Dynamical Qualitative Analysis of Evolutionary Systems

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Abstract

Kuipers' QSIM algorithm for tracking the monotonicity properties of solutions to differential equations has been revisited by Dordan by placing it in a rigorous mathematical framework. The Dordan QSIM algorithm provides the transition laws from one qualitative cell to the others.

We take up this idea and revisit it at the light of recent advances in the field of "hybrid systems" and, more generally, "impulse differential equations and inclusions".

Let us consider a family of "qualitative cells $Q(a)$" indexed by a parameter $a \in \mathcal{A}$: We introduce a dynamical system on the discrete set of qualitative states prescribing an order of visit of the qualitative cells *and an evolutionary system govening the "continuous" evolution of a system, such as a control system. The question arises to study and characterize the set of any pairs of qualitative and quantitative initial states from which start at least one order of visit of the qualitative cells and an continuous evolution visiting the qualitative cells in the prescribed order. This paper is devoted to the issues regarding this question using tools of set-valued analysis and viability theory.*

Keywords: impulse control, differential inclusion, viability, qualitative analysis, qualitative cell, qualitative viability, contingent cone, Marchaud map.

AMS Classification: 93C10, 93C15, 93C55, 49J24, 49J40, 49J53

Introduction

For many problems, we have an imperfect knowledge of the model and we may be interested by few features (often of a qualitative nature) of the solution, so that we see at once that the concept of partial knowledge involves two types of ideas:

1. require less precision in the results (for instance, signs of the components of vectors instead of their numerical values),

2. take into account a broader universality or robustness of these results with respect to uncertainty, disturbances and lack of precisions.

Qualitative physics takes up this issue by exploring the use of qualitative representations in controlling engineering problems. Those qualitative representations are, for instance, used for representing experimental and/or observational data. This is always the case when captors or sensors can only collect very shallow information like signs of the variables. In this particular case the number of qualitative values is finite. Those qualitative values will be labeled and the set of labels is denoted $\mathcal{A}$. The connection with quantitative values is obtain with qualitative cells defined by a finite family $\mathbf{Q} := \{Q_a\}_{a\in\mathcal{A}}$ where $\{Q_a\}_{a\in\mathcal{A}}$ is the a subset of a quantitative space denoted X. Variables take on different qualitative values at different times as they evolve under an "evolutionary system" $\mathcal{S} : X \rightsquigarrow \mathbf{C}(0,\infty;X)$ where $\mathcal{S}$ is set valued map associating with any initial state $x \in X$ a (possibly empty) set $\mathcal{S}(x)$ of evolutions $x(\cdot) : t \in \mathbf{R}_+ \mapsto x(t) \in X$. This covers ordinary differential equations, differential inclusions, controlled systems or parameterized systems, that may even be path-dependent.

This qualitative representation provides a sequence of discrete episodes that occur as the qualitative variable value changes and then generates a discrete dynamic process that we represent by a set-valued map $\Phi : \mathcal{A} \rightsquigarrow \mathcal{A}$ consistent with the qualitative cells in the sense that:

$$\forall\, a \in \mathcal{A},\ \forall\, b \in \Phi(a),\ \ Q(a) \cap Q(b)\ \neq\ \emptyset$$

In the framework of qualitative physics, each path of the discrete dynamic process is called a history, the network an "envisionment".

Several problems can be addressed :

- Finding the set of pairs (a,x) where $x \in Q(a)$ from which start a sequence $\vec{a} := \{a_n\}_{n\geq 0} \in \mathcal{S}_\Phi(a)$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ visiting the cells $Q(a_n)$ in the prescribed order until it possibly reaches a given target $C(a_N)$ at a finite step a_N.

  This subset will be called the qualitative viability kernel of the family $\mathbf{Q}$ with family $\mathbf{C}$ of targets $C(a)$ under the qualitative evolutionary system $(\mathcal{S},\Phi)$ and is denoted $\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$. This is a concept closely related to the concept of impulse viability kernel introduced and studied in [18, 19, Cruck] and [6, Aubin].

In the case when Φ is a single value map generating the sequence of qualitative values $\vec{a}$ starting from the initial qualitative state $a \in \mathcal{A}$, we shall find all the "quantitative initial conditions" in $Q(a)$ — if any — such that at least one "quantitative" evolution visits the qualitative cells $Q(a_n)$. This solves an observability problem, somewhat related to the detectability problem considered in [15, Aubin & Haddad].

- Finding the set of pairs (a, x) where $x \in Q(a)$ from which start an evolution $x(\cdot) \in \mathcal{S}(x)$ and $\vec{a} := \{a_n\}_{n \geq 0} \in \mathcal{S}_\Phi(a)$ visiting the cells $Q(a_n)$ in the prescribed order $\{a_n\}$ until it must reach a target $C(a_N)$ at a finite step a_N.

  This subset will be called the qualitative capture basin of the family $\mathbf{C}$ of targets qualitatively viable in the family $\mathbf{Q}$ under the qualitative evolutionary system $(\mathcal{S}, \Phi)$ and is denoted $\mathrm{QualCapt}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C})$.

  The target $C(a_N)$ can be viewed as a prediction based on a sequence of qualitative observations, the measure of the capture basin gives a confident measure of the prediction.

- A sequence of qualitative observations could be not continuous, for instance then we have missing qualitative data. It could happen that the underlying system has a chaotic behavior, for instance if for any sequence $a_0, a_1, \ldots$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ to the evolutionary system viable in $Q(a_{j-1})$ on $[t_{j-1}, t_j]$ and and a sequence of elements $t_j \geq 0$ such that $x(t_j) \in Q(a_j)$ for all $j \geq 0$. In this case we cannot obtain pertinent predictions.

In the first section we shall give all definitions relative to qualitative cells, in the second one we shall introduce qualitative viability domain and capture basins, characterize them in terms of minimax problems in the third section and in terms of capture basins in the forth section. The "Qualitative Viability Kernel Algorithm" is proposed to compute those kernels. We proceed by providing a geometric characterization of qualitative viability in terms of contingent cones in the sixth section. We use it in the seventh section to revisit the basic example of "monotonic cells", that are the sets $K(a)$ defined by

$$K(a) := \{x \in K \mid f(x) \in a\mathbf{R}^n_+\}$$

where $a \in \mathcal{A} := \{-,+\}^n$ K is a closed viability domain under $\mathcal{S}$ which is generated by an ordinary differential equation $x'(t) = f(x(t))$. The eight section relates the above study with the chaos à la Saari, a situation in which whatever a prescribing order of visits is given, there exists at least one evolution visiting it.

