Systems
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& Applications

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Set-Valued Analysis

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THIS BOOK IS DEDICATED TO C. OLECH
AND POLISH MATHEMATICIANS[1]
who contributed so much to set-valued analysis.

[1] "It was a common belief that cultivation of science and the growth of its potential would somehow guarantee the maintenance of the nation" wrote Kuratowski about the situation of Poland before 1918.

I will share all [my results] with you whenever you wish and do so without any ambition, from which I am more exempt and more distant than any man in the world.

Pierre de Fermat[a]

Fermat was one of the most important innovators in the history of mathematics. Newton himself recognized explicitly that he got the hint of the differential calculus from Fermat's method of building tangents devised half a century earlier.
This same method is used in our book to build a differential calculus for set-valued maps.
Fermat was also the one who discovered that the derivative of a (polynomial) function vanishes when it reaches an extremum. (This is Fermat's Rule, which remains the main strategy for obtaining necessary conditions of optimality, from mathematical programming to calculus of variations to optimal control). Fermat also was the first to discover the "principle of least time" in optics, the prototype of the variational principles governing so many physical and mechanical laws. He shared independently with Descartes the invention of analytic geometry and with Pascal the creation of the mathematical theory of probability. His achievements in number theory overshadowed his other contributions, and the Last Fermat Theorem remains a challenge. He was on top of that a poet, a linguist, ... and made his living as a lawyer!

[a]in his answer to a first letter from Mersenne inviting him to share his findings with the Parisian mathematicians, which put an end to Fermat's isolation in Toulouse, in 1636.

Epigraph

Who needs set-valued analysis?

Everyone, we are tempted to say, and we shall state our case.

This strong conviction — born out of accumulated experience in using it in control theory and differential games, mathematical economics and game theory, biomathematics, qualitative physics and viability theory — led us to devote time and effort to share some basic material which is used over and over.

One cannot afford anymore the luxury of studying only *well posed problems* in Hadamard's sense: *Ill posed problems, inverse problems* and many other unorthodox problems under other names are popping up in every domain of activity, whenever the existence of a solution may fail for some data, whenever uniqueness of the solution is at stake. Requiring that maps should be always single-valued and even, bijective, is too costly an attitude, above all in many applied fields, where we are not free to make such assumptions. This was indeed recognized during the three first decades of this century by the founders of "Functional Calculus": Painlevé, Hausdorff, Bouligand, Kuratowski to quote only a few. In his important book TOPOLOGIE, Kuratowski gave set-valued maps their proper status.

Set-valued map were abandoned by the authors of Bourbaki's volume TOPOLOGIE GÉNÉRALE, who chose to restrict their study to single-valued maps, regarding set-valued maps as single-valued maps from a set to the power set of another set, or factorizing single-valued maps to make them bijective.

This is not always the solution, for, by so doing, many important structural properties may be unfortunately lost, other ones are useless artifacts, making life more difficult rather than more simple. These points of view, which were widely disseminated all over the world after World War II, misled many of us into unnecessary detours (often towards culs de sac), encouraging the perception that direct routes were too arduous or worse, that they did not exist.

Hence, set-valued analysis inherited the undeserved image of being something difficult and mysterious, and consequently, was regarded as a mathematical curiosity, to be left in the hands of mathematicians who like to generalize for the sake of generalizing, without proper motivations.

In contrast, as it turned out, the need for set-valued analysis in solving problems arising in other fields of knowledge — control theory, economics and management, biology and systems sciences, artificial intelligence, etc. — was pressing enough to help mathematicians overcome the kind of recalcitrance felt towards set-valued analysis.

In views of such a wide variety of *motivating applications*, it is fortunate that most of the basic results of the chapters of "single-valued" analysis can be adapted to set-valued realm. These include:

- Limits and Continuity
- Linear Functional Analysis
- Nonlinear Functional Analysis (existence and approximation of solutions to equations and inclusions)
- Tangents and Normals
- Differentiation of Maps
- Gradients of Functions and the Fermat Rule
- Convergence of Maps
- Measures and Integration
- Differential Equations

The set-valued version of this list is nothing other than the outline of this book.

Our account of set-valued analysis is by no means exhaustive: it is just an introduction. The choice of the material has been dictated by our experience in applying these results in control and viability theory. However, we tried very hard to make this presentation as clear as possible, to help the reader become familiar with the main tools.

