$ between $\mathbf{C}$ and $\mathbf{K}$ satisfying the double fixed point property

$$\mathrm{Capt}_{\mathcal{S}}(\mathbf{K},\mathbf{D}) \;=\; \mathbf{D} \;=\; \mathrm{Capt}_{\mathcal{S}}(\mathbf{D},\mathbf{C})$$

Taking the southern borders of these subsets, we deduce that the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ is a extended function $\mathbf{v}$ satisfying (14):

$$\begin{cases} \mathbf{V}^{\inf}_{(\mathbf{v},\mathbf{c})}(T,\varphi) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V}_{(\mathbf{v},\mathbf{c})}((x(\cdot),u(\cdot)))(T,\varphi) \\ = \;\mathbf{v}(T,\varphi) \\ = \;\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{v})}(T,\varphi) \;:=\; \inf_{(x(\cdot),u(\cdot))\in\mathcal{C}(\varphi)} \mathbf{V}_{(\mathbf{b},\mathbf{v})}((x(\cdot),u(\cdot)))(T,\varphi) \end{cases}$$

Furthermore, since

$$\mathbf{b} \;\leq\; \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})} \;\leq\; \mathbf{c}$$

and since $(\mathbf{b},\mathbf{c}) \mapsto \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ is increasing, we deduce that any double fixed point

$$\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{v})} \;=\; \mathbf{v} \;=\; \mathbf{V}^{\inf}_{(\mathbf{v},\mathbf{c})}$$

is necessarily equal to $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$. Hence the uniqueness. $\square$

4.4 The Optimality Principle

We begin by stating and proving the following

Lemma 10 *We observe that*

$$\left\{\begin{array}{l} \forall\ (\kappa(\cdot)x, u(\cdot)) \in \mathcal{C}(x),\ \forall\ t \in [0,T], \\ \text{inequalities} \\ \mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi)\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi)\ \le\ y\ <\ \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) \\ \text{imply that} \\ y\ <\ \mathbf{J}_{\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}}(t;(x(\cdot),u(\cdot)))(T,\varphi) \end{array}\right. \tag{22}$$

Proof — Indeed, take $y \ge \mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi)$ such that $y < \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi)$. This means that for every $s \in [0,t]$, $y \ge \mathbf{J_b}(s;(x(\cdot),u(\cdot)))(T,\varphi)$, or, equivalently, that

$$\forall\ s \in [0,t],\ \ (T-s, \kappa(s)x, y(s))\ \in\ \mathcal{E}p(\mathbf{b})$$

where

$$y(t)\ :=\ e^{-\int_0^t M(x(s),u(s))ds}\left(y - \int_0^t e^{\int_0^\tau M(x(s),u(s))ds}\mathbf{l}(x(\tau),u(\tau))d\tau\right)$$

Since (ω,φ,η) does not belong to the viable capture basin, we infer that every evolution starting at (ω,φ,η) is viable in the subset $\mathbf{K}$ associated with the epigraph $\mathcal{E}p(\mathbf{b})$ of $\mathbf{b}$ before hitting the subset $\mathbf{D}$. Since the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$ is the southern border of $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$, we infer that

$$y(t)\ <\ \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T-t,\kappa(t)x)\ \le\ \mathbf{c}(T-t,\kappa(t)x)$$

that can be written in the form

$$y\ <\ \mathbf{J}_{\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}}(t;(x(\cdot),u(\cdot)))(T,\varphi)\ \le\ \mathbf{J_c}(t;(x(\cdot),u(\cdot)))(T,\varphi)$$