1 Definitions

Definition 1.1 *Let us consider a finite family* $\mathbf{Q} := \{Q(a)\}_{a \in \mathcal{A}}$ *of* "qualitative cells" $Q(a) \subset X$ *of a subset* X*. A set-valued map* $\Phi : \mathcal{A} \rightsquigarrow \mathcal{A}$ *is* consistent *with a family* $\mathbf{Q} := \{Q(a)\}_{a \in \mathcal{A}}$

of qualitative cells if

$$\forall\, a \in \mathcal{A},\ \forall\, b \in \Phi(a),\ \ Q(a) \cap Q(b)\ \neq\ \emptyset$$

We shall say that a family $\mathbf{C}$ *of qualitative cells* $C(a)$ *is contained in the family* $\mathbf{Q}$ *if for every* $a \in \mathcal{A}$*,* $C(a) \subset Q(a)$*. A family* $\mathbf{Q}$ *is closed if all the qualitative cells* $Q(a)$ *are closed.*

We shall say that $K \subset X$ *is* covered *by a finite family* $\mathbf{Q} := \{Q(a)\}_{a\in\mathcal{A}}$ *of qualitative cells* $Q(a) \subset K$ *if*

$$K\ =\ \bigcup_{a\in\mathcal{A}} Q(a)$$

We can always associate with a family $\mathbf{Q}$ of qualitative cells the largest consistent map $\widehat{\Phi}_{\mathbf{Q}}$ defined by

$$\forall\, a \in \mathcal{A},\ \ \widehat{\Phi}_{\mathbf{Q}}(a)\ :=\ \{b \mid Q(a) \cap Q(b) \neq \emptyset\}$$

Hence, Φ is consistent with $\mathbf{Q}$ if and only if $\Phi \subset \widehat{\Phi}_{\mathbf{Q}}$.

The set-valued map $\Phi : \mathcal{A} \rightsquigarrow \mathcal{A}$ defines the discrete dynamic process $\mathcal{S}_\Phi : \mathcal{A} \rightsquigarrow \mathcal{A}^{\mathbf{N}}$ associating with any $a \in \mathcal{A}$ the family of discrete evolutions $\vec{a} := \{a_n\}_{n\geq 0} \in \mathcal{S}_\Phi(a)$ starting at a and satisfying

$$\forall\, n \geq 0,\ \ a_{n+1}\ \in\ \Phi(a_n)$$

on the set of indexes (qualitative states).

It induces the corresponding order of visit of cells $Q(a_n)$ of continuous evolutions $t \mapsto x(t)$ in the following sense:

Definition 1.2 *We shall say that a continuous evolution* $t \mapsto x(t)$ visits *the cells* $Q(a) \in \mathbf{Q}$ *in the order prescribed by a discrete system* Φ *consistent with* $\mathbf{Q}$ *if there exist a qualitative state* $a \in \mathcal{A}$*, a sequence* $\vec{a} \in \mathcal{S}_\Phi(a)$ *and a nondecreasing sequence* $\mathcal{T}(x(\cdot)) := \{t_n\}_{n\geq 0}$ *of* impulse times $t_n \geq 0$ *such that*

$$\forall\, t \in [t_n, t_{n+1}],\ \ x(t) \in Q(a_n)\ \&\ x(t_{n+1})\ \in\ Q(a_{n+1})$$

for all $n \geq 0$ *or until an impulse time* t_N *when*

$$\forall\, t \in [t_N, +\infty[,\ \ x(t)\ \in\ Q(a_N)$$

Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $P : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\begin{cases} i) & x'(t)\ =\ f(x(t), u(t)) \\ ii) & u(t)\ \in\ P(x(t)) \end{cases}$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$ where the right hand side is defined by $F(x) := f(x, P(x)) := \{f(x,u)\}_{u \in P(x)}$.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.

We denote by

$$\text{Graph}(F) := \{(x,y) \in X \in Y \mid y \in F(x)\}$$

the graph of a set-valued map $F : X \rightsquigarrow Y$ and $\text{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its domain.

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0, \infty; X)$ the set of absolutely continuous functions $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t))$$

starting at time 0 at x: $x(0) = x$. The set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is called the solution map associated with F.

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud:

Definition 1.3 (Marchaud Map) *We shall say that F is a* Marchaud map *if*

$$\begin{cases} i) & \textit{the graph of } F \textit{ is closed} \\ ii) & \textit{the values } F(x) \textit{ of } F \textit{ are convex} \\ iii) & \textit{the growth of } F \textit{ is linear: } \exists\, c > 0 \mid \forall\, x \in X, \\ & \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

This covers the case of Marchaud control systems where $(x,u) \mapsto f(x,u)$ is continuous, affine with respect to the controls u and with linear growth and when P is Marchaud.

We recall the following version of the important Theorem 3.5.2 of **Viability Theory**, [1, Aubin]:

Theorem 1.4 *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}$ is upper semicompact with nonempty values: This means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.*

Actually, the basic results that we shall use only few properties of the solution map $\mathcal{S}$: Its upper semicompactness mentioned above, and the translation and concatenation properties that we now define:

Definition 1.5 *An evolutionary system is a set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ satisfying*

1. *the* translation property*: Let $x(\cdot) \in \mathcal{S}(x)$. Then for all $T \geq 0$, the function $y(\cdot)$ defined by $y(t) := x(t+T)$ is a solution $y(\cdot) \in \mathcal{S}(x(T))$ starting at $x(T)$,*

2. *the* concatenation property*: Let $x(\cdot) \in \mathcal{S}(x)$ and $T \geq 0$. Then for every $y(\cdot) \in \mathcal{S}(x(T))$, the function $z(\cdot)$ defined by*

$$z(t) \ := \left\{ \begin{array}{ll} x(t) & \text{if} \quad t \in [0,T] \\ y(t-T) & \text{if} \quad t \geq T \end{array} \right.$$

belongs to $\mathcal{S}(x)$.

The solution maps of differential inclusions, of differential inclusions with memory ([24, 25, 26, Haddad]), of partial differential inclusions (see [28, 30, 31, Shi Shuzhong]), of mutational equation $\overset{\circ}{x} \ni f(x)$ on metric spaces (see [4, Aubin]), of impulse differential equations (see [13, 15, 14, Aubin & Haddad]), etc. share these two properties and are examples of evolutionary systems.