We did not restrict our exposition to the framework of finite dimensional vector-spaces, since set-valued analysis is also useful for solving problems involving partial differential equations or inclusions. But whenever the proofs of the finite-dimensional and infinite-dimensional statements are quite different, two proofs are provided.

This book presents only the tools, without mentioning applications. Some applications (to control theory and viability theory in particular) are presented in companion texts[2].

[2] Viability Theory [**?**] by Aubin, and the forthcoming books Set-Valued Analysis and Control Theory [**?**] by Frankowska and Set-Valued Analysis and Subdifferential Calculus [**?**] by Rockafellar and Wets.

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We are happy to publish this monograph in the new series *Systems and Control: Foundations and Applications* of Birkhäuser.

Introduction

It is a fact that in mathematical sciences there has been a reluctance to deal with sequences of sets and set-valued maps. Despite the emergence of exciting new vistas for the applications of mathematics, our long familiarity with sequences (of elements) and with (single-valued) maps has perhaps been too deeply rooted in traditional mathematical conceptualizations that it has appeared easier to sacrifice the breadth of some problems or some simple underlying structure in order to avoid set-valued maps.

[3] and in particular, G. da Prato.

[4] especially, its head, A. Kurzhanski, as well as the dedicated and competent librarians of IIASA.

[5] C. Byrnes, R. Rockafellar and R. Wets among many other colleagues

[6] "...[a mathematician] does need a proper atmosphere; this proper atmosphere can be created only by the cultivation of common topics." Janiszewski wrote in 1918 in *On the Needs of Mathematics in Poland*, a program which in our mind can still be used all over the world.

[7] Finally, as it is customary, each author is grateful to her/his coauthor for his/her wonderful typing of the manuscript.

For this reason, we begin by providing examples of natural and/or general problems involving set-valued maps before giving a rough description of the results presented in the pages that follow.

Examples of Set-Valued Maps

1. First, we encounter set-valued maps each time we face *ill-posed problems* or *inverse problems*, i.e., problems for which either the existence of a solution or its uniqueness is not guaranteed for some data: Set-valued maps allow us to get away from the restriction that a map is bijective when we want to solve an equation.

   Indeed, the first natural instance when set-valued maps occur is the inverse f^{-1} of a single-valued map f from X to Y. We always can define f^{-1} as a set-valued map which associates with any data y the (possibly empty) set of solutions

   $$f^{-1}(y) \ := \ \{x \in X \mid f(x) = y\}$$

   to the equation $f(x) = y$.

   Of the three commandments of Hadamard's tablets, *existence, uniqueness and stability*, we shall only retain the stability requirements, which can be encapsulated in adequate definitions of *continuity* of f^{-1}: This is one of the topics of the first chapter.

2. Taking into account uncertainties, disturbances, modeling errors, etc., leads naturally to set-valued maps and inclusions. They also arise when we wish to treat a problem *qualitatively*, by looking for solutions common to a set of data, sharing the same (qualitative) properties. Set-valued analysis should play an important role in the new field of *qualitative physics*, a rapidly growing branch of Artificial Intelligence.

3. Problems with constraints also yield specific set-valued maps: Solving the equation $f(x) = y$, where the solution x is required to belong to a subset K, amounts to solving the inclusion $f|_K(x) = y$ where $f|_K$, the restriction of f to K, is regarded as the set-valued map associating with x the point $f(x)$ when $x \in K$ and the empty set when x is not in K.

4. Unilateral problems in mechanics were formulated in the framework of variational inequalities (also called "generalized equations" by some authors), which are again inclusions in disguise. Their solution by Stampacchia and J.-L. Lions in the sixties gave a new impetus to set-valued maps, with a different vocabulary.

5. Set-valued maps provide a useful framework for control theory, since the early contributions of Ważewski and Filippov in the beginning of the sixties.

   Such set-valued maps, called *parametrized maps*, are associated with a family of maps $x \mapsto f(x,u)$ from X to Y when u ranges over a set $U(x)$ of parameters.

   The (single-valued) map f describes the dynamics of the system: It associates with the state x of the system and the control u the velocity $f(x,u)$ of the system. The set-valued map U describes a *feedback map* assigning to the state x the subset $U(x)$ of admissible controls.