Therefore, Lemma 10 is proved. □

Since $(T,\varphi,\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi))$ belongs to $\mathcal{V}_{(\mathbf{b},\mathbf{c})}$ and since $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi) > \mathbf{b}(T,\varphi)$, it can be approximated by elements $y_n \in [\mathbf{b}(T,\varphi), \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi)[$. By assumption, for any $t < t^{**}$,

$$y\ <\ \mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi)\ <\ \mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}(T,\varphi)$$

and thus, there exists some N_t such that, for any $n \ge N_t$, y_n satisfies

$$y_n\ <\ \mathbf{I_b}(t;(x(\cdot),u(\cdot)))(T,\varphi)$$

But by Lemma 10, we know that in this case, $y_n \le \mathbf{J}_{\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}}(t;(x(\cdot),u(\cdot)))(T,\varphi)$. By letting n go to infinity, we deduce that $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})} = \mathbf{J}_{\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}}(t;(x(\cdot),u(\cdot)))(T,\varphi)$, i.e., the optimality principle (13). □

4.5 The Viability Theorem for History Dependent Systems

We shall use the following characterization of capture basin (see [6, Aubin]):

Theorem 11 *Let us assume that an evolutionary system $\mathcal{S}$ is upper semicompact*[18] *and that the subsets $\mathbf{C} \subset \mathbf{K}$ and $\mathbf{K}$ are closed. If $\mathbf{K}\backslash\mathbf{C}$ is a repeller (this is the case when $\mathbf{K}$ itself is a repeller), then the viable-capture basin $\mathrm{Capt}(\mathbf{K}, \mathbf{C})$ of the target $\mathbf{C}$ under $\mathcal{S}$ is the* **largest** *closed subset satisfying $\mathbf{C} \subset \mathbf{D} \subset \mathbf{K}$ such that $\mathbf{D}\backslash\mathbf{C}$ is* locally viable *under $\mathcal{S}$.*

The Viability Theorem for history dependent differential inclusions holds true whenever we assume that the dynamics governing the path-dependent evolution is Marchaud:

Definition 12 (Marchaud Map) *We shall say that $\mathbf{F} : \mathcal{H}(X) \rightsquigarrow X$ is a* Marchaud map *if*

$$\begin{cases} i) & \mathbf{F} \text{ is upper semicontinuous} \\ ii) & \text{the values } \mathbf{F}(\varphi) \text{ of } \mathbf{F} \text{ are convex} \\ iii) & \text{the growth of } \mathbf{F} \text{ is linear: } \exists\, c > 0 \mid \forall\, \varphi \in \mathcal{H}(X), \\ & \|\mathbf{F}(\varphi)\| := \sup_{v \in \mathbf{F}(\varphi)} \|v\| \ \le \ c(\|\varphi(0)\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(\varphi, u) \mapsto f(\varphi, u)$ is continuous, affine with respect to the controls u and with linear growth and when $U : \mathcal{H}(X) \rightsquigarrow \mathcal{U}$ is Marchaud.

We supply the history space $\mathcal{H}(X)$ with the compact convergence topology. We denote by $\mathcal{H}_\lambda$ the subset of Lipschitz functions with Lipschitz constant λ.

Observe that Ascoli's Theorem states that a *closed subset $\mathbf{K} \subset \mathcal{H}(X)_\lambda$ is compact if and only if $\mathbf{K}(0) := \{\varphi(0)\}_{\varphi \in \mathbf{K}}$ is bounded*, since it is closed and equicontinuous (by assumption) and pointwise bounded because, for all $\psi \in \mathbf{K}$ and $\tau \le 0$,

$$\|\psi(\tau)\| \le \|\psi(\tau) - \psi(0)\| + \|\psi(0)\| \le \lambda|\tau| + \|\mathbf{K}(0)\|$$

Theorem 13 *Assume that $\mathbf{F} : \mathcal{H}(X) \rightsquigarrow X$ is Marchaud. Then its solution map $\mathcal{S}_\mathbf{F}$ is upper semicompact with nonempty values on each $\mathcal{H}_\lambda(X)$.*

[18] This means that whenever $\varphi_n \in \mathcal{H}(X)$ converge uniformly on compact intervals to φ in $\mathcal{H}(X)$ and any history $\varphi_n(\cdot) := \kappa(\cdot)x_n \in \mathcal{S}_\mathbf{F}(\varphi_n)$ associated to a solution $x_n(\cdot)$ to the history-dependent differential inclusion $x'(t) \in \mathbf{F}(\kappa(t)x)$ starting at φ_n, there exists a subsequence (again denoted by) $\varphi_n(\cdot)$ converging uniformly on compact intervals to the history $\varphi(\cdot) := \kappa(\cdot)x$ of a solution $x(\cdot)$ to the path-dependent differential inclusion starting at φ.