We shall present our study for any evolutionary system $\mathcal{S} : X \rightsquigarrow \mathbf{C}(0,\infty;X)$ governing the evolution $x(\cdot) : t \mapsto x(t)$ of the state.

2 Qualitative Viability Kernels and Capture Basins

2.1 Definitions

Definition 2.1 *Let us consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathbf{C}(0,\infty;X)$, a set-valued map $\Phi : \mathcal{A} \rightsquigarrow \mathcal{A}$ and two families $\mathbf{Q}$ and $\mathbf{C} \subset \mathbf{Q}$.*

We shall denote by $\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ the set of pairs (a,x) where $x \in Q(a)$ from which start a sequence $\vec{a} := \{a_n\}_{n\geq 0} \in \mathcal{S}_\Phi(a)$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ visiting the cells $Q(a_n)$ in the prescribed order until it possibly *reach a target $C(a_N)$ at a finite step a_N. We shall say that it is the* qualitative viability kernel *of the family $\mathbf{Q}$ with family $\mathbf{C}$ of targets under the qualitative evolutionary system $(\mathcal{S},\Phi)$.*

If the subsets $C(a)$ are empty, we set

$$\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}) \ := \ \mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\emptyset)$$

and we say that $\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q})$ is the qualitative viability kernel *of the family $\mathbf{Q}$.*

We shall denote by $\mathrm{QualCapt}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ the set of pairs (a,x) where $x \in Q(a)$ from which start an evolution $x(\cdot) \in \mathcal{S}(x)$ and $\vec{a} := \{a_n\}_{n\geq 0} \in \mathcal{S}_\Phi(a)$ visiting the cells $Q(a_n)$ in the prescribed order $\{a_n\}$ until it reaches a target $C(a_N)$ at a finite step a_N. We shall say that it is the qualitative capture basin *of the family $\mathbf{C}$ qualitative viable in the family $\mathbf{Q}$ under the qualitative evolutionary system $(\mathcal{S},\Phi)$.*

We shall say that

1. $\mathbf{Q}$ *is* qualitatively viable outside $\mathbf{C}$ under the pair $(\mathcal{S},\Phi)$ *if*

$$\mathbf{Q} \ \subset \ \mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$$

2. $\mathbf{Q}$ qualitatively captures $\mathbf{C}$ *under the pair* $(\mathcal{S}, \Phi)$ *if*

$$\mathbf{Q} \subset \text{QualCapt}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C})$$

3. $\mathbf{C} \subset \mathbf{Q}$ *is* qualitatively isolated *in* $\mathbf{Q}$ *if*

$$\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C}) \subset \mathbf{C}$$

We observe at once:

Lemma 2.2 *The map* $(\mathbf{C}, \mathbf{P}) \mapsto \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C})$ *satisfies*

$$\mathbf{C} \subset \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C}) \subset \mathbf{Q}$$

and is increasing in the sense that

$$\text{If } \mathbf{C}_1 \subset \mathbf{C}_2 \ \& \ \mathbf{Q}_1 \subset \mathbf{Q}_2, \text{ then } \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}_1, \mathbf{C}_1) \subset \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}_2, \mathbf{C}_2)$$

Furthermore, the map $\mathbf{C} \mapsto \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C})$ *satisfies*

$$\text{QualViab}_{(\mathcal{S},\Phi)}\left(\mathbf{Q}, \bigcup_{i \in I} \mathbf{C}_i\right) = \bigcup_{i \in I} \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C}_i)$$

Example: Fluctuations For instance, in the case when $K := Q(a_1) \cup Q(a_2)$ and $Q(a_1) \cap Q(a_2) \neq \emptyset$, and when $\Phi(a_1) = a_2$ and $\Phi(a_2) = a_1$ and when $C(a_1) := C(a_2) := \emptyset$, the qualitative viability of $\mathbf{Q}$ under $(\mathcal{S}, \Phi)$ describes a property of fluctuation when starting from any $x \in K$, there exists at least one evolution $x(\cdot)$ visiting the two qualitative cells $Q(a_1)$ and $Q(a_2)$ alternatively. □

Example: Qualitative Oscillators and equilibria Assume that a family $\mathbf{Q}$ of qualitative cells is qualitatively viable under the qualitative evolutionary system $(\mathcal{S}, \Phi)$. Any periodic solution $\vec{a} := \{a_0, \ldots, a_{N-1}\}$ of period N of the discrete system $x_{n+1} \in \Phi(x_n)$ gives rise to a qualitative oscillator: From any initial state $x_0 \in Q(a_0)$ starts at least one evolution that visits the cells $Q(a_0), \ldots, Q(a_{N-1}), Q(a_0), \ldots$ periodically.

In particular, a cell associated with a fixed point $\overline{a} \in \Phi(\overline{a})$ of the discrete map Φ is viable under the evolutionary system $\mathcal{S}$, and thus, can be regarded as a qualitative equilibrium.

3 Minimax Characterization

We denote by $\mathcal{D}(\mathbf{Q},\mathbf{C})$ the set of families $\mathbf{P}$ of qualitative cells $P(a)$ contained in $\mathbf{Q}$ and containing $\mathbf{C}$.

Theorem 3.1 *The qualitative viability kernel* $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ *of a family* $\mathbf{Q}$ *of qualitative cells with family* $\mathbf{C}\subset\mathbf{Q}$ *of targets is*

1. *the* largest *family* $\mathbf{P}\in\mathcal{D}(\mathbf{Q},\mathbf{C})$ *viable outside the family* $\mathbf{C}$ *under the qualitative evolutionary system* $(\mathcal{S},\Phi)$*,*
2. *the* smallest *subset* $\mathbf{P}\in\mathcal{D}(\mathbf{Q},\mathbf{C})$ *isolated in* $\mathbf{Q}$ *under the qualitative evolutionary system* $(\mathcal{S},\Phi)$*,*
3. *the* unique *minimax* $\mathbf{P}\in\mathcal{D}(\mathbf{Q},\mathbf{C})$ *in the sense that*

$$\mathbf{P} \;=\; \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{P}) \;=\; \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{P},\mathbf{C})$$

The same properties hold true for the qualitative viable-capture map $\text{QualCapt}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ *of a target* $\mathbf{C}$ *viable in* $\mathbf{Q}$*.*

Proof — We prove that $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ is qualitatively viable in $\mathbf{Q}$ outside $\mathbf{C}$ and qualitatively isolated in $\mathbf{Q}$ under the qualitative evolutionary system $(\mathcal{S},\Phi)$.