   Hence the map F which associates with each state x the subset $F(x)$ of feasible velocities is defined by:

$$F(x) \;:=\; f(x,U(x)) \;=\; \{f(x,u)\}_{u\in U(x)}$$

   So, the control system governed by the family of parametrized differential equations

$$x'(t) \;=\; f(x(t),u(t)) \;\text{ where }\; u(t)\in U(x(t))$$

   is actually governed by the *differential inclusion*

$$x'(t) \;\in\; F(x(t))$$

6. Optimization provides examples of problems where uniqueness of the solution is naturally lacking:

   Let W be a function from $X \times Y$ to $\mathbf{R}$. We consider the family of minimization problems

$$\forall\, y\in Y,\;\; V(y) \;:=\; \inf_{x\in X} W(x,y)$$

   parametrized by parameters y.

   The function V is called the *marginal* (or *performance* or *value*) function. For every $y \in Y$, let

$$G(y) \;:=\; \{x\in X \mid W(x,y) = V(y)\}$$

   be the subset of solutions to our minimization problem.

   One of the main issues of optimization theory is to study the set-valued map G (nonvacuity, continuity and differentiability in a suitable sense, and so on). We shall call G the *marginal map*. It is no wonder that game theory and mathematical economics use set-valued maps in a natural way.

7. Another source of strong motivations came from optimization and mathematical programming, when necessary conditions (the Fermat rule[8], stating that the derivative of a function vanishes at points where it achieves an extremum) were needed to replace optimization problems by the resolution of equations.

   The Fermat rule is indeed one idea which was revisited and enhanced again and again under different names:

   — The Euler-Lagrange equations, when dealing with problems of the calculus of variations;

   — Lagrange and Kuhn-Tucker multipliers, when *state constraints* were added to optimization problems;

   — The Pontriagin principle when dealing with optimal control problems.

   After the advances of Functional Analysis, it was time to uncover the common fact behind all these results. It is still and always the *Fermat rule*, provided we are able to "differentiate" larger and larger classes of functions beyond differentiable functions.

   The crucial revolution in the history of the concept of gradients happened in the sixties when J.- J. Moreau and Rockafellar proposed in the framework of convex analysis the notion of *subdifferential* of a convex function, *which is no longer an element, but a set of "subgradients"*.

8. The use of set-valued maps in mathematical economics and game theory started when von Neumann asked for an extension of the Brouwer fixed point theorem to set-valued maps, which was needed for finding noncooperative equilibria for n-person games, for instance. This was achieved with the famous Kakutani Fixed-Point Theorem, in the forties. It has been used by Arrow and Debreu in the early fifties to provide the long-expected proof of the existence of a Walrasian equilibrium price.

   While this achievement made set-valued maps popular among mathematical economists, it was not until the challenges raised by optimization, control theory and unilateral problems in mechanics at the beginning of the sixties that renewed motivations arose to study set-valued maps, as an important subject in its ownrite.

[8] "Je désire seulement qu'il [Descartes] sache que nos questions *de Maximis et Minimis* et *de Tangentibus linearum curvarum* sont parfaites depuis huit ou dix ans et que plusieurs personnes qui les ont vues depuis cinq ou six ans le peuvent témoigner", Fermat wrote in 1638 when Descartes accused that the *Methodus de Maxima et Minima* was just due to luck and trial and error! ("à tâtons et par rencontre").

This was the time when Zarantonello introduced the *monotone maps*, which cover many important nonlinear single-valued or set-valued maps of the Calculus of Variations.

Properties of Set-Valued Maps

Having briefly indicated the importance of set-valued maps in a wide spectrum of applications and of fundamental mathematics, we paint a broad picture of their properties. We follow in doing so the outline of the book.

- LIMITS AND CONTINUITY

  Limits of sets have been introduced by Painlevé in the first years of this century, just after Fréchet axiomatized in 1906 the concept of $\mathcal{L}$-spaces (on which a notion of limit is defined[9]). Studying limits of sets together with limits of elements may have been very natural in this context.

  The topological ideas are, indeed, quite simple and straightforward. In the same way that topological concepts are based on the notions of limits and cluster points of sequences of elements, their set-valued analogues are rooted in the concepts of *lower and upper limits* of sequences of sets, which are, so to speak, *"thick" limits and cluster points respectively*: The lower limit of a sequence of subsets K_n is the set of limits of sequences of elements $x_n \in K_n$ and the upper limit is the set of cluster points of such sequences.

  We mentioned already that *stability* is the only requirement that we retain to study *ill posed or inverse problems*. *Stability* is a catch word which means that the set of solutions depends continuously upon the data.

  How can we proceed to define continuity of set-valued maps?

  If we try to adapt to the set-valued case the two equivalent definitions of continuity of single-valued maps, we obtain two notions which are no longer equivalent!