When an evolutionary system $\mathcal{B} := \mathcal{B}_{\mathbf{F}}$ is defined from an history differential inclusion $x'(t) \in \mathbf{F}(\kappa(t)x)$, we introduce the following definitions due to Haddad.

Definition 14 *Let $\mathbf{K} \subset \mathcal{H}(X)$ be a subset of histories and $\varphi \in \mathcal{H}$. We denote by $\mathcal{D}_{\mathbf{K}}(\varphi)$ the set of vectors $v \in X$ such that there exist a sequence $h_n > 0$ converging to 0 and a sequence $\psi_n \in \mathcal{A}(X)$ satisfying*

$$\left\{ \begin{array}{ll} i) & \forall\, n \geq 0, \;\; \varphi \diamond_{h_n} \psi_n \;\in\; \mathbf{K} \\ ii) & \dfrac{\psi_n(h_n)}{h_n} \to v \end{array} \right. \tag{23}$$

We quote the Haddad History-Dependent Viability Theorems of [29, 30, 31, Haddad] (see also Theorem 12.4.1 of [3, Aubin]):

Theorem 15 *Let us assume that $\mathbf{F}$ is Marchaud, that $\mathbf{K} \subset \mathcal{H}_\lambda(X)$ is closed and that a closed subset $\mathbf{C}$ satisfies $\mathrm{Viab}_{\mathbf{F}}(\mathbf{K}\backslash\mathbf{C}) = \emptyset$. Then the viable-capture basin $\mathrm{Capt}_{\mathbf{F}}^{\mathbf{K}}(\mathbf{C})$ is the* **largest** *closed subset $\mathbf{D}$ satisfying $\mathbf{C} \subset \mathbf{D} \subset \mathbf{K}$ and*

$$\forall\, \varphi \in \mathbf{D}\backslash\mathbf{C}, \;\; \mathbf{F}(\varphi) \cap \mathcal{D}_{\mathbf{D}}(\varphi) \;\neq\; \emptyset \tag{24}$$

4.6 Clio Hamilton-Jacobi-Bellman Equations

We prove now Theorem 7.

The subset $\mathbf{D}$ being equal to the capture basin $\mathrm{Capt}_{\mathcal{G}_\leq}(\mathbf{K}, \mathbf{C})$, it is the largest subset $\mathbf{D}$ between $\mathbf{C}$ and $\mathbf{K}$ such that $\mathbf{D}\backslash\mathbf{C}$ is locally viable, i.e., such that for any $(\omega, \varphi, \eta) \in \mathbf{D}\backslash\mathbf{C}$, there exists a control $u \in U(\varphi)$ and $v \in [0, \lambda + \mu y]$ such that

$$(-1, f(\varphi, u), -\mathbf{m}(\varphi, u)y - \mathbf{l}(\varphi, u) - v) \;\in\; \mathcal{D}_{\mathbf{D}}(\omega, \varphi, \eta) \tag{25}$$

By the Haddad Theorem 15, this means that there exists $h_n > 0$ converging to 0, $\psi_n \in \mathcal{A}(X)$ such that $\dfrac{\psi_n(h_n)}{h_n}$ converges to $f(\varphi, u)$ and λ_n converging to $-\mathbf{m}(\varphi, u)y - \mathbf{l}(\varphi, u) - v$ such that

$$(\omega(t) \diamond_{h_n} (-1), \varphi(t) \diamond_{h_n} \psi_n, \eta(t) \diamond_{h_n} \lambda_n) \;\in\; \mathbf{D}$$

and thus, by Lemma 9, such that

$$(t - h_n, \kappa(t)x \diamond_{h_n} \psi_n, y + h_n\lambda_n) \;\in\; \mathcal{V}_{(\mathbf{b},\mathbf{c})}$$

where $t := \omega(0)$, $x(\cdot) \in \mathcal{C}(\varphi)$ and $y := \eta(0) \geq \mathbf{v}(t, \varphi)$.