1. The qualitative viability kernel $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ is viable outside $\mathbf{C}$ under the qualitative evolutionary system $(\mathcal{S},\Phi)$:

   $$\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C}) \;\subset\; \text{QualViab}_{(\mathcal{S},\Phi)}(\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C}))$$

   Take $a\in\mathcal{A}$ and $x_0\in\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})(a)$ and prove that there exist a sequence $\vec{a}\in\mathcal{S}_\Phi(a)$ and $x(\cdot)\in\mathcal{S}(x_0)$ starting at x_0 qualitatively viable in $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ until it possibly reaches $\mathbf{C}$.

   Indeed, there exist a sequence $\vec{a}\in\mathcal{S}_\Phi(a)$ and an evolution $x(\cdot)\in\mathcal{S}(x_0)$ qualitatively viable in $\mathbf{Q}$ until some time $T\geq 0$ either infinite or finite when it reaches a target $C(a_p)$ at some time T where p is the index such that $T\in[t_p,t_{p+1}]$. Let $\mathcal{T}(x(\cdot)):=\{t_n\}$ the nonincreasing sequence of impulse times when the solution enters the cell $C(a_n)$.

   Then for all $t\in[0,T[$, setting $b_n:=a_{n+p}$, the evolution $y(\cdot)$ defined by $y(\tau):=x(t+\tau)$ is an evolution $y(\cdot)\in\mathcal{S}(x(t))$ starting at $x(t)$ and visiting the cells $Q(b_n)$ until it possibly reaches $C(b_{n-p})$ at time $T-t$. Hence $x(t)$ does belong to $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ for every $t\in[0,T[$.

2. The qualitative viability kernel $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$ is isolated in $\mathbf{Q}$:

$$\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})) \subset \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$$

Let $a \in \mathcal{A}$ and x belongs to $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C}))(a)$. There exist at least one solution $\vec{a} \in \mathcal{S}_\Phi(a)$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ that would either be qualitatively viable in $\mathbf{Q}$ forever or reach the cell $\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})(a_N)$ in finite time. In this case, it can be concatenated with an evolution either qualitatively viable in

$$\text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C}) \subset \mathbf{Q}$$

or reaching the family $\mathbf{C}$ in finite time. This implies that $x \in \text{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q},\mathbf{C})$.

The same proof shows that qualitative viable-capture basins always enjoy these properties. □

4 A Representation Theorem

If $Q(a) \cap Q(b) \neq \emptyset$, we shall set

$$\begin{cases} \text{Viab}_{\mathcal{S}}(Q(a),Q_b) := \text{Viab}_{\mathcal{S}}(Q(a),Q_b \cap Q(a)) \\ \text{Capt}_{\mathcal{S}}(Q(a),Q_b) := \text{Capt}_{\mathcal{S}}(Q(a),Q_b \cap Q(a)) \end{cases}$$

Theorem 4.1 *A family $\mathbf{Q}$ is qualitatively viable outside a qualitative family $\mathbf{C} \subset \mathbf{Q}$ under $(\mathcal{S},F)$ if and only if*

$$\forall\, a \in \mathcal{A},\ \ Q(a) \subset \mathcal{B}(\mathbf{Q},\mathbf{C})(a) := C(a) \cup \bigcup_{b \in \Phi(a)} \text{Capt}_{\mathcal{S}}(Q(a),Q(b))$$

Consequently, the qualitative viability kernel of the family $\mathbf{Q}$ with target $\mathbf{C}$ under the qualitative evolutionary system $(\mathcal{S},\Phi)$ is the largest family $\mathbf{P}$ between the families $\mathbf{C}$ and $\mathbf{Q}$ satisfying the above property.

Proof — Indeed, if $\mathbf{Q}$ is qualitatively viable, then, starting from $x \in Q(a)$, either $x \in C(a)$ and the target is reached, or else, there exists a solution $x(\cdot) \in \mathcal{S}(x)$ viable in $Q(a)$ until it reaches a cell $Q(b)$ in finite time for some $b \in \Phi(a)$. This means that x belongs to $C(a) \cup \bigcup_{b\in\Phi(b)} \text{Capt}_{\mathcal{S}}(Q(a),Q(b))$.

Conversely, if $\mathbf{Q}$ satisfies the above property, we deduce that for any $a \in \mathcal{A}$ and for any $x \in Q(a)$, either x belongs to $C(a)$ or else, that there exists $b \in \Phi(a)$ such that x belongs to the capture basin $\text{Capt}_{\mathcal{S}}(Q(a),Q(b))$ of the target $Q(a) \cap Q(b)$ viable in $Q(a)$. In this case, there exist an evolution $x(\cdot) \in \mathcal{S}(x)$ and a finite time $t_1 \geq 0$ such that $x(t) \in Q(a)$ for $t \in [0,t_1]$ and $x(t_1)$ belongs to $Q(b)$. Iterating this procedure, we find an evolution $x(\cdot) \in \mathcal{S}(x)$ visiting the cells $Q(a_n)$ where $a_{n+1} \in \Phi(a_n)$ for $n \geq 0$ until it possibly reaches a target $C(a_N)$ for the first step a_N. □

4.1 Prerequisite from Viability Theory

We shall need the following definitions and results from Viability Theory:

Definition 4.2 *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set.*

1. *The subset* $\mathrm{Viab}(K,C)$ *of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

   A subset $C \subset K$ is said to be isolated in K by $\mathcal{S}$ *if it coincides with its viability kernel:*

$$\mathrm{Viab}(K,C) \;=\; C$$

2. *The subset* $\mathrm{Capt}^K(C)$ *of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K and*

$$\mathrm{Capt}(C) \;:=\; \mathrm{Capt}^X(C)$$

   is said to be the capture basin *of C.*

3. *When the target $C = \emptyset$ is the empty set, we set*

$$\mathrm{Viab}(K) \;:=\; \mathrm{Viab}(K,\emptyset) \;\&\; \mathrm{Capt}^K(\emptyset) \;=\; \emptyset$$

   and we say that $\mathrm{Viab}(K)$ *is the viability kernel of K.*

   A subset K is a repeller *under $\mathcal{S}$ if its viability kernel is empty, or, equivalently, if the empty set is isolated in K.*

In other words, the viability kernel $\mathrm{Viab}(K)$ is the subset of initial states $x_0 \in K$ such that at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$. Furthermore, we observe that

$$\mathrm{Viab}(K,C) \;=\; \mathrm{Viab}(K \backslash C) \cup \mathrm{Capt}^K(\emptyset) \tag{1}$$

Consequently, the viability kernel $\mathrm{Viab}(K,C)$ of K with target C coincides with the capture basin $\mathrm{Capt}^K(C)$ of C viable in K whenever the viability kernel $\mathrm{Viab}(K \backslash C)$ is empty, i.e., whenever $K \backslash C$ is a repeller:

$$\mathrm{Viab}(K \backslash C) \;=\; \emptyset \;\Rightarrow\; \mathrm{Viab}(K,C) \;=\; \mathrm{Capt}^K(C) \tag{2}$$

This happens in particular when K is a repeller, or when the viability kernel $\mathrm{Viab}(K)$ of K is contained in the target C.