  This unfortunate situation led to two concepts of *semicontinuity* of set-valued maps, introduced at the beginning of the thirties by Bouligand and Kuratowski: Lower and upper semicontinuity. These issues were developed in the monographs

[9] In his famous thesis, judged at the time far too much abstract; it was published in the Rendiconti Cir. Mat. di Palermo!
Lebesgue wrote: "....set theory was placed outside the pale of mathematics by the high priests of analytic functions... Set theory, which developed from the theory of analytical functions, could prove useful to its elder sister and could show people of good will its qualities and richness".

of Hausdorff, Kuratowski and thoroughly investigated at this time by many (but not exclusively) Polish and French mathematicians.

For some reason, just a while later, set-valued maps yielded the way to single-valued maps: A set-valued map was viewed at the time as a single-valued map from a set to the power set of another set. However, as it turned out, the structures exported to power sets were too poor and specific information was indeed wasted by doing so.

For instance, when we regard a set-valued map as a single-valued map from one set to the power set of the other (supplied with any one of the topologies we can think of), we arrive at continuity concepts which are stronger than both lower and upper semicontinuity, introducing parasitic artifacts. For example, using such topologies to differentiate set-valued maps, lead to so strong requirements, than most set-valued maps would become nondifferentiable.

This is the reason why we shall start this book with the study of limits and leave aside the examination of topologies on power sets.

Furthermore, we shall renew history, by regarding a map not ....as a map, but as a graph (a subset of the product of the departure and the arrival sets), reestablishing some symmetry by putting these two sets on the same footing. This brings us back to the source of analytical geometry, at the time of F. Viète, P. de Fermat and R. Descartes, before the concept of function and map evolved from the one of curves and graphs.

To regard a map as a graph is our constant and basic point of view throughout this book (which has been called the *graphical approach*).

For instance, *closed maps*, that is maps with closed graph, shall play a starring role in this book. It is a weaker property than continuity or even, upper semicontinuity, very familiar and thus easy to check, common to both set-valued maps and their inverses.

- Linear Functional Analysis

What is the set-valued version of a continuous linear operator?

Remembering that the graph of a continuous linear operator is a closed vector subspace, we are tempted to single out the maps whose graphs are closed linear subspaces (called *linear processes*).

This generalization is not bold enough, since dealing exclusively with closed vector subspaces is still too restrictive: We need to use the notion of closed convex cone, which is a kind of vector subspace in which it is forbidden to use the subtraction. They enjoy many properties of the vector subspaces.

For this reason, we select the *closed convex processes*, i.e., the maps whose graphs are closed convex cones, as the candidates to play the part of set-valued linear maps.

We shall see later that derivatives of some set-valued maps are closed convex processes, which is a desirable property for a derivative[10]. Indeed, the two basic theorems on continuous linear operators due to Banach, the Closed Graph Theorem (equivalent to the Open Mapping Principle) and the Banach-Steinhauss Theorem, can be adapted to closed convex processes. The first one states that a closed convex process defined on the whole space is continuous and the second states that pointwise bounded families of closed convex processes are bounded — a prerequisite for studying the convergence of closed convex processes. But most important of all, one can *transpose* closed convex processes and use the benefits of duality theory, based on the Bipolar Theorem.

Closed convex processes also possess eigenvectors and invariant spaces and cones. They truly deserve the status of linear set-valued maps.

- NONLINEAR FUNCTIONAL ANALYSIS

We are convinced that many problems can be regarded as *inclusions*

$$\text{given } F : X \rightsquigarrow Y \text{ and } y \in Y, \text{ find } x \in X \text{ such that } \; F(x) \ni y$$

Most theorems on existence of solutions to nonlinear equations can be extended to the case of inclusions.

For example, this is the case for the Brouwer Fixed Point Theorem, whose generalization to set-valued maps is the famous Kakutani Fixed-Point Theorem. We shall prove an equivalent statement, called the *Equilibrium Theorem*, which provides the existence of an equilibrium of a set-valued map, a solution to the inclusion $F(x) \ni 0$.

Of course, for applications, we need not only to solve such a problem, but also to approximate its solutions by solutions to approximate problems:

$$\text{given } F_n : X_n \rightsquigarrow Y_n,\; y_n \in Y_n, \text{ find } x_n \in X_n \text{ with } \; F_n(x_n) \ni y_n$$

[10]keeping us in line with Hadamard's linearity bill enforced by Fréchet that a derivative should be linear with respect to the increment.

where X_n and Y_n are subspaces of X and Y.