Since the viable-capture basin is closed by Theorems 11 and 13, then the valuation subset is also closed and thus, equal to the epigraph of its southern border, which is the valuation function $\mathbf{V}^{\inf}_{(\mathbf{b},\mathbf{c})}$. Hence the valuation set is the epigraph of the smallest lower semicontinuous extended function $\mathbf{v}$ satisfying

$$(t - h_n, \kappa(t)x\Diamond_{h_n}\psi_n, y + h_n\lambda_n) \ \in \ \mathcal{E}p(\mathbf{v})$$

or again, since $y \geq \mathbf{v}(\mathbf{t}, \varphi)$, such that

$$\lambda_n \ \geq \ \frac{\mathbf{v}(t - h_n, \kappa(t)x\Diamond_{h_n}\psi_n) - y}{h_n} \ \geq \ \frac{\mathbf{v}(t - h_n, \kappa(t)x\Diamond_{h_n}\psi_n) - \mathbf{v}(t, \varphi)}{h_n}$$

If $y = \mathbf{v}(t, \varphi)$, this is equivalent to

$$\mathbf{D}_{\uparrow}\mathbf{v}(t, \varphi)(-1, f(\varphi, u)) + \mathbf{l}(\varphi, u) + \mathbf{m}(\varphi, u)\mathbf{v}(t, \varphi) \ \leq \ -v \ \leq \ 0$$

This implies that

$$\mathbf{b}(t, \varphi) \ \leq \ \mathbf{v}(t, \varphi) \ \leq \ \mathbf{c}(t, \varphi)$$

and, whenever $\mathbf{v}(t, \varphi) < \mathbf{c}(t, \varphi)$,

$$\inf_{u \in U(\varphi)} (\mathbf{D}_{\uparrow}\mathbf{v}(t, \varphi)(-1, f(\varphi, u)) + \mathbf{l}(\varphi, u) + \mathbf{m}(\varphi, u)\mathbf{v}(t, \varphi)) \ \leq \ 0$$

Conversely, if $y \geq \mathbf{v}(t, \varphi)$, property

$$\mathbf{D}_{\uparrow}\mathbf{v}(t, \varphi)(-1, f(\varphi, u)) \ \leq \ -\mathbf{m}(\varphi, u)\mathbf{v}(t, \varphi) - \mathbf{l}(\varphi, u)$$

implies that

$$(t-h_n, \kappa(t)x\Diamond_{h_n}\psi_n, y+h_n\lambda_n) \ = \ (t-h_n, \kappa(t)x\Diamond_{h_n}\psi_n, \mathbf{v}(t, \varphi)+h_n\lambda_n)+(0, h_n(\mathbf{v}(t, \varphi)-y)) \ \in \ \mathcal{E}p(\mathbf{v})$$

for n large enough. Hence, when $y \geq \mathbf{v}(t, \varphi)$, we have shown that

$$(t - h_n, \kappa(t)x\Diamond_{h_n}\psi_n, y + h_n\lambda_n) \ \in \ \mathcal{V}_{(\mathbf{b},\mathbf{c})} \ = \ \mathcal{E}p(\mathbf{v})$$

and thus, that

$$(\omega(t)\Diamond_{h_n}(-1), \varphi(t)\Diamond_{h_n}\psi_n, \eta(t)\Diamond_{h_n}\lambda_n) \ \in \ \mathbf{D}$$

This concludes the proof. $\square$

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