It will also be useful to handle hitting and exit functions and Theorem 4.4 below:

Definition 4.3 *Let $C \subset K \subset X$ be two subsets. The functional $\tau_K : \mathcal{C}(0,\infty;X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its* exit time *$\tau_K(x(\cdot))$ defined by*

$$\tau_K(x(\cdot)) \ := \ \inf\{t \in [0,\infty[\mid x(t) \notin K\}$$

is called the exit functional.

The (constrained) hitting (or minimal time) functional *$\varpi_{(K,C)}$ defined by*

$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \ \& \ \forall s \in [0,t], \ x(s) \in K \ \}$$

has been introduced in [17, Cardaliaguet, Quincampoix & Saint-Pierre]). We set

$$\varpi_C(x(\cdot)) := \varpi_{(X,C)}(x(\cdot))$$

If $\mathcal{S}$ is the solution map associated with the map F, the function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\sharp(x) \ := \ \sup_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot))$$

the upper exit function *and the function $\varpi^\flat_{(K,C)} : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\varpi^\flat_{(K,C)}(x) \ := \ \inf_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$

is called the lower (constrained) hitting function

We shall need the following:

Theorem 4.4 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and C and K be two closed subsets such that $C \subset K$. Then the hitting function $\varpi^\flat_{(K,C)}$ is lower semicontinuous and the exit function $\tau_K^\sharp$ is upper semicontinuous. Furthermore, for any $x \in \mathrm{Dom}(\varpi^\flat_{(K,C)})$, there exists one solution $x^\flat(\cdot) \in \mathcal{S}(x)$ which hits C as soon as possible before possibly leaving K*

$$\varpi^\flat_{(K,C)}(x) \ = \ \varpi_{(K,C)}(x^\flat(\cdot))$$

and for any $x \in \mathrm{Dom}(\tau_K^\sharp)$, there exists one solution $x^\sharp(\cdot) \in \mathcal{S}(x)$ which remains viable in K as long as possible:

$$\tau_K^\sharp(x) \ = \ \tau_K(x^\sharp(\cdot))$$

(see Proposition 4.2.4 of [1, 5, Aubin], for instance)

5 The Qualitative Viability Kernel Algorithm

Since the qualitative viability kernel map is the largest fixed point of the map $\mathbf{Q} \mapsto \mathcal{B}(\mathbf{Q}, \mathbf{C})$, we can use the Qualitative Viability Kernel Algorithm defined in the following way :

Starting with $\mathbf{Q}_0 := \mathbf{Q}$, we define recursively the families $\mathbf{Q}_n$ by

$$\forall\, a \in \mathcal{A},\;\; \forall\, n \geq 0,\;\; Q_{n+1}(a) \;:=\; C(a) \cup \bigcup_{b \in \Phi(a)} \mathrm{Capt}_{\mathcal{S}}(Q_n(a), Q_n(b))$$

Theorem 5.1 *Let us assume that the families* $\mathbf{Q}$ *and* $\mathbf{C} \subset \mathbf{Q}$ *are closed and that the evolutionary system* $\mathcal{S}$ *is upper semi-compact. Then the family* $\mathbf{Q}_n$ *are closed and*

$$\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C}) \;=\; \bigcap_{n \geq 0} \mathbf{Q}_n$$

Proof — We already know that

$$\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C}) \;\subset\; \bigcap_{n \geq 0} \mathbf{Q}_n$$

Adequate topological assumptions imply the equality: Indeed, we need first to prove recursively that the subsets $Q_n(a)$ are closed. For that purpose, it is enough to prove that $Q_{n+1}(a) := C(a) \cup \bigcup_{b \in \mathcal{A}} \mathrm{Capt}_{\mathcal{S}}(Q_n(a), Q_n(b))$ is closed whenever the subsets $Q_n(a)$ and $Q_n(b)$ are closed. But this results from the fact that the evolutionary system $\mathcal{S}$ is upper semicompact.

Let x belongs to $\mathbf{Q}_\infty := \bigcap_{n \geq 0} \mathbf{Q}_n$. If $a \in \mathcal{A}$ and $x \in Q_\infty(a) \backslash C(a)$, then for every $n \geq 0$, there exists $b_n \in \Phi(a)$ such that $x \in \mathrm{Capt}_{\mathcal{S}}(Q_n(a), Q_n(b_n))$. Since $\mathcal{A}$ is a finite set, we deduce that there exists a subsequence (again denoted by) b_n, $b \in \mathcal{A}$ and N such that $b_n = b$ for $n \geq N$. Hence, for such $n \geq N$, x belongs to $\mathrm{Capt}_{\mathcal{S}}(Q_n(a), Q_n(b))$. Since the sequence of closed subsets $Q_n(a)$ and $Q_n(b)$ is not increasing, their Painlevé-Kuratowski upper limits are respectively equal to $Q_\infty(a)$ and $Q_\infty(b)$. On the other hand, we know that whenever the evolutionary system $\mathcal{S}$ is upper semicompact,

$$\mathrm{Limsup}_{n \to +\infty} \mathrm{Capt}_{\mathcal{S}}(Q_n(a), Q_n(b)) \;\subset\; \mathrm{Capt}_{\mathcal{S}}(Q_\infty(a), Q_\infty(b))$$

Therefore,

$$\forall\, a \in \mathcal{A},\;\; Q_\infty(a) \;\subset\; C(a) \cup \bigcup_{b \in \Phi(a)} \mathrm{Capt}_{\mathcal{S}}(Q_\infty(a), Q_\infty(b))$$

so that the family Q_∞ is qualitative viable, and thus, contained in the qualitative viability kernel $\mathrm{QualViab}_{(\mathcal{S},\Phi)}(\mathbf{Q}, \mathbf{C})$. □