The famous Lax's principle of numerical analysis states that *convergence* of the data y_n to y, *consistency* of F_n to F and *stability* of the F_n's imply the convergence of approximate solutions. As an important special case, when the spaces $X_n = X$, $Y_n = Y$ and the maps $F_n = F$ are constant, this principle boils down to the statement of the Inverse-Function Theorem.

Convergence, consistency and stability, which were originally defined in the framework of linear equations, can be extended to this general case by introducing adequate notions of *convergence* and *derivatives* of set-valued maps. For instance, the concept of consistency is nothing other than the fact that the graph of F is the lower limit of the graphs of the approximate maps F_n, while *stability is the boundedness of the inverses of the derivatives of the maps F_n.*

This provides a first motivation for devising a set-valued differential calculus. For that purpose, we need to begin with the study of tangents.

- Tangents and Normals

The concept of tangency has been overshadowed in some sense by the requirement that the space of tangent vectors must be a vector space, so that the original idea became concealed after its formal implementation in differential geometry.

If we come back to the idea underlying the notion of tangency to a subset K at some point $x \in K$, we are tempted to form *"thick" differential quotients*

$$\frac{K - x}{h}$$

and to take (in various ways) their limits when $h > 0$ goes to 0.

We obtain in this way a variety of closed cones made of what we call tangent vectors. The most popular of these tangent cones is for the time the *contingent cone* introduced in the thirties by Bouligand, (which is the upper limit of these differential quotients). Some of these tangent cones are closed convex cones, and then enjoying a property which is the natural extension of linearity (without subtraction).

These tangent cones possess a pretty rich calculus which justifies their use in many questions, mainly in problems *with state constraints*: They are involved in the sufficient conditions for the existence of an equilibrium and for the stability of solutions to equations with constraints. They also appear in the formulation of necessary conditions in optimization problems with constraints and play a key role in viability theory.

In order to define space of normals, which in differential geometry consists of vectors othogonal to the tangent vector space, we are led to introduce the dual concept of *normal cones* to any subset.

- Differentiation of Maps

We already mentioned that the concept of stability in the Inverse-Function Theorem requires the notion of derivative of a set-valued map, leading to the question: *How to formulate this concept*?

The idea is very simple, and goes back to the prehistory of the differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of tangent to the graph of a function: *The tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \mapsto f'(x)u$.*

It is possible to implement this idea *for any set-valued map F*, since we have introduced a way to implement the tangency for any subset of a normed space. Therefore, in the framework of a given problem, *we can regard a tangent cone to the graph of the set-valued map F at some point (x, y) of its graph as the graph of the associated "derivative" of F at this point (x, y).*

Derivatives built in this way from the various choices of tangent cones are called *graphical derivatives* and the calculus of tangent cones can be transferred to a set-valued differential calculus, including chain rules.

With such derivatives of set-valued maps in our hands, we can *linearize* set-valued problems for approximating them by linearized ones. The latter involving closed convex processes, this strategy provides ways for transferring some properties of linear set-valued maps to nonlinear maps.

- Gradients of Functions and the Fermat Rule

The particular case of real-valued functions deserves a study by itself for taking into account the order relation of real numbers. We are led to do so whenever we look for a minimizer of a function or when we study the monotone behavior of a function along a solution to a differential equation or inclusion (Lyapunov property).

The set-valued approach indicates the route: We associate with a function V the set-valued map $\mathbf{V}_{\uparrow}$ defined by

$$\mathbf{V}_{\uparrow}(x) \;:=\; [V(x), +\infty[$$

whose graph is the *epigraph of* V.

The graphs of the derivatives of such set-valued maps $\mathbf{V}_{\uparrow}$ are the epigraphs of functions which are called *epiderivatives*. We discover that they are close relatives of the directional derivatives introduced by Dini, who was among the first to revolt again the rigidity imposed by the heirs of Cauchy[11].

As far as optimization is concerned, the Fermat Rule can be extended to any function by using these epiderivatives. Since they enjoy a pretty rich calculus, we obtain in this way many necessary conditions for a minimum. This can be done by transferring the set-valued differential calculus to what can be called an *epidifferential calculus*.

By duality, we associate with each of the epiderivatives a concept of *generalized gradient*: It is in general a subset of elements, reduced to the usual gradient whenever the function is differentiable in the usual way. In this framework, the Fermat Rule becomes: *If a point achieves the minimum of a function, then it is an equilibrium of the generalized gradient*, i.e., the generalized gradient at an optimal point contains 0.