6 Characterization of Qualitative Viability

We now provide another characterization of the qualitative viability kernel of a family of qualitative cells:

Theorem 6.1 *Let us assume that the evolutionary system $\mathcal{S}$ is upper semicompact, that the family $\mathbf{Q}$ is closed (i.e., the subsets $Q(a)$ are closed), that for every $b \in \Phi(a)$, $Q(a) \backslash C(b)$ is a repeller and that Φ is consistent with $\mathbf{Q}$.*

Then the family $\mathbf{Q}$ is qualitatively viable under $(\mathcal{S}, \Phi)$ outside $\mathbf{C}$ if and only if for every $a \in \mathcal{A}$, the subsets $Q(a) \backslash \left(C(a) \cup \bigcup_{b \in \Phi(a)} Q(a) \right)$ are locally viable.

Proof — We have to prove that from any $x \in Q(a)$, there exists $b \in \Phi(a)$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in $Q(a)$ until it reaches some $C(a)$ in finite time or until it reaches some $Q(b)$ where $b \in \Phi(a)$ in finite time. There exists an evolution $x^\sharp(\cdot) \in \mathcal{S}(x)$ that maximizes the exit time $t^\sharp := \tau^\sharp_{Q(a)}(x) := \tau_{Q(a)}(x^\sharp(\cdot))$ on $\mathcal{S}(x)$. We deduce that $x^\sharp(t^\sharp)$ belongs to $M(a) := C(a) \cup \bigcup_{b \in \Phi(a)} \Phi(b)$. Otherwise, $x^\sharp(t^\sharp)$ would belong to $Q(a) \backslash M(a)$. Since this set is locally viable by assumption, there exists at least one evolution $y(\cdot) \in \mathcal{S}(x^\sharp(t^\sharp))$ viable in $Q(a) \backslash M(a) \subset Q(a)$ on a nonempty interval $[0, T]$. Concatenating $y(\cdot)$ with $x(\cdot)$, we would obtain an evolution $\widetilde{x}(\cdot) \in \mathcal{S}(x)$ viable in $Q(a)$ on an interval $[0, t^\sharp + T]$, a contradiction. □

When the evolutionary system $\mathcal{S} := \mathcal{S}_F$ comes from a differential inclusion $x' \in F(x)$, we can characterize the local viability thanks to the Viability Theorem[1]. We recall that the contingent cone $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in K$ for every n (see for instance [12, Aubin & Frankowska]) or [27, Rockafellar & Wets] for more details).

Theorem 6.2 *Let us assume that the set-valued map $F : X \rightsquigarrow X$ is Marchaud, that the family $\mathbf{Q}$ is closed, that for every $b \in \Phi(a)$, $Q(a) \backslash C(b)$ is a repeller and that Φ is consistent with $\mathbf{Q}$.*

Then the family $\mathbf{Q}$ is qualitatively viable under (F, Φ) outside $\mathbf{C}$ if and only if for every $a \in \mathcal{A}$,

$$\forall\, a \in \mathcal{A},\ \forall\, x \in Q(a) \backslash \left(C(a) \cup \bigcup_{b \in \Phi(a)} Q(b) \right),\ \ F(x) \cap T_{Q(a)}(x) \neq \emptyset$$

[1]See for instance Theorems 3.2.4, 3.3.2 and 3.5.2 of [1, Aubin].

7 Applications to Monotonic Cells

We posit the assumptions of the Viability Theorem for differential equations (called the Nagumo Theorem):

$$\begin{cases} i) & f \text{ is continuous with linear growth} \\ ii) & K \text{ is a closed viability domain} \end{cases} \tag{3}$$

Therefore, from every initial state $x_0 \in K$ stars a solution to the differential equation

$$x'(t) \;=\; f(x(t)) \tag{4}$$

viable (remaining) in K.

7.1 Monotonic Behavior of the Components of the State

For studying the qualitative behavior of the differential equation (4), i.e., the evolution of the functions $t \mapsto \text{sign}(x'(t))$ associated with solutions $x(\cdot)$ of the differential equation, we split the viability domain K of the differential equation into 2^n "monotonic cells" $K(a)$ defined by

$$K(a) \;:=\; \{x \in K \mid f(x) \in a\mathbf{R}^n\}$$

where $a \in \mathcal{A} := \{-,+\}^n$

Indeed, the quantitative states $x(\cdot)$ evolving in a given monotonic cell $K(a)$ share the same monotonicity properties because, as long as $x(t)$ remains in $K(a)$,

$$\forall\, i = 1, \ldots, n, \;\; \text{sign of } \frac{dx_i(t)}{dt} \;=\; a_i$$

These monotonic cells are examples of qualitative cells

7.2 Monotonic Behavior of Observations of the State

But before proceeding further, we shall generalize our problem — free of any mathematical cost — to take care of physical considerations.

Instead of studying the monotonicity properties of each component $x_i(\cdot)$ of the state of the system under investigation, which can be too numerous, we shall only study the monotonicity properties of m functionals $V_j(x(\cdot))$ on the state (for instance, energy or entropy functionals in physics, observations in control theory, various economic indexes in economics) which do matter.

The previous case is the particular case when we take the n functionals V_i defined by $V_i(x) := x_i$.

We shall assume for simplicity that these functionals V_j are continuously differentiable around the viability domain K.

We denote by $\mathbf{V}$ the map from X to $Y := \mathbf{R}^m$ defined by

$$\mathbf{V}(x) \ := \ (V_1(x), \ldots, V_m(x))$$

Since the derivative of the observation $\mathbf{V}(x(\cdot))$ is equal to $\mathbf{V}'(x(\cdot))x'(\cdot) = \mathbf{V}'(x(\cdot))f(x(\cdot))$, it will be convenient to set

$$\forall\, x \in K, \ \ g(x) \ := \ \mathbf{V}'(x)f(x)$$

Hence, we associate with each qualitative state a the qualitative cells $K(a)$ and the large qualitative cells $\overline{K}_a$ defined by

$$K(a) \ := \ \{x \in K \mid g(x) \in a\mathbf{R}^m_+\}$$

In other words, the quantitative states $x(\cdot)$ evolving in a given monotonic cell $K(a)$ share the same monotonicity properties of their observations because, as long as $x(t)$ remains in $K(a)$,

$$\forall\, j = 1, \ldots, m, \ \ \text{sign of } \frac{d}{dt}V_j(x(t)) \ = \ a_j$$

In particular, the m functions $V_j(x(t))$ remain constant while they evolve in the qualitative cell K_0.

By using observation functionals chosen in such a way that many qualitative cells are empty, the study of transitions may be drastically simplified: this is a second reason to carry our study in this more general setting.

This is the case for instance when the observation functionals are "Lyapunov functions" $V_j : K \mapsto \mathbf{R}$. We recall that V is a Lyapunov function if $< V'(x), f(x) > \leq 0$ for all $x \in K$, so that $V(x(\cdot))$ decreases along the solutions to the differential equation.

Hence, if the observation functionals are Lyapunov functions, the qualitative cells $K(a)$ are empty whenever a component a_i is positive. In this case, we have at most 2^m nonempty qualitative cells. (In some sense, one can say that Lyapunov was the originator of qualitative simulation a century ago).

Naturally, we would like to know directly the laws which govern the transition from one qualitative cell $K(a)$ to other qualitative cells, without solving the differential equation, and therefore, without knowing the state of the system, but only some of its properties.

For that purpose, we shall set

$$h(x) \ := \ g'(x)f(x) \ = \ V''(x)(f(x), f(x)) + V'(x)f'(x)f(x)$$

Lemma 7.1 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K.*

If v belongs to the contingent cone to the $\overline{K}_a$ at x, then condition

$$v \in T_K(x) \ \& \ \forall\, i \in I_0(x), \ \text{sign of } \ (g'(x)v)_i = a_i \ \text{ or } \ 0$$

is satisfied.

The converse is true if we posit the transversality assumption[2]*:*

$$\forall\, x \in \overline{K}_a, \ \ g'(x)C_K(x) - a\mathbf{R}_+^{I_0(x)} \ = \ \mathbf{R}^m$$

Proof — Since the large qualitative cell $\overline{K}_a$ is the intersection of K with the inverse image by g of the convex cone $a\mathbf{R}_+^m$, we know that the contingent cone to $\overline{K}_a$ at some $x \in \overline{K}_a$ is contained in

$$T_K(x) \cap g'(x)^{-1} T_{a\mathbf{R}_+^m}(g(x))$$

and is equal to this intersection provided that the "transversality assumption"

$$g'(x)C_K(x) - C_{a\mathbf{R}_+^m}(g(x)) \ = \ \mathbf{R}^m$$

is satisfied. On the other hand, we know that $a\mathbf{R}_+^m$ being convex,

$$C_{a\mathbf{R}_+^m}(y) \ = \ T_{a\mathbf{R}_+^m}(y) \ = \ aT_{\mathbf{R}_+^m(ay)} \ \supset a\mathbf{R}_+^m$$

and that $v \in T_{\mathbf{R}_+^m}(z)$ if and only if

$$\text{whenever } z_j = 0, \text{ then } \ v_j \geq 0$$

Consequently, $v \in T_{a\mathbf{R}_+^m}(g(x))$ if and only if

$$\text{whenever } g(x)_j = 0, \text{ then sign of } \ v_j \ = \ a_j \ \text{ or } \ 0$$

i.e., $T_{a\mathbf{R}_+^m}(g(x)) = a\mathbf{R}_+^{I_0(x)}$.

Hence v belongs to the contingent cone to $\overline{K}_a$ at x if and only if v belongs to $T_K(x)$ and $g'(x)v$ belongs to $T_{a\mathbf{R}_+^m}(g(x))$, i.e., the sign of $(g'(x)v)_j$ is equal to a_j or 0 whenever j belongs to $I_0(x)$. □

We denote by $I(a,b)$ the set of indexes $i \in \{1,\ldots,m\}$ such that $b_i = -a_i$ and by $I_\Phi(a) := \bigcup_{b\in\Phi(a)} I(a,b)$. Hence, to say that x belongs to $K(a)\backslash \left(\bigcup_{b\in\Phi(a)} K(b)\right)$ means that the sign of $g_i(x)$ is equal to a_i for all $i \notin I_\Phi(a)$.

We introduce the notation

$$\overline{K}_a^i \ := \ \{ \ x \in \overline{K}_a \mid g(x)_i = 0 \ \}$$

[2] The cone $C_K(x)$ denotes the Clarke tangent cone to K at x. See for instance [12, Aubin & Frankowska]

Theorem 7.2 *Let us assume that f is continuously differentiable and that the m functions V_j are twice continuously differentiable around the viability domain K.*

We posit the transversality assumption:

$$\forall\, x \in \overline{K}_a,\ \ g'(x)C_K(x) - a\mathbf{R}_+^{I_0(x)} \ = \ \mathbf{R}^m$$

Let Φ be a set-valued map consistent with $\mathbf{Q}$.

Then $\mathbf{Q}$ is qualitative viable under $(\mathcal{S}_f, \Phi)$ if and only if for any $a \in \{-,+\}^m$, for any $i \notin I_\Phi(a)$, for any $x \in K(a)^i$, the sign of $h_i(x)$ is equal to a_i.

Proof — This is a consequence of Theorem 6.2 and Lemma 7.1. □

We shall denote by Γ the set-valued map from $\mathcal{R}^m$ to itself defined by

$$\forall\, a \in \mathcal{R}^m,\ (\Gamma(a))_i \text{ is the set of signs of } h_i(x) \text{ when } x \in \overline{K}_a^i$$

Hence the necessary and sufficient condition for the qualitative viability of monotonic cells can be written in the symbolic form:

$$\forall\, a \in \{-,+\}^n,\ \ \Gamma(a)|_{I_\Phi(a)} \ \subset\ a|_{I_\Phi(a)}$$

Hence, the knowledge of the map Ψ allows us to characterize the qualitative viability of a monotonic cells under a differential equation.