- CONVERGENCE OF MAPS

  What about the convergence of a sequence of set-valued maps F_n?

  The first idea which comes to the mind is to extend the various notions of uniform convergence of single-valued maps, regarded as a map from one space to another one.

  Since we know how to deal with limits of sets, it is again natural to use the *graphical approach* and to study the upper and lower limits of graphs.

  We follow this hint and study *graphical upper and lower limits* of a sequence of set-valued maps as the maps whose graphs are the upper and lower limits of the graphs.

[11] Cauchy, however, had the merit to formalize the concept of limits, continuity and differentiability. His definitions have been canonized ever since: A function was allowed to be differentiated only if the differential quotients were converging to the derivative for the pointwise convergence topology. The need to use nondifferentiable functions has been felt several times. By Bouligand, with the notions of contingent and paratingent, by Dini who also break Hadamard's linearity law, by L. Schwartz and S. Sobolev, with the discovery of weak derivatives of functions and distributions. But each of these extensions was devised for specific purposes (solving partial differential equations, for instance).

When we deal with real-valued functions, we are led to use the set-valued maps $\mathbf{V}_{n\uparrow}$ associated with functions V_n, whose graphs are epigraphs of the functions V_n. The graphical limits of the set-valued maps $\mathbf{V}_{n\uparrow}$ induce what we call *epigraphical limits* of the functions V_n. This concept is closely related to G-convergence introduced by de Giorgi and has been extensively used in the study of stability of optimization problems[12].

An interesting question arises: What are the connections between the (epigraphical) convergence of a sequence of functions and the (graphical) convergence of their gradients? We shall answer such questions.

- MEASURES AND INTEGRATION

We encounter measurable maps whenever we deal with models of systems having measurable data, and in particular when we deal with random set-valued variables (an issue we shall not address in this book).

Another important instance where measurable set-valued maps do arise is in the linearization of a control system (or a differential inclusion) along a solution.

Hence, we cannot escape the burden of studying measurable maps, which are the maps whose graphs are measurable, and checking in particular that all the standard operations preserve measurability.

We also need measurability for defining integrals of set-valued maps. Integrals of set-valued maps are involved in many convexification (also called relaxation) problems, since roughly speaking *the integral of a measurable set-valued map is always convex*.

This property was in fact the original motivation to introduce the integral of set-valued maps in mathematical economics and game theory (with a continuum of players).

We shall also address the basic questions of ergodic theory, extending to set-valued maps the Poincaré Recurrence Theorem and the existence of invariant measures.

- DIFFERENTIAL INCLUSIONS

Control Theory on one hand, and the evolution of macro-systems under uncertainty on the other hand, constitute very strong motivations for extending dif-

[12]See the forthcoming book SET-VALUED ANALYSIS AND SUBDIFFERENTIAL CALCULUS [?] by Rockafellar and Wets.

ferential and partial differential equations to differential and partial differential inclusions.

We shall provide only an introductory survey of some basic results, since covering this material requires books by themselves[13].

We shall state some existence theorems, show that the set of solutions depends continuously upon the initial data, describe some properties of Lyapunov functions (which, thanks to the concept of epiderivatives, can even be taken lower semicontinuous), relate the graphical derivatives of the solution map to the solution map of variational inclusions (which are linearizations of the differential inclusion along a solution) and state some applications of the Viability Theorem.

- SELECTIONS AND PARAMETRIZATION

  We cannot escape *in fine* answering two natural questions: Can we find selections of set-valued maps inheriting their regularity properties? Are set-valued maps *parametrizable*?

  We shall be able to show that measurable set-valued maps do have measurable selections and that continuous (Carathéodory, Lipschitz) maps do have continuous (Carathéodory, Lipschitz) selections under severe restrictions: The images of the set-valued map must be convex.

  Actually, a Lipschitz set-valued map F with closed convex images is parametrizable in the sense that there exists a "control space" U and a Lipschitz map $f : X \times U \mapsto X$ such that

$$\forall\, x, \;\; F(x) \;=\; \{f(x,u)\}_{u\in U}$$

  Therefore, this limited class of set-valued maps is in essence made of families of single-valued maps.

[13] See for instances the books DIFFERENTIAL INCLUSIONS [?] by Aubin & Cellina, VIABILITY THEORY [?] by Aubin and SET-VALUED ANALYSIS AND CONTROL THEORY [?] by Frankowska.