8 Chaos à la Saari

We associate with the sequence $a_0, a_1, \ldots$ the subsets $\mathbf{C}_{a_0 a_1 \cdots a_n}$ defined by induction by $\mathbf{C}_{a_n} := Q(a_n)$,

$$\mathbf{C}_{a_{n-1}a_n} \ :=\ \text{Capt}(Q(a_{n-1}), Q(a_n))$$

which is the subset of $x \in Q(a_{n-1})$ such that there exists $x(\cdot) \in \mathcal{S}(x)$ viable in $Q(a_{n-1})$ until it reaches $Q(a_n)$ in finite time. For $j = n-2, \ldots, 0$, we define recursively the cells:

$$\mathbf{C}_{a_j a_{j+1} \cdots a_n} \ :=\ \text{Capt}(Q(a_j), \mathbf{C}_{a_{j+1} \cdots a_n})$$

Lemma 8.1 *Let us assume that $\mathcal{S}$ is upper semicompact and that the cells $Q(a)$ $(a \in \mathcal{A})$ of the family Q are closed repellers. Given a sequence of qualitative indexes $a_0, \ldots, a_n, \ldots$, such that $Q(a_n) \cap Q(a_{n+1}) \neq \emptyset$, the set $\mathbf{C}_{a_0 a_1 \cdots a_n \cdots}$ is the set of initial states $x_0 \in Q(a_0)$ from which at least one evolution visits the cells $Q(a_j)$ is the prescribed order $a_0, \ldots, a_n, \ldots,$.*

Proof — Let us consider an element x in the intersection $\mathbf{C}_{a_0 a_1 \cdots a_n \cdots} := \bigcap_{n=0}^{\infty} \mathbf{C}_{a_0 a_1 \cdots a_n}$ is nonempty.

Let $T := \sup_{a \in \mathcal{A}} \sup_{x \in Q(a)} \tau^{\sharp}_{Q(a)}(x)$, which is finite since the cells $Q(a)$ are repellers.

Let us take an initial state x in Q_∞ and fix n. Hence there exists $x_n(\cdot) \in \mathcal{S}(x)$ and a sequence of $t_n^j \in [0, jT]$ such that

$$\forall\, j = 1, \ldots, n, \;\; x_n(t_n^j) \in \mathbf{C}_{a_j \cdots a_n} \subset Q(a_j)$$

Indeed, there exist $y_1(\cdot) \in \mathcal{S}(x)$ and $\varpi^{\flat}_{\mathbf{C}_{a_1 \cdots a_n}}(y_1(\cdot)) \in [0, T]$ such that $y_1(\varpi^{\flat}_{\mathbf{C}_{a_1 \cdots a_n}}(y_1(\cdot)))$ belongs to $\mathbf{C}_{a_1 \cdots a_n}$. We set $t_n^1 := \varpi^{\flat}_{\mathbf{C}_{a_1 \cdots a_n}}(y_1(\cdot))$, $x_n^1 = y_1(t_n^1)$ and $x_n(t) := y_1(t)$ on $[0, t_n^1]$.

Assume that we have built $x_n(\cdot)$ on the interval $[0, t_n^k]$ such that $x_n(t_n^j) \in \mathbf{C}_{a_j \cdots a_n} \subset Q(a_j)$ for $j = 1, \ldots, k$. Since $x_n(t_n^k)$ belongs to $\mathbf{C}_{a_k \cdots a_n}$, there exist $y_{k+1}(\cdot) \in \mathcal{S}(x_n(t_n^k))$ and $\varpi^{\flat}_{\mathbf{C}_{a_{k+1} \cdots a_n}}(y_{k+1}(\cdot)) \in [0, T]$ such that

$$y_{k+1}(\varpi^{\flat}_{\mathbf{C}_{a_{k+1} \cdots a_n}}(y_{k+1}(\cdot))) \;\in\; \mathbf{C}_{a_{k+1} \cdots a_n}$$

We set

$$t_n^{k+1} := t_n^k + \varpi^{\flat}_{\mathbf{C}_{a_{k+1} \cdots a_n}}(y_{k+1}(\cdot)) \;\;\&\;\; x_n(t) := y_{k+1}(t + \varpi^{\flat}_{\mathbf{C}_{a_{k+1} \cdots a_n}}(y_{k+1}(\cdot)))$$

on $[t_n^k, t_n^{k+1}]$. When $k = n$, we extend $x_n(\cdot)$ to $[t_n^n, \infty[$ by any solution to the evolutionary system starting at $x_n(t_n^n)$ at time t_n^n.

Since the sequence $x_n(\cdot) \in \mathcal{S}(x)$ is compact in the space $\mathbf{C}(0, \infty; X)$, a subsequence (again denoted $x_n(\cdot)$) converges to some solution $x(\cdot) \in \mathcal{S}(x)$ to the evolutionary system. By extracting successive converging subsequences of $t_{n_1}^1, \ldots, t_{n_j}^j, \ldots$, we infer the existence of t_j's in $[0, jT]$ such that $x_{n_j}(t_{n_j}^j)$ converges to $x(t_j) \in Q(a_j)$, because the functions $x_n(\cdot)$ remain in an equicontinuous subset. □

A situation in which all the subsets $\mathbf{C}_{a_0 a_1 \cdots a_n}$ are nonempty should generate a kind of chaos, which was introduced by Donald Saari:

Definition 8.2 *We shall say that the covering of K by a family of closed subsets $Q(a)$ $(a \in \mathcal{A})$ is* chaotic à la Saari *under $\mathcal{S}$ if for any sequence $a_0, a_1, \ldots$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ to the evolutionary system is viable $Q(a_{j-1})$ on $[t_{j-1}, t_j]$ and and a sequence of elements $t_j \geq 0$ such that $x(t_j) \in Q(a_j)$ for all $j \geq 0$.*

Theorem 8.3 (Chaotic Behavior) *Let us assume that a compact viability domain K of the upper semicompact map $\mathcal{S}$ is covered by a family of closed repellers $Q(a)$ $(a \in \mathcal{A})$ satisfying the following* controllability assumption*:*

$$K \;\subset\; \bigcap_{a \in \mathcal{A}} \bigcup_{t \geq 0} \vartheta_{\mathcal{S}}(t, Q(a))$$

Then, for any sequence $a_0, a_1, \ldots, a_n, \ldots$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ to and a sequence of elements $t_j \geq 0$ such that $x(\cdot)$ is viable $Q(a_{j-1})$ on $[t_{j-1}, t_j]$ and $x(t_j) \in Q(a_j)$ for all $j \geq 0$.

Proof — The controllability assumption implies that subsets $\mathbf{C}_{a_0 a_1 \cdots a_n}$ are nonempty. They are closed since the evolutionary system is assume to be upper semicompact. Therefore the intersection $\mathbf{C}_{a_0 a_1 \cdots a_n \cdots} := \bigcap_{n=0}^{\infty} \mathbf{C}_{a_0 a_1 \cdots a_n}$ is nonempty.

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